FINDING LIE SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS WITH `MATHEMATICA`®: APPLICATIONS TO NONLINEAR FIBER OPTICS

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Abstract. A `MATHEMATICA`® package for finding Lie symmetries of partial differential equations is presented. The package has been applied to perform a full Lie group analysis of basic models of nonlinear fiber optics. As a result of this group invariant solutions have been obtained. Comparisons with earlier published computer algebra implementations of the Lie group method are discussed.

1. Introduction

The Lie group method for establishing the transformations leaving a system of partial differential equations (PDEs) invariant can be found in many books on this subject [8, 11, 12]. The key to finding a Lie group of symmetry transformations is the infinitesimal generator of the group. In order to provide a bases of group generators one has to create and then to solve the so called determining system of equations (DSEs). The operations are straightforward but nonetheless formidably tedious to be done by hand. It is very frequent occurrence that hundreds of equations are manipulated when PDEs of order higher than two are considered and the independent variables are more than about two. In situations like this it is essential in our days the use of a contemporary computer algebra system, such as `Reduce`, `MATHEMATICA`®, `Maple`, etc.

The aim of this paper is to present a computer algebra implementation of the Lie method – the MATHEMATICA® package LieSymm-PDE. The package is designed to create and solve the DSEs of an arbitrary number of simultaneous PDEs. It works without any restrictions on the number of the equations, on the number of the variables, either independent, or dependent, and on the highest order of the derivatives that may be involved. To the authors knowledge other programs related to Lie symmetries have been developed in various packages like Reduce [16], MATHEMATICA® [3], Maple®. The algorithm of LieSymm-PDE (Maple®10 standard release) for solving the DSEs is closely related to the solving technique of [16].

2. Finding Lie Groups of PDEs: Formulation of the Problem

Following the terms and notations in [11] we give a brief outline of the basic concepts of the Lie theory. Let be given a system of PDEs in p independent $x = (x^1, \ldots, x^p) \in X \equiv \mathbb{R}^p$ and q dependent variables $u = (u^1, \ldots, u^q) \in U \equiv \mathbb{R}^q$ involving derivatives up to order n

$$F_m(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(n)}) = 0, \qquad m = 1, 2, \ldots, l \tag{1}$$

where the notation $u^{(s)}$ stands for a vector in the Euclidean space $U^{(s)}$ having as coordinates the derivatives $u^\alpha_{j_1 \ldots j_s} \equiv \partial u^\alpha / \partial x^{j_1} \ldots \partial x^{j_s}$, $s = 1, \ldots, n$, $\alpha = 1, \ldots, q$, $j_\nu = 1, \ldots, p$, $\nu = 1, \ldots, s$. It is said that the system (1) admits a one-parameter local Lie group of point-symmetry transformations of the space $Z = X \times U$

$$\begin{aligned} x' &= f(a, x, u) \\ u' &= \varphi(a, x, u) \end{aligned} \tag{2}$$

(a is the group parameter, $a \in \Delta \subset \mathbb{R}$, $0 \in \Delta$), if each solution after the transformation of the group remains a solution of the system. Finding the admitted Lie groups of PDEs is based on the fundamental correspondence between the Lie groups and their Lie algebras of infinitesimal generators

$$V = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \tag{3}$$

with coefficients $\xi^i(x, u) = \partial f^i(0, x, u)/\partial a$, $\eta^\alpha(x, u) = \partial \varphi^\alpha(0, x, u)/\partial a$, $f = (f^1, \ldots, f^p)$, $\varphi = (\varphi^1, \ldots, \varphi^q)$. From a geometrical point of view V is a tangent vector field on Z, which flow coincides with a one-parameter group of transformations.

The milestone of the Lie method is the infinitesimal criterion which is based on a special technique for prolongation of the groups and their infinitesimal generators.

The system of PDEs (1) is viewed as a sub-manifold Δ_F in the prolonged space $Z^{(n)} = Z \times U^{(1)} \times \cdots \times U^{(n)}$

$$\Delta_F = \left\{ z^{(n)} \in Z^{(n)} \,;\, F_m(z^{(n)}) = 0,\, m = 1, 2, \ldots, l \right\} \subset Z^{(n)}. \qquad (4)$$

If the rank of the Jacobi matrix of $F(z^{(n)}) \equiv (F_1(z^{(n)}), \ldots, F_l(z^{(n)}))$ is assumed to be l whenever the point $z^{(n)}$ belongs to the sub-manifold Δ_F, then the system (1) admits a one-parameter group of transformations (2) with the infinitesimal generator V if and only if the following infinitesimal condition holds

$$\mathrm{pr}^{(n)} V \left[F(z^{(n)}) \right] = 0 \qquad \text{for} \qquad z^{(n)} \in \Delta_F \qquad (5)$$

where

$$\mathrm{pr}^{(n)} V = V + \sum_{i=1}^{p} \sum_{\alpha=1}^{q} \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \sum_{j_1=1}^{p} \cdots \sum_{j_n=1}^{p} \sum_{\alpha=1}^{q} \zeta_{j_1 \ldots j_n}^\alpha \frac{\partial}{\partial u_{j_1 \ldots j_n}^\alpha} \qquad (6)$$

is the n-th prolongation of the infinitesimal generator V. The coefficients $\zeta_{j_1 \ldots j_k}^\alpha$, $k = 1, \ldots, n$ depend on the functions $\xi(x, u)$, $\eta(x, u)$ and can be obtained by the recursive formulae

$$\zeta_i^\alpha = D_i(\eta^\alpha) - \sum_{s=1}^{p} u_s^\alpha D_i(\xi^s)$$

$$\zeta_{j_1 \ldots j_k}^\alpha = D_{j_k}(\zeta_{j_1 \ldots j_{k-1}}^\alpha) - \sum_{s=1}^{p} u_{j_1 \ldots j_{k-1} s}^\alpha D_{j_k}(\xi^s) \qquad (7)$$

where D_i is the operator of total differentiation with respect to the variable x^i

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} u_i^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{j=1}^{p} \sum_{\alpha=1}^{q} u_{ji}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots$$

$$+ \sum_{j_1=1}^{p} \cdots \sum_{j_{n-1}=1}^{p} \sum_{\alpha=1}^{q} u_{j_1 \ldots j_{n-1} i}^\alpha \frac{\partial}{\partial u_{j_1 \ldots j_{n-1}}^\alpha} \cdot \qquad (8)$$

Since the variables x^i, u^α, $u_{j_1 \ldots j_s}^\alpha$ are supposed to be independent, the equation (5) can be facilitated by equating to zero all the coefficients of the monomials in the partial derivatives $u_{j_1 \ldots j_s}^\alpha$. Thus, a large number of linear homogeneous partial differential equations are obtained. They are known as the DSEs of the symmetry group admitted by (1) for they serve to determine the unknown coefficients $\xi^i(x, u)$, $\eta^\alpha(x, u)$ of the respective group generator. The solutions of the DSEs constitute the widest admitted Lie algebra.

3. Algorithm of the Package `LieSymm-PDE`

The algorithm of the package (see Fig. 1) follows strictly the theoretical formulae in the preceding section. It consists of the following steps:

i) Basic Setup. In full accordance with the definitions (3), (4), (6)–(8) by using the data input some basic symbolic expressions, rules and operators are generated. They include the submanifold Δ_F represented by a list of rules, the prolonged group generator $\mathrm{pr}^{(n)}V$ defined as an operator and the coefficients $\xi^i(x, u)$ and $\eta^\alpha(x, u)$ of the group generator determined as two lists of $p + q$ arbitrary functions.

ii) Determining Equations. The prolonged infinitesimal generator $\mathrm{pr}^{(n)}V$ is applied to the functions $F_1, F_2, \ldots, F_l$. Then the resultant expressions are recalculated on the submanifold Δ_F and all the coefficients of the monomials in $u_{j_1 \ldots j_s}^\alpha$ are equated to zero. It means that the infinitesimal criterion (5) is completed and the determining equations are created.

iii) Solving procedure. An automatic procedure for solving of the DSEs is carried out. It is based on a repetition of several programming modules capable of solving some distinct types of equations with known solutions

$$C_1 x + C_2 = 0, \qquad C_1 x + C_2 y = 0$$
$$C_1 y_x + C_2 = 0, \qquad C_1 y_{xx} + C_2 = 0, \qquad C_1 y_{xxx} + C_2 = 0 \tag{9}$$

where C_1 and C_2 are arbitrary constants, $y_x \equiv \partial y / \partial x$, etc. If any such equation does exist in the list of the DSEs, its solution is substituted for the respective variable in the remainder of the equations. As a result the functions $\xi^i(x, u)$ and $\eta^\alpha(x, u)$ change getting closer to the exact explicit solution and the number of the equations in the DSEs diminishes. The solving process is completed when, either the number of the determining equations has been reduced to zero, or all of the remaining equations have become unsolvable by the existing modules. In this latter case the solution $(\xi^1, \ldots, \xi^p, \eta^1, \ldots, \eta^q)$ generated at the package output is expressed in terms of some unknown functions satisfying certain differential and algebraic equations. If this happens, two additional programming tools named `Rules` and `Hints` (see Fig. 1) are available in the package providing a possibility for the user to solve these equations in a partly automatic way. `Rules` collects together special modules for making transformations such as for adding, subtracting, and differentiating of equations. One special module is designated to carry out a search for functionally independent parts of the equations that after being equated to zero are added to the list of the DSEs. `Hints` is a list of substitutions specifying the functions being sought.
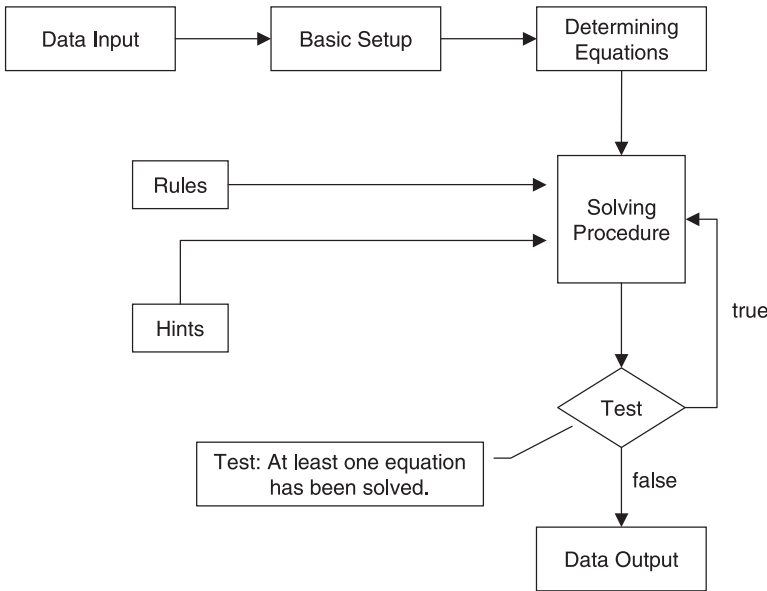
Figure 1. Algorithm's flowchart.

4. Using the Tools of **LieSymm-PDE**

Following standard MATHEMATICA® conventions [19] the function that finds the solution of the determining system is named **LieInfGen**. LieInfGen[{lhs1, lhs2, ...}, {rhs1, rhs2, ...}, {iv1, iv2, ...}, {dv1, dv2, ...}] gives the coefficients $\xi^i(x,u)$, $\eta^\alpha(x,u)$ of the infinitesimal generator (3) admitted by the system of PDEs lhs1 = rhs1, lhs2 = rhs2, ... with independent variables iv1, iv2, ... and dependent variables dv1, dv2, ... The package displays an usage message that tells the user all that is needed to execute the program. For instance, the original equations must be solved in regard to either one independent or dependent variable, or any of the derivatives, and then these single variables must be substituted for the left-hand sides of the equations lhsi. The message also explains that the derivatives must be typed as dvi [ivj, ivk, ...], which means the derivative of the i-th dependent variable in regard to the independent variables ivj, ivk, ... Notice that the package contains private context specification, which protects the objects from getting confused with other objects defined outside the package and having the same names.

If the functions generated at the package output are not in their full explicit form the user is advised to proceed with applying the package in interactive mode. This

mode gives a possibility to the user to effectively participate in the solving process by giving hints to the solutions and by using some user-level commands for making transformations of the determining equations. This is needed in view of the fact that no general solution scheme of the DSEs has been known yet. First, by using the command CreateDSE[{lhs1, lhs2, ...}, {rhs1, rhs2, ...}, {iv1, iv2, ...}, {dv1, dv2, ...}] the determining equations are being created, and second, the solving process is started up by applying the iterative function SolveDSE. By using this command special solving modules are applied repeatedly in sequence to determining equations in order to identify and solve those of them that match any of the pre-defined types of equations (9). There are also two commands **DetSysEqs** and **LieInfGen** used to display any current state of the DSEs and its solution.

In most cases the coefficients of the infinitesimal generator are expressed in terms of some unknown functions. These functions must satisfy certain differential equations that are not handled by the package modules. Instead of trying to solve them by hand the user can take advantage of the additional tools of LieSymm−PDE – the commands **SplitDSE**[ ], **DiffDSE**[ ], **AddDSE**[ ]. They provide automatic equivalent transformations of the DSEs that are, respectively, for splitting up of polynomials to functionally independent terms, for differentiating of equations, for adding and subtracting of pairs of equations.

It is very frequent occurrence that the solving modules and the transformation rules available by the package are not enough to solve all of the determining equations. In cases like this it suffices that the user could derive some additional information from the returned equations that to be fed back as hints to the solving modules. This is achieved by the special command **Hints**[{subs1, subs2, ...}, {newfun1, newfun2, ...}], which input consists of a list of substitutions specifying some of the undefined functions by other functions – those given in the second curly brackets, that are considered by LieSymm−PDE as new functions to be determined. By following this semi-automatic way of giving hints and applying transformation rules the DSEs is completely solved.

5. Application of **LieSymm−PDE** to Basic Models of Nonlinear Fiber Optics

We are going to present the results of the package application to a) equation describing light pulses propagation in single-mode nonlinear fibers at zero-dispersion wavelength [1]

$$\mathrm{i}A_x + \frac{1}{2}A_{tt} + |A|^2 A = \mathrm{i}\beta A_{ttt} \tag{10}$$

($\beta = $ const), and b) two coupled nonlinear Schrödinger equations (CNSEs)

$$iA_x + \frac{1}{2}A_{tt} + (|A|^2 + \gamma|B|^2 - \theta(|A|^2)_t - \theta(|B|^2)_t)A + kB = 0$$
$$iB_x + \frac{1}{2}B_{tt} + (\gamma|A|^2 + |B|^2 - \theta(|A|^2)_t - \theta(|B|^2)_t)B + kA = 0 \tag{11}$$

which are the basic mathematical model of two polarization modes propagating in weak [10] ($k \neq 0$, $\gamma \neq 0$) and strong [1] ($k = 0$, $\gamma = 2/3$) birefringent fibers (WBF and SBF), of two waves at different carrier wavelengths in two-mode fibers (TMF) [6] ($k = 0$, $\gamma = 2$), and of nonlinear directional couplers (NLDC) [18] ($k \neq 0$, $\gamma = 0$). The terms with the parameter θ account for the parallel Raman gain [7]. The functions $A(x, t)$ and $B(x, t)$ represent the normalized electric field components depending on the dimensionless time t and the longitudinal coordinate x.

The admitted Lie symmetries of equations (10) and (11) that we found by the help of the package `LieSymm-PDE` were used to prepare a full Lie group analysis of all physically relevant cases. It means that the optimal set of one-dimensional subalgebras and the corresponding optimal set of ordinary reduced differential equations (RDEs) have been obtained. In all of the considered cases the determining equations were solved automatically by applying the `LieSymm-PDE` tools for making equivalent transformations. There were no needs of giving hints.

a) Pulse propagation at zero-dispersion wavelength. The DSEs consists of 94 equations, which solution reveals that the equation (10) admits a four-dimensional Lie algebra with the following bases of group generators

$$V_1 = \partial_t, \; V_2 = \partial_x, \; V_3 = \partial_\varphi, \; V_4 = (x - 6\beta t)\partial_t - 18\beta x\partial_x + t\partial_\varphi + 9\beta z\partial_z \tag{12}$$

where $A = ze^{i\varphi}$; $\partial_t \equiv \partial/\partial t$, etc. The corresponding one-parameter Lie groups of transformations are: for V_1, $t' = t + a_1$, for V_2, $x' = x + a_2$, for V_3, $\varphi' = \varphi + a_3$, and for V_4

$$t' = te^{-6\beta a_4} + \frac{x}{12\beta}e^{-6\beta a_4} - \frac{x}{12\beta}e^{-18\beta a_4}$$
$$x' = xe^{-18\beta a_4}$$
$$\varphi' = \varphi - \frac{t}{6\beta}e^{-6\beta a_4} - \frac{x}{72\beta^2}e^{-6\beta a_4} + \frac{x}{216\beta^2}e^{-18\beta a_4}$$
$$z' = ze^{9\beta a_4}.$$

The inner automorphisms $A_i(V_j) \equiv \mathrm{Ad}(\exp(\varepsilon V_i))V_j$, $\varepsilon \in \mathbb{R}$, generated by the basic vectors V_i are acting on V_j according to

$$A_1(V_j) = V_j, \qquad j = 1, 2, 3, \qquad A_1(V_4) = V_4 + 6\beta\varepsilon V_1 - \varepsilon V_3$$
$$A_2(V_j) = V_j, \qquad j = 1, 2, 3, \qquad A_2(V_4) = V_4 - \varepsilon V_1 + 18\beta\varepsilon V_2$$
$$A_3(V_j) = V_j, \qquad j = 1, 2, 3, 4 \tag{13}$$
$$A_4(V_1) = V_1 \mathrm{e}^{-6\beta\varepsilon} + \frac{V_3}{6\beta}(1 - \mathrm{e}^{-6\beta\varepsilon})$$
$$A_4(V_2) = \frac{V_1}{12\beta}\left(\mathrm{e}^{-6\beta\varepsilon} - \mathrm{e}^{-18\beta\varepsilon}\right) + V_2 \mathrm{e}^{-18\beta\varepsilon} + \frac{V_3}{216\beta^2}\left(\mathrm{e}^{-18\beta\varepsilon} - 3\mathrm{e}^{-6\beta\varepsilon} + 2\right)$$
$$A_4(V_j) = V_j, \qquad j = 3, 4.$$

Each one of the Lie group generators (12) and their various linear combinations can be useful for yielding group invariant solutions. For this purpose group invariant quantities are substituted for the independent and the dependent variables so that a simpler system of ordinary RDEs is obtained. The adjoint representations (13) allow introducing a conjugate relation in the set of all subalgebras of the same dimension, which leads to a classification of all cases of reduction. By taking one representative from each family of conjugate subalgebras an optimal set of subalgebras is created. We built up the optimal set of one-dimensional subalgebras and the corresponding optimal set of RDEs, which we present here in three unified cases by using two auxiliary parameters ε and δ.

Case A. The subalgebras of this case are represented by the group generators $\delta V_3 + V_4$, $\delta \in \mathbb{R}$. They lead to the invariant solutions

$$A(x, t) = \sqrt{\frac{p(y)}{x}} \exp \mathrm{i}\left\{f(y) - \frac{t}{6\beta} - \frac{x}{108\beta^2} - \frac{\delta \ln|x|}{18\beta}\right\}$$
$$y = tx^{-1/3} + \frac{x^{2/3}}{12\beta}$$

and the RDEs for the unknown functions $p(y)$ and $f(y)$ (here and hereafter prime denotes differentiation)

$$12\beta p^2 p''' - 18\beta pp'p'' + 9\beta(p')^3 - 36\beta p^2 p'(f')^2 + 4yp^2 p'$$
$$- 72\beta p^3 f'f'' + 12p^3 = 0$$
$$36\beta^2 p^2 f''' + 54\beta^2 pp'f'' - 36\beta^2 p^2(f')^3 + 54\beta^2 pp''f' - 27\beta^2 f'(p')^2$$
$$+ 12\beta yp^2 f' + 36\beta p^3 + 2\delta p^2 = 0.$$

Case B. The subalgebras representatives are $V_1 + \varepsilon V_2 + \delta V_3$ with $\varepsilon, \delta \in \mathbb{R}$. They imply the group invariant solutions $A = p(y) \exp \mathrm{i}\{f(y) + \delta t\}$, $y = \varepsilon t - x$. The

new functions $p(y)$ and $f(y)$ satisfy the RDEs

$$2\varepsilon^3\beta p''' - 6\varepsilon^3\beta p'(f')^2 - 2\varepsilon^2(6\delta\beta + 1)p'f' - 2(3\varepsilon\delta^2\beta + \varepsilon\delta - 1)p'$$
$$- 6\varepsilon^3\beta pf'f'' - \varepsilon^2(6\delta\beta + 1)pf'' = 0$$

$$2\varepsilon^3\beta pf''' + 6\varepsilon^3\beta p'f'' - 2\varepsilon^3\beta p(f')^3 - \varepsilon^2(6\delta\beta + 1)p(f')^2 + 6\varepsilon^3\beta p''f'$$
$$- 2(3\varepsilon\delta^2\beta + \varepsilon\delta - 1)pf' + \varepsilon^2(6\delta\beta + 1)p'' + 2p^3 - \delta^2(2\delta\beta + 1)p = 0.$$

Case C. The set of subalgebras in this case is given by $V_2 + \delta V_3$, $\delta \in \mathbb{R}$. The corresponding invariant solutions $A = p(t)\exp i\{f(t) + \delta x\}$ depend on the functions $p(t)$ and $f(t)$ through the equations

$$2\beta p''' - 6\beta p'(f')^2 - 2p'f' - pf'' - 6\beta pf'f'' = 0$$
$$2\beta pf''' + 6\beta p'f'' - 2\beta p(f')^3 - p(f')^2 + 6\beta p''f' + p'' + 2p^3 - 2\delta p = 0.$$

There are not invariant solutions related to the subalgebra V_3.

b) Pulse propagation governed by two CNSEs. If the Raman terms in (11) are not taken into account ($\theta = 0$) the DSEs consists of 139 equations but when $\theta \neq 0$ this number increases to 173 equations. `LieSymm-PDE` gives the solution for each one of the considered cases: for TMF and SBF with $\theta = 0$

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi$$
$$V_5 = x\partial_t + t(\partial_\phi + \partial_\psi), \qquad V_6 = -t\partial_t - 2x\partial_x + z\partial_z + \zeta\partial_\zeta \tag{14}$$

for WBF and NLDC with $\theta = 0$

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi + \partial_\psi, \qquad V_4 = x\partial_t + t(\partial_\phi + \partial_\psi) \tag{15}$$

and for SBF with parallel Raman gain ($\theta \neq 0$)

$$V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = \partial_\phi, \quad V_4 = \partial_\psi, \quad V_5 = x\partial_t + t(\partial_\phi + \partial_\psi) \tag{16}$$

where we have used the notation $A = ze^{i\phi}$, $B = \zeta e^{i\psi}$. The operators (15) coincide with those obtained in [2] by the help of the symbolic computer language `Reduce`. The infinitesimal generators (14), (15) and (16) were applied to perform a full Lie symmetry classification of one-parameter group invariant solutions to all physically relevant cases [13–15]. Most of the exact solutions found in literature for WBF are invariant for some of the subgroups having the generators $V_2 + \delta V_3 = \partial_x + \delta(\partial_\phi + \partial_\psi)$, $\delta \in \mathbb{R}$ (see, e.g., [5,9,17]). Here we present a stationary solution valid for NLDC

$$A = \sqrt{\frac{E + U(x)}{2}}\exp i\left\{\frac{3E}{4}x - \frac{\Psi(x)}{2}\right\}$$

$$B = \sqrt{\frac{E - U(x)}{2}}\exp i\left\{\frac{3E}{4}x + \frac{\Psi(x)}{2}\right\} \tag{17}$$

with the functions $U(x)$ and $\Psi(x)$ defined, either by

$$U(x) = E\operatorname{cn}(2kx|h^2), \qquad \Psi(x) = \arcsin(\operatorname{dn}(2kx|h^2)), \qquad h \leq 1$$

or by

$$U(x) = E\operatorname{dn}(2khx|1/h^2), \qquad \Psi(x) = \arcsin(\operatorname{cn}(2khx|1/h^2)), \qquad h \geq 1$$

where $\operatorname{cn}(\cdot|m)$, $\operatorname{dn}(\cdot|m)$ are the Jacobian elliptic functions with parameter m, $h = E/4k$, $E = \text{const}$. We obtained the solution (17) as a result of the symmetry reduction process that lead us to the RDEs

$$p' + kq\sin{(g-f)} = 0, \qquad f' = p^2 + k\frac{q}{p}\cos(g-f)$$

$$q' + kp\sin{(f-g)} = 0, \qquad g' = q^2 + k\frac{p}{q}\cos(f-g)$$

satisfied by the functions $p(x)$, $q(x)$, $f(x)$ and $g(x)$ related to the original unknown functions through the substitutions $A = p(x)\exp\mathrm{i}\{f(x)\}$, $B = q(x)\exp\mathrm{i}\{g(x)\}$.

6. Discussions and Conclusion

The MATHEMATICA® package LieSymm-PDE has been presented. The package is developed for automatic determination of Lie point symmetries of PDEs, either directly in one step, or by taking advantage of an elaborate interactive mode. In comparison with the MATHEMATICA® program in [4] the package described here does not require a polynomial ansatz for the infinitesimals and needs less external advice (hints) to fulfill the task. We compared the functions of LieSymm-PDE with those available by the package liesymm of Maple®. We revealed that the LieSymm-PDE function CreateDSE for creating of the DSEs can be used as an alternative of the Maple® command liesymm[determine](). We found also that Maple® does not provide special tools for solving of the DSEs as it is done by LieSymm-PDE.

The method of LieSymm-PDE for solving DSEs is based on several programming modules for dealing with some pre-determined types of equations. This method is generally allied with the approach applied in the Reduce package [16]. Finally, it should be noted that LieSymm-PDE is open to adding new solving modules and transformation rules so that its capabilities can be constantly enhanced. This leads to reducing the needs of user's hints and makes the program flexible and self-contained. As a result new larger and more complicated systems of PDEs become manageable. The package has been tested to a large number of PDEs with known symmetries and has been successfully applied to different models of nonlinear fiber optics. All these prove the effectiveness of the package LieSymm-PDE in solving practical problems and justify this presentation.

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