Eighth International Conference on
Geometry, Integrability and Quantization
June 9–14, 2006, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2007, pp 264–278

OLD AND NEW STRUCTURES ON THE TANGENT BUNDLE

MARIAN IOAN MUNTEANU

*"Al. I. Cuza" University of Iaşi, Faculty of Mathematics, Blvd. Carol I, n. 11
700506 Iaşi, Romania*

Abstract. In this paper we study a Riemanian metric on the tangent bundle $T(M)$ of a Riemannian manifold M which generalizes Sasakian metric and Cheeger–Gromoll metric along a compatible almost complex structure which together with the metric confers to $T(M)$ a structure of locally conformal almost Kählerian manifold. This is the natural generalization of the well known almost Kählerian structure on $T(M)$. We found conditions under which $T(M)$ is almost Kählerian, locally conformal Kählerian or Kählerian or when $T(M)$ has constant sectional curvature or constant scalar curvature.

1. A Brief History

A Riemannian metric g on a smooth manifold M gives rise to several Riemannian metrics on the tangent bundle $T(M)$ of M. Maybe the best known example is the Sasakian metric g_S introduced in [18]. Although the Sasakian metric is *naturally* defined, it is very *rigid* in the following sense. For example, Kowalski [11] has shown that the tangent bundle $T(M)$ with the Sasakian metric is never locally symmetric unless the metric g on the base manifold is flat. Then, Musso and Tricerri [13] have proved a more general result, namely, that the Sasakian metric has constant scalar curvature if and only if (M, g) is locally Euclidean. In the same paper, they have given in explicit form a positive definite Riemannian metric introduced by Cheeger and Gromoll [9] and called this metric *the Cheeger–Gromoll metric*. In [19] Sekizawa computed the Levi-Civita connection, the curvature tensor, the sectional curvatures and the scalar curvature of this metric. These results are completed in 2002 by Gudmundson and Kappos [10]. They have also shown that the scalar curvature of the Cheeger–Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, Abbassi and Sarih have proved that $T(M)$ with the Cheeger–Gromoll metric is never a space of constant sectional curvature (cf. [2]). A more general metric is given by

Anastasiei [6] which generalizes both of the two metrics mentioned above in the following sense: it preserves the orthogonality of the two distributions, on the horizontal distribution it is the same as on the base manifold, and finally the Sasakian and the Cheeger–Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and hence $T(M)$ becomes a locally conformal almost Kählerian manifold.

Oproiu and his collaborators constructed a family of Riemannian metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties (cf. [14–17]). In particular, the scalar curvature of $T(M)$ can be constant also for a non-flat base manifold with constant sectional curvature. Then Abbassi and Sarih [3] proved that the metrics considered by Oproiu form a particular subclass of the so-called *g-natural metrics* on the tangent bundle (see also [1,3–5,12]).

2. Introduction

By thinking of $T(M)$ as a vector bundle associated with $O(M)$ (the space of orthonormal frames on M), namely $T(M) \equiv O(M) \times \mathbb{R}^n/\mathrm{O}(n)$ (where the orthogonal group $\mathrm{O}(n)$ acts on the right on $O(M)$), Musso and Tricerri construct some natural metrics on $T(M)$ (see [13, §4]). The idea is to consider a symmetric, semi-positive definite tensor field Q, of type $(2,0)$ and rank $2n$ on $O(M) \times \mathbb{R}^n$. Assuming that Q is *basic* for $\psi : O(M) \times \mathbb{R}^n \longrightarrow T(M)$, $(\mathbf{u}, \zeta) \mapsto (p, \zeta^i \mathrm{u}_i)$, where $\mathbf{u} = (p, \mathrm{u}_1, \dots, \mathrm{u}_n)$ and $\zeta = (\zeta^1, \dots, \zeta^n)$ (i.e., Q is $\mathrm{O}(n)$-invariant and $Q(X,Y) = 0$ for all X tangent to a fiber of ψ) there is a unique Riemannian metric g_Q on $T(M)$ such that $\psi^* g_Q = Q$. In this paper we will show that the metric introduced in [6] can be constructed by using the method of Musso and Tricerri and we study it. After a compatible almost complex structure is introduced, we give the conditions under which $T(M)$ is almost Kählerian (Theorem 1). We also obtain a locally conformal Kähler structure on $T(M)$ (cf. Example 2) and Kähler structures on portions of $T(M)$ (cf. Theorem 2). These results extend the known result saying that $T(M)$ endowed with the Sasakian metric and the canonical almost complex structure is Kählerian if and only if the base manifold is locally Euclidean.

Next we want to have constant sectional curvature and respectively constant scalar curvature on $T(M)$. With this end in view, we compute the Levi-Civita connection, the curvature tensor, the sectional curvature and the scalar curvature of this metric. We found relations between the sectional curvature (respectively scalar curvature) on $T(M)$ and the corresponding curvature on the base M. We give an example of metric on $T(M)$ of Cheeger–Gromoll type which is flat. (Recall the fact that Cheeger–Gromoll metric can not have constant sectional curvature.) See also Proposition 6. We give some examples of metrics on $T(M)$ (when M is a space form) having constant scalar curvature. See Examples 3, 4 and 5.

3. On the Geometry of the Tangent Bundle $T(M)$

Let (M, g) be a Riemannian manifold and let ∇ be its Levi-Civita connection. Let $\tau : T(M) \longrightarrow M$ be its tangent bundle. If $\mathrm{u} \in T(M)$ it is well known that we have the following decomposition of the tangent space $T_{\mathrm{u}}T(M)$

$$T_{\mathrm{u}}T(M) = V_{\mathrm{u}}T(M) \oplus H_{\mathrm{u}}T(M)$$

where $V_{\mathrm{u}}T(M) = \ker \tau_{*,\mathrm{u}}$ is the vertical space and $H_{\mathrm{u}}T(M)$ is the horizontal space obtained by using ∇. (A curve $\widetilde{\gamma} : I \longrightarrow T(M)$, $t \mapsto (\gamma(t), V(t))$ is *horizontal* if the vector field $V(t)$ is parallel along $\gamma = \widetilde{\gamma} \circ \tau$. A vector field on $T(M)$ is *horizontal* if it is tangent to a horizontal curve and *vertical* if it is tangent to a fiber. Locally, if (U, x^i), $i = 1, \ldots, m$, where $m = \dim M$, is a local chart at $p \in M$, consider a local chart $(\tau^{-1}(U), x^i, y^i)$ on $T(M)$. If $\Gamma^k_{ij}(x)$ are the Christoffel symbols, then $\delta_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k}$ at u, $i = 1, \ldots, m$ span the space $H_{\mathrm{u}}T(M)$, while $\frac{\partial}{\partial y^i}$, $i = 1, \ldots, m$ span the vertical space $V_{\mathrm{u}}T(M)$.) We have obtained the horizontal (vertical) distribution HTM (VTM) and a direct sum decomposition

$$TTM = HTM \oplus VTM$$

of the tangent bundle of $T(M)$. If $X \in \chi(M)$, denote by X^H (and respectively X^V) the horizontal (vertical) lift of X to $T(M)$.

If $\mathrm{u} \in T(M)$ then we consider the energy density at u on $T(M)$, namely

$$t = \frac{1}{2}\, g_{\tau(\mathrm{u})}(\mathrm{u}, \mathrm{u}).$$

3.1. The Sasakian Structure

The **Sasakian metric** is defined uniquely by the following relations

$$g_S(X^H, Y^H) = g_S(X^V, Y^V) = g(X, Y) \circ \tau, \quad g_S(X^H, Y^V) = 0 \tag{1}$$

for each $X, Y \in \chi(M)$.

On $T(M)$ we also define an almost complex structure J_S by

$$J_S X^H = X^V, \quad J_S X^V = -X^H \quad \text{for all } X \in \chi(M). \tag{2}$$

It is known that $(T(M), J_S, g_S)$ is an almost Kählerian manifold. Moreover, the integrability of the almost complex structure J_S implies that (M, g) is locally flat (see, e.g., [7]).

3.2. The Cheeger–Gromoll Structure

The **Cheeger–Gromoll metric** on $T(M)$ is given by

$$\begin{aligned} g_{CG(p,\mathrm{u})}(X^H,Y^H) &= g_p(X,Y), \quad g_{CG(p,\mathrm{u})}(X^H,Y^V)=0 \\ g_{CG(p,\mathrm{u})}(X^V,Y^V) &= \frac{1}{1+2t}\,\left(g_p(X,Y)+g_p(X,\mathrm{u})g_p(Y,\mathrm{u})\right) \end{aligned} \tag{3}$$

for any vectors X and Y tangent to M.

Since the almost complex structure J_S is no longer compatible with the metric g_{CG}, one defines on $T(M)$ another almost complex structure J_{CG}, compatible with the Chegeer–Gromoll metric, by the formulas

$$\begin{aligned} J_{CG}X^H_{(p,\mathrm{u})} &= \mathfrak{r}X^V - \frac{1}{1+\mathfrak{r}}\,g_p(X,\mathrm{u})\mathrm{u}^V \\ J_{CG}X^V_{(p,\mathrm{u})} &= -\frac{1}{\mathfrak{r}}\,X^H - \frac{1}{\mathfrak{r}(1+\mathfrak{r})}\,g_p(X,\mathrm{u})\mathrm{u}^H \end{aligned} \tag{4}$$

where $\mathfrak{r}=\sqrt{1+2t}$ and $X\in T_p(M)$. Remark that $J_{CG}\mathrm{u}^H=\mathrm{u}^V$ and $J_{CG}\mathrm{u}^V=-\mathrm{u}^H$. We get an almost Hermitian manifold $(T(M),J_{CG},g_{CG})$. Moreover, if we denote by Ω_{CG} the Kähler two-form (namely $\Omega_{CG}(U,V)=g_{CG}(U,J_{CG}V)$, for all $U,V\in\chi(T(M))$) it is quite easy to prove the following

Proposition 1. We have

$$\mathrm{d}\Omega_{CG}=\omega\wedge\Omega_{CG} \tag{5}$$

where $\omega\in\Lambda^1(T(M))$ is defined by

$$\omega_{(p,\mathrm{u})}(X^H)=0 \quad\text{and}\quad \omega_{(p,\mathrm{u})}(X^V)=-\left(\frac{1}{\mathfrak{r}^2}+\frac{1}{1+\mathfrak{r}}\right)g_p(X,\mathrm{u}),\ X\in T_p(M).$$

Proof: A simple computation gives

$$\begin{aligned} \Omega_{CG}(X^H,Y^H) &= \Omega(X^V,Y^V)=0 \\ \Omega_{CG}(X^H,Y^V) &= -\frac{1}{\mathfrak{r}}\left(g(X,Y)+\frac{1}{1+\mathfrak{r}}\,g(X,\mathrm{u})g(Y,\mathrm{u})\right). \end{aligned}$$

(From now on we will omit the point (p,u).)

The differential of Ω_{CG} is given by

$$\begin{aligned} &\mathrm{d}\Omega_{CG}(X^H,Y^H,Z^H)=\mathrm{d}\Omega_{CG}(X^H,Y^H,Z^V)=\mathrm{d}\Omega_{CG}(X^V,Y^V,Z^V)=0 \\ &\mathrm{d}\Omega_{CG}(X^H,Y^V,Z^V)=\frac{1}{\mathfrak{r}}\left(\frac{1}{\mathfrak{r}^2}+\frac{1}{1+\mathfrak{r}}\right)[g(X,Y)g(Z,\mathrm{u})-g(X,Z)g(Y,\mathrm{u})] \end{aligned}$$

for any $X,Y,Z\in\chi(M)$.

Hence the statement. □

Remark 1. The almost Hermitian manifold $(T(M), J_{CG}, g_{CG})$ is never almost Kählerian (i.e., $\mathrm{d}\Omega_{CG} \neq 0$).

Finally, a necessary condition for the integrability of J_{CG} is that the base manifold (M, g) is locally Euclidian.

3.3. The General Structure

A general metric, let us call it g_A, is in fact a family of Riemannian metrics (depending on two parameters) and the Sasakian metric and the Cheeger–Gromoll metric are obtained by taking particular values for the two parameters. It is defined (cf. [6]) by the following formulas

$$\begin{aligned} &g_{A(p,\mathrm{u})}(X^H, Y^H) = g_p(X,Y), \qquad g_{A(p,\mathrm{u})}(X^H, Y^V) = 0 \\ &g_{A(p,\mathrm{u})}(X^V, Y^V) = a(t) g_p(X,Y) + b(t) g_p(X,\mathrm{u}) g_p(Y,\mathrm{u}) \end{aligned} \tag{6}$$

for all $X, Y \in \chi(M)$, where $a, b : [0,+\infty) \longrightarrow [0,+\infty)$ and $a > 0$. For $a = 1$ and $b = 0$ one obtains the Sasakian metric and for $a = b = \frac{1}{1+2t}$ one gets the Cheeger–Gromoll metric.

Proposition 2. The metric defined above can be constructed by using the method described in Musso and Tricerri [13].

Proof: If we denote by $\theta = (\theta^1, \ldots, \theta^n)$ the canonical one-form on the frame bundle $O(M)$ (namely, if $\mathbf{p} : O(M) \longrightarrow M$, θ is defined by $\mathrm{d}\mathbf{p}_{\mathbf{u}}(X) = \theta^i(X)\mathrm{u}_i$, for $\mathbf{u} = (p, \mathrm{u}_1, \ldots, \mathrm{u}_n)$ and $X \in T_p(M)$) we have $R^*_{\mathbf{a}}(\theta^i) = (\mathbf{a}^{-1})^i_h \theta^h$ for each $\mathbf{a} \in \mathrm{O}(n)$. The vertical distribution of ψ is defined by

$$\theta^i = 0, \quad D\zeta^i := \mathrm{d}\zeta^i + \zeta^j \omega^i_j$$

where $\omega = (\omega^j_i)$ denotes the $\mathfrak{so}(n)$-valued connection one-form defined by the Levi-Civita connection of g. Since $R^*_{\mathbf{a}}(\omega^i_j) = (\mathbf{a}^{-1})^i_h \omega^h_k a^k_j$ we can also write $R^*_{\mathbf{a}}(D\zeta^i) = (\mathbf{a}^{-1})^i_h D\zeta^h$, for all $\mathbf{a} \in \mathrm{O}(n)$.

Consider now the following bilinear form on $O(M)$

$$Q_A = \sum_{i=1}^n (\theta^i)^2 + a\left(\frac{1}{2}\|\zeta\|^2\right) \sum_{i=1}^n (D\zeta^i)^2 + b\left(\frac{1}{2}\|\zeta\|^2\right) \left(\sum_{i=1}^n \zeta^i D\zeta^i\right)^2 . \tag{7}$$

It is symmetric, semi-positive definite and basic. Moreover, since the following diagram

$$\begin{array}{ccc} O(M) \times \mathbb{R}^n & \xrightarrow{\ \psi\ } & T(M) \\ {\scriptstyle \mathrm{proj}_1}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ O(M) & \xrightarrow{\ \ \mathbf{p}\ \ } & M \end{array}$$

is commutative, we have $\psi^* g_A = Q_A$. (For details see [13, §4] and [4, §3].) $\square$

Again, we have to find an almost complex structure on $T(M)$, call it J_A, which is compatible with the metric g_A. Inspired from the previous cases we look for the almost complex structure J_A of the following form

$$\begin{aligned} J_A X^H_{(p,\mathrm{u})} &= \alpha X^V + \beta g_p(X,\mathrm{u})\mathrm{u}^V \\ J_A X^V_{(p,\mathrm{u})} &= \gamma X^H + \rho g_p(X,\mathrm{u})\mathrm{u}^H \end{aligned} \tag{8}$$

where $X \in \chi(M)$ and α, β, γ and ρ are smooth functions on $T(M)$ which will be determined from $J_A^2 = -I$ and from the compatibility conditions with the metric g_A. Following the computations made in [6] we get first $\alpha = \pm\frac{1}{\sqrt{a}}$ and $\gamma = \mp\sqrt{a}$. Without lost of the generality we can take

$$\alpha = \frac{1}{\sqrt{a}} \quad \text{and} \quad \gamma = -\sqrt{a}.$$

Then one obtains

$$\beta = -\frac{1}{2t}\left(\frac{1}{\sqrt{a}} + \epsilon\frac{1}{\sqrt{a+2bt}}\right) \quad \text{and} \quad \rho = \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right)$$

where $\epsilon = \pm 1$.

Thus we have the almost complex structure J_A

$$\begin{aligned} J_A X^H &= \frac{1}{\sqrt{a}}X^V - \frac{1}{2t}\left(\frac{1}{\sqrt{a}} + \epsilon\frac{1}{\sqrt{a+2bt}}\right) g(X,\mathrm{u})\mathrm{u}^V \\ J_A X^V &= -\sqrt{a}X^H + \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right) g(X,\mathrm{u})\mathrm{u}^H \end{aligned} \tag{9}$$

and the almost Hermitian manifold $(T(M), g_A, J_A)$.

Remark 2. In this case J_A is defined on $T(M)\backslash\{0\}$ (the bundle of nonzero tangent vectors), but if we consider $\epsilon = -1$ the previous relations define J_A on the whole $T(M)$.

Remark 3. If we take $\epsilon = -1$, $a = 1$ and $b = 0$ we get the manifold $(T(M), g_S, J_S)$ and for $\epsilon = -1$, $a = b = \frac{1}{1+2t}$ we obtain the manifold $(T(M), g_{CG}, J_{CG})$.

If we denote by Ω_A the Kählerian two-form (i.e., $\Omega_A(U,V) = g_A(U, J_A V)$, for all $U, V \in \chi(T(M))$) one obtains

Proposition 3 (see [6]). The almost Hermitian manifold $(T(M), g_A, J_A)$ is locally conformal almost Kählerian, that is

$$\mathrm{d}\Omega_A = \omega \wedge \Omega_A \tag{10}$$

where ω is a closed and globally defined one-form on $T(M)$ given by

$$\omega(X^H) = 0 \quad \text{and} \quad \omega(X^V) = \frac{1}{\sqrt{a}}\left(\frac{a'}{\sqrt{a}} + \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right)\right) g(X, \mathrm{u}).$$

As a consequence of the above one can state also the following

Theorem 1. *The almost Hermitian manifold* $(T(M), g_A, J_A)$ *is almost Kählerian if and only if*

$$b(t) = \frac{2a'(t)\,(ta'(t) + a(t))}{a(t)}$$

and for $\epsilon = -1$*,* $a(t)$ *is an increasing function, while for* $\epsilon = +1$*,* $ta(t)$ *is a decreasing function.*

Proof: The condition $\omega = 0$ is equivalent to

$$2ta'(t) + a(t) = -\epsilon\sqrt{a(t)}\sqrt{a(t) + 2tb(t)}.$$

From here, we get $b(t)$. Moreover it follows that $a(t)\sqrt{t}$ is a monotone function, namely it is increasing if $\epsilon = -1$ and decreasing for $\epsilon = +1$. Since $b(t)$ is positive we conclude

- if $\epsilon = -1$: $2a't + a > 0 \longleftrightarrow 2(a't + a) > a \longrightarrow a't + a > 0 \longrightarrow a' > 0 \longrightarrow a$ increases (this implies $a\sqrt{t}$, at are also increasing functions)
- if $\epsilon = +1$: $2a't + a < 0 \longleftrightarrow a't + a < -a't \longrightarrow a't + a < 0 \longrightarrow at$ decreases (this implies that $a\sqrt{t}$, a are also decreasing functions).

□

3.4. The Integrability of J_A

In order to have an integrable complex structure J_A on $T(M)$ we have to compute the **Nijenhuis tensor** N_{J_A} of J_A and to check that it vanishes identically.
We have the following relations for N_{J_A}

$$N_{J_A}(X^H, Y^H) = \left(-\frac{a'}{2a^2} + \frac{a + ta'}{a\sqrt{a}}A(t)\right)\left(g(X,\mathrm{u})Y - g(Y,\mathrm{u})X\right)^V + (R_{XY}\mathrm{u})^V$$

$$\begin{aligned} &N_{J_A}(X^V, Y^V) \\ &\quad = \big(-aR_{XY}\mathrm{u} + \sqrt{a}B(t)g(Y,\mathrm{u})R_{X\mathrm{u}}\mathrm{u} - \sqrt{a}B(t)g(X,\mathrm{u})R_{Y\mathrm{u}}\mathrm{u}\big)^V \\ &\quad\quad - \frac{1}{\sqrt{a}}\left(\frac{a'}{2\sqrt{a}} + B(t)\right)\left(g(Y,\mathrm{u})X - g(X,\mathrm{u})Y\right)^V \end{aligned} \tag{11}$$

where $A(t) = \frac{1}{2t}\left(\frac{1}{\sqrt{a}} + \epsilon\frac{1}{\sqrt{a+2bt}}\right)$ and $B(t) = \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right)$.
(The expression for $N_{J_A}(X^H, Y^V)$ is very complicated and will be omitted.)

Thus if J_A is integrable then

$$R_{XY}\mathrm{u} = \left(-\frac{a'}{2a^2} + \frac{a+ta'}{a\sqrt{a}}A(t)\right)(g(Y,\mathrm{u})X - g(X,\mathrm{u})Y)$$

for every $X, Y \in \chi(M)$ and for every point $\mathrm{u} \in T(M)$. It follows that M is a space form $M(c)$ (c is the constant sectional curvature of M). Consequently,

$$-\frac{a'}{2a^2} + \frac{a+ta'}{a\sqrt{a}}\,A(t) = c. \tag{12}$$

Example 1. In the Sasakian case ($a(t) = 1$, $b(t) = 0$, $\epsilon = -1$) it follows that $c = 0$, i.e., the manifold M is flat.

Example 2. Looking for a locally conformal Kähler structure on $T(M)$ for which the metric is of Cheeger–Gromoll type, namely $a(t) = b(t)$, we obtain

$$a(t) = b(t) = \frac{\mathrm{e}^{2\sqrt{1+2t}}}{2\left(c\mathrm{e}^{2\sqrt{1+2t}}t + (1+t+\sqrt{1+2t})k\right)}$$

with k being a positive real constant and c must be nonnegative.

Question: *Can $(T(M), g_A, J_A)$ be a Kählerian manifold?*

If this happens then the base manifold is a space form $M(c)$ and the functions a and b satisfy

$$b = \frac{2a'(ta'+a)}{a} \tag{13}$$

and

$$a' = 2ca(2ta'+a). \tag{14}$$

If $c = 0$ (M is flat) then a is a positive constant and b vanishes.

If $c \neq 0$ the ODE (14) has general solutions

$$a_{1,2}(t) = \frac{1 \pm \sqrt{1+\kappa t}}{4ct} \tag{15}$$

in which κ a real constant. Taking into account that a and b are positive functions and using (13) one gets:

Case 1.

$$a = \frac{1+\sqrt{1+\kappa t}}{4ct} \quad \text{and} \quad b = -\frac{\kappa(1+\sqrt{1+\kappa t})}{8ct(1+\kappa t)}. \tag{16}$$

Here $c > 0$, $t > 0$, $\kappa < 0$, $t < -\frac{1}{\kappa}$ and $\epsilon = +1$.

Case 2.

$$a = -\frac{\kappa}{4c(1+\sqrt{1+\kappa t})} \quad \text{and} \quad b = \frac{\kappa^2}{8c(1+\kappa t)(1+\sqrt{1+\kappa t})}. \tag{17}$$

Here $\kappa c < 0$, $c < 0$ (then $\kappa > 0$), $t < -\frac{1}{\kappa}$ and $\epsilon = -1$.

Consider $B_\kappa = \left\{ \mathrm{v} \in T(M)\,;\ g_{\tau(\mathrm{v})}(\mathrm{v},\mathrm{v}) < -\frac{2}{\kappa} \right\}$ and $\dot{B}_\kappa = B_\kappa \backslash M$.

Theorem 2. *The manifolds B_κ in Case 1 and $\dot{B}_\kappa$ in Case 2 are Kähler manifolds.*

Now we give

Proposition 4. Let (M, g) be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the metric g_A. Then, the corresponding Levi-Civita connection $\tilde{\nabla}^A$ satisfies the following relations:

$$\begin{aligned}
\tilde{\nabla}^A_{X^H} Y^H &= (\nabla_X Y)^H - \frac{1}{2}\,(R_{XY}\mathrm{u})^V \\
\tilde{\nabla}^A_{X^H} Y^V &= (\nabla_X Y)^V + \frac{a}{2}\,(R_{\mathrm{u}Y}X)^H \\
\tilde{\nabla}^A_{X^V} Y^H &= \frac{a}{2}\,(R_{\mathrm{u}X}Y)^H \\
\tilde{\nabla}^A_{X^V} Y^V &= \mathtt{L}\left(g(X,\mathrm{u})Y^V + g(Y,\mathrm{u})X^V\right) + \mathtt{M}g(X,Y)\mathrm{u}^V \\
&\quad + \mathtt{N}g(X,\mathrm{u})g(Y,\mathrm{u})\mathrm{u}^V
\end{aligned} \tag{18}$$

where

$$\mathtt{L} = \frac{a'(t)}{2a(t)}, \qquad \mathtt{M} = \frac{2b(t) - a'(t)}{2(a(t) + 2tb(t))} \qquad \text{and} \qquad \mathtt{N} = \frac{a(t)b'(t) - 2a'(t)b(t)}{2a(t)(a(t) + 2tb(t))}.$$

Proof: The statement follows from Koszul formula by usual computations. □

Having determined Levi-Civita connection, we can compute now the Riemannian curvature tensor $\tilde{R}^A$ on $T(M)$. We have

Proposition 5. The curvature tensor is given by

$$\begin{aligned}
\tilde{R}^A_{X^H Y^H} Z^H &= (R_{XY}Z)^H + \frac{a}{4}\,[R_{\mathrm{u}R_{XZ}\mathrm{u}}Y - R_{\mathrm{u}R_{YZ}\mathrm{u}}X + 2R_{\mathrm{u}R_{XY}\mathrm{u}}Z]^H \\
&\quad + \frac{1}{2}\,[(\nabla_Z R)_{XY}\mathrm{u}]^V \\
\tilde{R}^A_{X^H Y^H} Z^V &= \left[R_{XY}Z + \frac{a}{4}(R_{YR_{\mathrm{u}Z}X}\mathrm{u} - R_{XR_{\mathrm{u}Z}Y}\mathrm{u})\right]^V + \mathtt{L}g(Z,\mathrm{u})(R_{XY}\mathrm{u})^V \\
&\quad + \mathtt{M}g(R_{XY}\mathrm{u}, Z)\mathrm{u}^V + \frac{a}{2}\,[(\nabla_X R)_{\mathrm{u}Z}Y - (\nabla_Y R)_{\mathrm{u}Z}X]^H \\
\tilde{R}^A_{X^H Y^V} Z^H &= \frac{a}{2}\,[(\nabla_X R)_{\mathrm{u}Y}Z]^H \\
&\quad + \frac{1}{2}\left[R_{XZ}Y - \frac{a}{2}R_{XR_{\mathrm{u}Y}Z}\mathrm{u} + \mathtt{L}g(Y,\mathrm{u})R_{XZ}\mathrm{u} + \mathtt{M}g(R_{XZ}\mathrm{u}, Y)\mathrm{u}\right]^V
\end{aligned}$$

$$\begin{aligned}
\tilde{R}^A_{X^H Y^V} Z^V &= -\frac{a}{2}(R_{YZ}X)^H - \frac{a^2}{4}(R_{\mathrm{u}Y} R_{\mathrm{u}Z} X)^H \\
&\quad + \frac{a'}{4}\left[g(Z,\mathrm{u})(R_{\mathrm{u}Y}X)^H - g(Y,\mathrm{u})(R_{\mathrm{u}Z}X)^H\right] \\
\tilde{R}^A_{X^V Y^V} Z^H &= a(R_{XY}Z)^H + \frac{a'}{2}\left[g(X,\mathrm{u})R_{\mathrm{u}Y}Z - g(Y,\mathrm{u})R_{\mathrm{u}X}Z\right]^H \\
&\quad + \frac{a^2}{4}\left[R_{\mathrm{u}X}R_{\mathrm{u}Y}Z - R_{\mathrm{u}Y}R_{\mathrm{u}X}Z\right]^H \\
\tilde{R}^A_{X^V Y^V} Z^V &= F_1(t) g(Z,\mathrm{u})\left[g(X,\mathrm{u})Y^V - g(Y,\mathrm{u})X^V\right] \\
&\quad + F_2(t)\left[g(X,Z)Y^V - g(Y,Z)X^V\right] \\
&\quad + F_3(t)\left[g(X,Z)g(Y,\mathrm{u}) - g(Y,Z)g(X,\mathrm{u})\right]\mathrm{u}^V
\end{aligned} \tag{19}$$

where $F_1 = \mathtt{L}' - \mathtt{L}^2 - \mathtt{N}(1+2t\mathtt{L})$, $F_2 = \mathtt{L} - \mathtt{M}(1+2t\mathtt{L})$ and $F_3 = \mathtt{N} - (\mathtt{M}' + \mathtt{M}^2 + 2t\mathtt{M}\mathtt{N})$.

Remark 4. a) In the case of the Sasakian metric we have: $\mathtt{L} = \mathtt{M} = \mathtt{N} = 0$, $F_1 = F_2 = F_3 = 0$ (cf. also [8]).

b) In the case of the Cheeger–Gromoll metric we have (see also [10, 19]):

$$\mathtt{L} = -\frac{1}{\mathfrak{r}^2}, \quad \mathtt{M} = \frac{\mathfrak{r}^2+1}{\mathfrak{r}^4}, \quad \mathtt{N} = \frac{1}{\mathfrak{r}^4}, \quad \mathtt{L}' = \frac{2}{\mathfrak{r}^4}, \quad \mathtt{M}' = -\frac{2(\mathfrak{r}^2+2)}{\mathfrak{r}^6}$$

$$1 + 2t\mathtt{L} = \frac{1}{\mathfrak{r}^2}, \quad F_1 = \frac{\mathfrak{r}^2-1}{\mathfrak{r}^6}, \quad F_2 = -\frac{\mathfrak{r}^4+\mathfrak{r}^2+1}{\mathfrak{r}^6}, \quad F_3 = \frac{\mathfrak{r}^2+2}{\mathfrak{r}^6}$$

where $\mathfrak{r} = \sqrt{1+2t}$.

In the following let $\tilde{Q}^A(U,V)$ denote the square of the area of the parallelogram with sides U and V for $U, V \in \chi(T(M))$

$$\tilde{Q}^A(U,V) = g_A(U,U) g_A(V,V) - g_A(U,V)^2.$$

We have

Lemma 1. *Let $X, Y \in T_pM$ be two orthonormal vectors. Then*

$$\begin{aligned}
&\tilde{Q}^A(X^H, Y^H) = 1 \\
&\tilde{Q}^A(X^H, Y^V) = a(t) + b(t) g(Y,\mathrm{u})^2 \\
&\tilde{Q}^A(X^V, Y^V) = a(t)^2 + a(t)b(t)\left(g(X,\mathrm{u})^2 + g(Y,\mathrm{u})^2\right).
\end{aligned} \tag{20}$$

We compute now the sectional curvature of the Riemannian manifold $(T(M), g_A)$, namely $\tilde{K}^A(U,V) = \frac{g_A(\tilde{R}^A_{UV}V,U)}{\tilde{Q}^A(U,V)}$ for $U, V \in \chi(T(M))$.

Denote by $T_0(M) = T(M) \setminus \{\mathbf{0}\}$ the bundle of non-zero tangent vectors tangent of M. For a given point $(p,\mathrm{u}) \in T_0(M)$ consider an orthonormal basis $\{e_i\}_{i=\overline{1,m}}$

in the tangent space $T_p(M)$ of M such that $e_1 = \frac{\mathrm{u}}{|\mathrm{u}|}$. Consider on $T_{(p,\mathrm{u})}T(M)$ the following vectors

$$\begin{aligned} E_i &= e_i^H, \quad i = 1, \ldots, m \\ E_{m+1} &= \frac{1}{\sqrt{a+2tb}}\, e_1^V \\ E_{m+k} &= \frac{1}{\sqrt{a}}\, e_k^V, \quad k = 2, \ldots, m. \end{aligned} \tag{21}$$

It is easy to check that $\{E_1, \ldots, E_{2m}\}$ is an orthonormal basis in $T_{(p,\mathrm{u})}T(M)$ (with respect to the metric g_A). We will write the expressions for the sectional curvature $\tilde{K}^A$ in terms of this basis. We have

$$\begin{aligned} &\tilde{K}^A(E_i, E_j) = K(e_i, e_j) - \frac{3a(t)}{4}\left|R_{e_ie_j}\mathrm{u}\right|^2, \quad \tilde{K}^A(E_i, E_{m+1}) = 0 \\ &\tilde{K}^A(E_i, E_{m+k}) = \frac{1}{4}\left|R_{\mathrm{u}e_k}e_i\right|^2, \quad \tilde{K}^A(E_{m+1}E_{m+k}) = -\frac{F_2 + 2tF_3}{a(t)} \\ &\tilde{K}^A(E_{m+k}E_{m+l}) = -\frac{F_2}{a(t)}, \quad i, j = 1, \ldots, m, \quad k, l = 2, \ldots, m. \end{aligned} \tag{22}$$

Here $|\cdot|$ denotes the norm of the vector with respect to the metric g (in a point).

Question: *Can we have constant sectional curvature $\tilde{c}$ on $T(M)$?*

If this happens, then it must be 0, so $T(M)$ is flat. First, one gets easily that M is locally Euclidean. Then, we should also have $F_2(t) = 0$ and $F_3(t) = 0$ for any t. It follows that $\mathtt{M} = \frac{\mathtt{L}}{1+2t\mathtt{L}}$ and $\mathtt{N} = \frac{\mathtt{L}'-\mathtt{L}^2}{1+2t\mathtt{L}}$. (Hence $F_1(t) = 0$.) These equalities yield two ordinary differential equations (involving a and b), namely

$\diamond$) $t(a')^2 + 2aa' - 2ab = 0$

$\diamond\diamond$) $\dfrac{ab' - 2a'b}{a + 2tb} = \dfrac{2a''a - 3(a')^2}{2(a + ta')}$.

A simple computation shows that $\diamond\diamond$) is a consequence of $\diamond$). So, we must have

$$b(t) = a'(t)\left(1 + \frac{ta'(t)}{2a(t)}\right). \tag{23}$$

It is interesting to fix our attention to the following special cases

Case i) $b(t) = \mathtt{k}a'(t)$, where $\mathtt{k}$ is a real constant.

If $a' = 0$ then $b = 0$ and a is constant, so g_A is homothetic to Sasakian metric.

If $a' \neq 0$ then $a(t) = a_0 t^{2(\mathtt{k}-1)}$, ($\mathtt{k} > 1$ or $\mathtt{k} \leq 0$, $a_0 > 0$) and in this case we have to consider $T_0(M)$.

Case ii) $b(t) = a(t)$. We obtain $\frac{a'}{a} = \frac{-1\pm\sqrt{1+2t}}{t}$ which gives

$$a(t) = a_0 \frac{\mathrm{e}^{2\sqrt{1+2t}}}{(1+\sqrt{1+2t})^2}, \quad a_0 > 0 \tag{$*$}$$

or

$$a(t) = a_0 \frac{\mathrm{e}^{-2\sqrt{1+2t}}}{1+t-\sqrt{1+2t}}$$

and in this case we have to deal with non-zero vectors.

Remark 5. The manifold $T(M)$ equipped with the Cheeger–Gromoll metric has a non constant sectional curvature.

Putting $a_0 = 1$ in $(*)$, we can state the following

Proposition 6. Consider g_1 on $T(M)$ given by

$$\begin{aligned} &g_1(X^H, Y^H) = g(X,Y) \\ &g_1(X^H, Y^V) = 0 \\ &g_1(X^V, Y^V) = \frac{\mathrm{e}^{2\sqrt{1+2t}}}{(1+\sqrt{1+2t})^2}\,(g(X,Y) + g(X,\mathrm{u})g(Y,\mathrm{u})) \end{aligned} \tag{24}$$

The manifold $(T(M), g_1)$ is flat.

We compare now the scalar curvatures of (M,g) and $(T(M), g_A)$.

Proposition 7. Let (M,g) be a Riemannian manifold and endow the tangent bundle $T(M)$ with the metric g_A. Let scal and $\widetilde{\mathrm{scal}}^A$ be the scalar curvatures of g and g_A respectively. The following relation holds

$$\widetilde{\mathrm{scal}}^A = \mathrm{scal} + \frac{2-3a}{2}\sum_{i<j}\left|R_{e_ie_j}\mathrm{u}\right|^2 + \frac{1-m}{a}\,(mF_2 + 4tF_3) \tag{25}$$

where $\{e_i\}_{i=1,\dots,m}$ is a local orthonormal frame of $T(M)$.

Proof: Using that $\mathrm{scal} = \sum_{i\neq j} K(e_i, e_j)$ and the formula

$$\sum_{i,j=1}^{m} |R_{e_i\mathrm{u}}e_j|^2 = \sum_{i,j=1}^{m}\left|R_{e_ie_j}\mathrm{u}\right|^2$$

we get the conclusion. □

Let us consider the case when $M = M(c)$ is a real space form.

Question: *Could we find functions a and b such that $T(M)$ equipped with the metric g_A has a constant scalar curvature?*

First of all let us consider the case when $a = k$ (a positive real constant). After some computations we obtain that $b(t)$ should satisfy the following ODE

$$\begin{aligned} c^2(2-3k)k^3t + b(t)(k(m+4c^2(2-3k)kt^2) \\ + 2t(-2+m+2c^2(2-3k)kt^2)b(t)) + 2ktb'(t) = \text{constant.} \end{aligned} \tag{26}$$

Let us give some examples:

Example 3. If we take $a = \frac{2}{3}$ and $b = 0$ we obtain that $(T(M), g_A)$ has a constant scalar curvature $\widetilde{\text{scal}}^A = m(m-1)c$.

Example 4. For $a = k = \frac{2}{3}$ and if the constant in (26) vanishes (and $b \neq 0$) we can integrate the ODE obtaining

$$b = \mathrm{e}^{-\frac{3}{2}\left[(m-2)t + \frac{m \log t}{3}\right]}.$$

Then, $(T_0(M), g_A)$ has a constant scalar curvature $\widetilde{\text{scal}}^A = m(m-1)c$.

Example 5. If we take $a = k \in \left(0, \frac{2}{3}\right)$ and $b = \dfrac{c^2k^2(3k-2)t}{2+m+2c^2(2-3k)kt^2}$ then $(T(M), g_A)$ has a constant scalar curvature $\widetilde{\text{scal}}^A = m(m-1)c$.

Let us consider now $b(t) = a(t)$, as in the case of Cheeger–Gromoll metric. Then $(T(M), g_A)$ has a constant scalar curvature if and only if a satisfies the following ODE

$$\begin{aligned} &-\frac{1}{2(1+2t)^2a(t)^3}\big[-2(m+2(-2+m)t)a(t)^2 - 4t(c+2ct)^2a(t)^3 \\ &+ 6t(c+2ct)^2a(t)^4 + (-6+m)t(1+2t)a'(t)^2 \\ &+ 2a(t)((m+2(-1+m)t)a'(t) + 2t(1+2t)a''(t))\big] = \text{const}. \end{aligned}$$

which seems to be very complicated to solve.

Acknowledgements

This work was done during my stay at the Istituto di Matematica "G. Castelnuovo", Università degli Studi di Roma "La Sapienza" in the period June–November 2005. I would like to thank professor Stefano Marchiafava for inviting me there and for his constant encouragement. I am also grateful to CNR (Consiglio Nazionale delle Ricerche), Italy, for the financial support. The participation at the conference was partially supported by CNCSIS (Romania) 1463/#18/2005.

References

[1] Abbassi M., *Note on the Classification Theorems of g-Natural Metrics on the Tangent Bundle of a Riemannian Manifold* (M, g), Comment. Math. Univ. Carolinae **45** (2004) 591–596.

[2] Abbassi M. and Sarih M., *Killing Vector Fields on Tangent Bundles with Cheeger–Gromoll Metric*, Tsukuba J. Math. **27** (2003) 295–306.

[3] Abbassi M. and Sarih M., *On Some Hereditary Properties of Riemannian g-Natural Metrics on Tangent Bundles of Riemannian Manifolds*, Differential Geometry and Its Applications **22** (2005) 19–47.

[4] Abbassi M. and Sarih M., *On Natural Metrics on Tangent Bundles of Riemannian Manifolds*, Archivum Mathematicum **41** (2005) 71–92.

[5] Abbassi M. and Sarih M., *On Riemannian g-Matural Metrics of the Form* $a.g^s + b.g^h + c.g^v$ *on the Tangent Bundle of a Riemannian Manifold* (M, g), Mediterranian J. Math. **2** (2005) 19–43.

[6] Anastasiei M., *Locally Conformal Kaehler Structures on Tangent Manifold of a Space Form*, Libertas Math. **19** (1999) 71–76.

[7] Blair D., *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, Birkhäuser, Boston, 2002.

[8] Boeckx E. and Vanhecke L., *Harmonic and Minimal Vector Fields on Tangent and Unit Tangent Bundles*, Differential Geometry and Its Applications **13** (2000) 77–93.

[9] Cheeger J. and Gromoll D., *On the Structure of Complete Manifolds of Nonnegative Curvature*, Ann. Math. **96** (1972) 413–443.

[10] Gudmundson S. and Kappos E., *On the Geometry of the Tangent Bundle with the Cheeger–Gromoll Metric*, Tokyo J. Math. **25** (2002) 75–83.

[11] Kowalski O., *Curvature of the Induced Riemannian Metric of the Tangent Bundle of a Riemannian Manifold*, J. Reine Angew. Math. **250** (1971) 124–129.

[12] Kowalski O. and Sekizawa M., *Natural Transformations of Riemannian Metrics on Manifolds to Metrics on Tangent Bundles – a Classification*, Bull. Tokyo Gakuei Univ. **40** (1988) 1–29.

[13] Musso E. and Tricerri F., *Riemannian Metrics on Tangent Bundles*, Ann. Mat. Pura Appl. **150** (1988) 1–19.

[14] Oproiu V., *Some new Geometric Structures on the Tangent Bundle*, Publ. Math Debrecen **55** (1999) 261–281.

[15] Oproiu V., *A Locally Symmetric Kaehler Einstein Structure on the Tangent Bundle of a Space Form*, Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) **40** (1999) 363–372.

[16] Oproiu V., *A Kähler Einstein Structure on the Tangent Bundle of a Space Form*, Int. J. Math. Math. Sci. **25** (2001) 183–195.

[17] Oproiu V. and Papaghiuc N., *Some Classes of Almost Anti-Hermitian Structures on the Tangent Bundle*, Mediterranian J. Math. **1** (2004) 269–282.

[18] Sasaki S., *On the Differential Geometry of Tangent Bundles of Riemannian Manifolds*, Tôhoku Math. J. **10** (1958) 338–354.

[19] Sekizawa M., *Curvatures of Tangent Bundles with Cheeger–Gromoll Metric*, Tokyo J. Math. **14** (1991) 407–417.