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# A NEW FORMULA FOR THE EXPONENTS OF THE GENERATORS OF THE LORENTZ GROUP

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**Abstract.** Here we derive a new formula for the exponent of an arbitrary matrix in the Lie algebra of the Lorentz group. Our considerations are based on the fact that for each constant electromagnetic field there exist an inertial system in which one can easily solve the respective Lorentz equation and therefore to find the explicit formulas describing the trajectories of the charged particles in this field.

## 1. Introduction

Many mathematical models of processes in Physics, Biology and Chemistry are based on systems of linear, ordinary differential equations with constant coefficients. For example, by rewriting the classical Lorentz force law equation in the relativistic form (see [1], [3] and [7] for more details about this issue), the motion of a charged particle in a constant electromagnetic field can be described by a system of four linear differential equations – the so called **Lorentz equations**

$$\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = a\mathcal{F}^\alpha_\beta U^\beta, \qquad \alpha, \beta = x, y, z, t. \tag{1}$$

Here and below $U$ denotes the particle four-velocity (column) vector with respect to the fixed inertial system

$$U = {}^t(U^x, U^y, U^z, U^t), \qquad {}^t = \text{transpose} \tag{2}$$

with

$$U^\gamma = \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \qquad \gamma = x, y, z \quad \text{and} \quad U^t = \frac{E}{m} \tag{3}$$

being its space, respectively time components, $\tau$ is the proper time on the world line, while $E$ is the particle energy, $m$ is its mass at rest and a summation over repeated indices is implicitly assumed. The real parameter $a$ embodies the physical constants, more precisely $a = e/m$, where $e$ is the electric charge of the particle.

Finally, the time independent electromagnetic field with respect to the same fixed inertial reference system is represented by a second order tensor $\mathcal{F}$

$$\mathcal{F} = \begin{bmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix} \tag{4}$$

where $E_1$, $E_2$, $E_3$ and $B_1$, $B_2$, $B_3$ are the components of the electric, respectively magnetic fields, measured in the fixed inertial system [8], and we suppose units chosen so that the velocity of light is unity, $c = 1$.

Introducing

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5}$$

it is a trivial matter to check that any $\mathcal{F}$ from (4) satisfies the identity

$$^t\mathcal{F}\eta + \eta\,\mathcal{F} = 0 \tag{6}$$

and the latter means that $\mathcal{F}$ belongs to the Lie algebra $\mathfrak{so}(3,1)$ of the **Lorentz group**

$$\mathrm{SO}(3,1) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R});\ ^t\Lambda\eta\Lambda = \eta, \quad \det\Lambda = 1\}. \tag{7}$$

If we choose another reference system, the matrix which represents the electromagnetic field there, transforms into a new matrix by an inner automorphism $\zeta \in \mathrm{Int}(\mathfrak{so}(3,1))$ of the Lie algebra $\mathfrak{so}(3,1)$.

On the other hand the general solution of equation (1) is

$$U(\tau) = \mathrm{Exp}(a\mathcal{F}\tau)\overset{\circ}{U} \tag{8}$$

where Exp is the exponential map for the Lorentz group and $\overset{\circ}{U} = U(0)$ is the initial value of four-velocity vector (at the proper time $\tau = 0$).

Before we solve the problem of finding out the exponent of an arbitrary matrix $\mathcal{F}$ of type (4), we will show that any such matrix can be mapped via a proper inner automorphism $\zeta$ into a matrix $\zeta(\mathcal{F}) = \widetilde{\mathcal{F}}$, which is of much simpler form. Physically this is equivalent to the statement that with respect to a proper reference system the electromagnetic field which the observer sees, consists of either parallel or perpendicular electric and magnetic fields. In Section 4 we will determine the trajectories of a charged particle in terms of the coordinates in such a properly chosen inertial system.

Before that in Section 2 we will prove that for any matrix from (4) there exists a reference frame in which it can takes just one of the two possible (in some sense canonical) forms and in Section 3 we will derive two explicit formulas for the matrix exponentials for the Lorentz group – one of them being new, while the other one was known for some time in a slightly different setting (cf. [16]).

## 2. Canonical Form of $\mathfrak{so}(3,1)$ Matrices

First of all we will fix an isomorphism between Lie algebras $\mathfrak{so}(3,1)$ and $\mathfrak{sl}(2,\mathbb{C})$. The matrices

$$e_1 = \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \quad e_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_3 = \frac{1}{2}\begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}$$
$$e_4 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad e_5 = \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad e_6 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9}$$

form a basis of $\mathfrak{sl}(2,\mathbb{C})$, treated as a real Lie algebra. The commutators of the matrices in (9) are

$$\begin{array}{llll} [e_1,e_2]=e_3 & [e_1,e_3]=-e_2 & [e_1,e_4]=0 & [e_1,e_5]=e_6 \\ [e_1,e_6]=-e_5 & [e_2,e_3]=e_1 & [e_2,e_4]=-e_6 & [e_2,e_5]=0 \\ [e_2,e_6]=e_4 & [e_3,e_4]=e_5 & [e_3,e_5]=-e_4 & [e_3,e_6]=0 \\ [e_4,e_5]=-e_3 & [e_4,e_6]=e_2 & [e_5,e_6]=-e_1. & \end{array} \tag{10}$$

The set of matrices

$$X_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$
$$X_4 = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&1&0&0 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{bmatrix} \tag{11}$$

form a basis of $\mathfrak{so}(3,1)$. The corresponding commutators are

$$\begin{array}{llll}
[X_1,X_2]=X_3 & [X_1,X_3]=-X_2 & [X_1,X_4]=0 & [X_1,X_5]=X_6 \\
[X_1,X_6]=-X_5 & [X_2,X_3]=X_1 & [X_2,X_4]=-X_6 & [X_2,X_5]=0 \\
[X_2,X_6]=X_4 & [X_3,X_4]=X_5 & [X_3,X_5]=-X_4 & [X_3,X_6]=0 \\
[X_4,X_5]=-X_3 & [X_4,X_6]=X_2 & [X_5,X_6]=-X_1. &
\end{array} \tag{12}$$

Comparing (12) and (10) we find out, that the linear map $\theta$, defined by

$$\theta : \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1), \qquad \theta(e_i)=X_i, \qquad i=1,2,\ldots,6 \tag{13}$$

provides an isomorphism between the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,1)$. This fact and the Lemma below are indispensable for the main result of the paper.

**Lemma 1.** *For each nonzero element $X \in \mathfrak{sl}(2,\mathbb{C})$ there exists an inner automorphism $\zeta \in \mathrm{Int}(\mathfrak{sl}(2,\mathbb{C}))$, such that either*

$$\text{I)} \qquad \zeta(X) = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \qquad a \in \mathbb{C}^*$$

*or*

$$\text{II)} \qquad \zeta(X) = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}, \qquad c \in \mathbb{C}^*$$

*and there is not an inner automorphism that maps some element of type* I) *into an element of type* II).

**Proof:** Let $X$ be an arbitrary element in $\mathfrak{sl}(2,\mathbb{C})$, i.e.,

$$X = \begin{bmatrix} r & p \\ q & -r \end{bmatrix}, \qquad p,q,r \in \mathbb{C}. \tag{14}$$

Then obviously

$$X^2 = (pq + r^2)\mathrm{I}_2 \tag{15}$$

and there are two possible cases.

First case: $pq + r^2 \neq 0$. In this case the element $X$ is semisimple. Hence, there exists an invertible matrix $g \in \mathrm{GL}(2,\mathbb{C})$ such that $gXg^{-1}$ is a diagonal matrix. Let us factorize the above matrix $g$ in the form $\sqrt{\det g}\,\widetilde{g}$ with $\widetilde{g} \in \mathrm{SL}(2,\mathbb{C})$. Since $gXg^{-1} = \widetilde{g}X\widetilde{g}^{-1}$, the map

$$\zeta : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C}) \qquad \zeta(Y) = \widetilde{g}Y\widetilde{g}^{-1}$$

is an inner automorphism of $\mathfrak{sl}(2,\mathbb{C})$. Moreover, because $gXg^{-1}$ is a diagonal matrix, $\zeta(X)$ is a diagonal matrix too. Hence, there exists some complex number $a \in \mathbb{C}^*$ such that

$$\zeta(X) = \widetilde{g}X\widetilde{g}^{-1} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}. \tag{16}$$

It is convenient to choose

$$2\Re(a) = \sigma \qquad -2\Im(a) = \alpha \qquad \text{where} \quad \alpha, \sigma \in \mathbb{R}$$

and with this notation (16) can be rewritten in the form

$$\zeta(X) = \frac{\sigma}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{\alpha}{2}\begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} = \alpha(-e_3) + \sigma e_6 = -\alpha e_3 + \sigma e_6.$$

Second case: $pq + r^2 = 0$. In this case the element $X$ is nilpotent and therefore there exists an invertible matrix $g \in \mathrm{GL}(2,\mathbb{C})$ such that $gXg^{-1}$ is a strictly upper triangular matrix. Let us use again the factorization of $g$ in the form $\sqrt{\det g}\,\widetilde{g}$, with $\widetilde{g} \in \mathrm{SL}(2,\mathbb{C})$. Since $gXg^{-1} = \widetilde{g}X\widetilde{g}^{-1}$ the inner automorphism

$$\zeta : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C}) \qquad \zeta(Y) = \widetilde{g}Y\widetilde{g}^{-1}$$

maps $X$ this time into a strictly upper triangular matrix $\zeta(X)$, i.e., there exists a complex number $c \in \mathbb{C}^*$ such that

$$\zeta(X) = \widetilde{g}X\widetilde{g}^{-1} = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}.$$

Since $\det \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = 0$, and $\det \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \neq 0$, it follows that an inner automorphism of $\mathfrak{sl}(2,\mathbb{C})$, which maps an element of type $\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ into an element of type $\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$ does not exist. □

**Lemma 2.** *For each strictly upper triangle matrix* $X = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$, $X \in \mathfrak{sl}(2,\mathbb{C})$ *there exists an inner automorphism* $\zeta \in \mathrm{Int}(\mathfrak{sl}(2,\mathbb{C}))$ *such that*

$$\zeta(X) = |c|^2 \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{bmatrix}.$$

**Proof:** If $g = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C})$, then $g^{-1} = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$ and

$$gXg^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s & -q \\ -r & p \end{bmatrix} = c\begin{bmatrix} -pr & p^2 \\ -r^2 & pr \end{bmatrix}. \tag{17}$$

If we take the following values for $p$, $q$, $r$, $s$

$$p = \sqrt{\mathrm{i}\bar{c}} \qquad q = 0 \qquad r = \mathrm{i}\sqrt{\mathrm{i}\bar{c}} \qquad s = 1/p$$

and replace them in (17) we end up with the desired result

$$gXg^{-1} = |c|^2 \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{bmatrix}.$$

Making use of (9) this can be rewritten finally as

$$gXg^{-1} = 2|c|^2(e_6 - e_1).$$

□

Combining Lemma 1 and Lemma 2 produces the following

**Proposition 3.** *For each element* $X \in \mathfrak{sl}(2,\mathbb{C})$ *there exists an inner automorphism* $\zeta \in \mathrm{Int}(\mathfrak{sl}(2,\mathbb{C}))$ *such that either*

$$\text{I)} \qquad \zeta(X) = -\alpha e_3 + \sigma e_6 \qquad \alpha, \sigma \in \mathbb{R}$$

*or*

$$\text{II)} \qquad \zeta(X) = \nu(e_6 - e_1) \qquad \nu \in \mathbb{R}^+.$$

*There does not exist an inner automorphism which maps an element of type* I) *into an element of type* II).

The set of the inner automorphisms of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ coincides with the set of inner automorphisms of $\mathfrak{sl}(2,\mathbb{C})$, treated as a real Lie algebra.

Furthermore, the map $\theta$ in (13) is an isomorphism between real Lie algebras and therefore we can reformulate Proposition 3 as

**Proposition 4.** *For each element* $X \in \mathfrak{so}(3,1)$ *there exists an inner automorphism* $\zeta \in \mathrm{Int}(\mathfrak{so}(3,1))$ *such that either*

$$\text{I)} \qquad \zeta(X) = -\alpha X_3 + \sigma X_6 = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix}, \qquad \alpha, \sigma \in \mathbb{R}$$

*or*

$$\text{II)} \qquad \zeta(X) = \nu(X_6 - X_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & -\nu & 0 & \nu \\ 0 & 0 & \nu & 0 \end{bmatrix}, \qquad \nu \in \mathbb{R}^+.$$

*There does not exist an inner automorphism that maps an element of type* I) *to an element of type* II).

## 3. Two Formulas for the Exponents of $\mathfrak{so}(3,1)$ Matrices

In this section, using the results from the previous, we will derive two formulas for the exponential map Exp: $\mathfrak{so}(3,1) \to \mathrm{SO}(3,1)$. By its very definition the exponential map for any square matrix $X$ of dimension $n$ is given by the convergent power series

$$\mathrm{Exp} : \mathfrak{gl}(n,\mathbb{K}) \to \mathrm{GL}(n,\mathbb{K}), \quad \mathrm{Exp}(X) = \mathrm{I}_n + \sum_{k=1}^{\infty} \frac{X^k}{k!}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C} \tag{18}$$

where $\mathrm{I}_n$ is the identity matrix in the respective dimension. We will deal with an arbitrary nonzero matrix $\mathcal{F}$ from $\mathfrak{so}(3,1)$ in a form specified as in (4), where the triples $(E_1, E_2, E_3)$ and $(B_1, B_2, B_3)$ are considered as vectors and denoted respectively by **E** and **B**. The standard notation **E**.**B** mean the scalar product of the vectors **E** and **B**, i.e.,

$$\mathbf{E}.\mathbf{B} = E_1B_1 + E_2B_2 + E_3B_3 \quad \mathbf{E}^2 = E_1^2 + E_2^2 + E_3^2 \quad \mathbf{B}^2 = B_1^2 + B_2^2 + B_3^2. \tag{19}$$

The characteristic polynomial $P(z)$ of the matrix $\mathcal{F}$ is calculated following the elegant procedure explained in [6] to be

$$P(z) = z^4 + (\mathbf{B}^2 - \mathbf{E}^2)z^2 - (\mathbf{E}.\mathbf{B})^2 \tag{20}$$

and hence

$$\det(\mathcal{F}) = -(\mathbf{E}.\mathbf{B})^2. \tag{21}$$

From Proposition 4 we deduce, that there exists an inner automorphism $\zeta$ such that $\zeta(\mathcal{F})$ is a matrix either of type I) or type II). As one can see, the matrices of the type II) are nilpotent, hence if $\zeta(\mathcal{F})$ is of that type, $\mathcal{F}$ must be nilpotent as well and conversely if $\mathcal{F}$ is nilpotent then $\zeta(\mathcal{F})$ must be of type II) since a matrix of type I) is not nilpotent. The last statement implies that $\zeta(\mathcal{F})$ is of type II) if and only if $\mathcal{F}$ is nilpotent, and the latter is equivalent to $\mathbf{E}.\mathbf{B} = \mathbf{B}^2 - \mathbf{E}^2 = 0$, (see (20) and (21)) so we have proved

**Proposition 5.** *Let $\mathcal{F} \in \mathfrak{so}(3,1)$ be an arbitrary element in the form* (4) *which satisfies*

$$\mathbf{E}.\mathbf{B} = \mathbf{B}^2 - \mathbf{E}^2 = 0. \tag{22}$$

*Then there exists $\zeta \in \mathrm{Int}(\mathfrak{so}(3,1))$ such that $\zeta(\mathcal{F})$ is of type* II).
*If either $\mathbf{E}.\mathbf{B} \neq \mathbf{0}$ or $\mathbf{B^2} - \mathbf{E^2} \neq \mathbf{0}$ then there exists $\zeta \in \mathrm{Int}(\mathfrak{so}(3,1))$ such that $\zeta(\mathcal{F})$ is of type* I).

Thereby we may consider the following two cases.

First case: Let either $\mathbf{E}.\mathbf{B} \neq 0$ or $\mathbf{B}^2 - \mathbf{E}^2 \neq 0$. In this case as we have explained in Proposition 5 $\zeta(\mathcal{F})$ must be of type I), i.e., we have

$$\zeta(\mathcal{F}) = \widetilde{\mathcal{F}} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix}, \qquad \alpha, \sigma \in \mathbb{R}. \tag{23}$$

Since we exclude the trivial case $\mathcal{F} \equiv 0$ from our considerations for obvious reasons, then either $\alpha \neq 0$ or $\sigma \neq 0$.

The characteristic polynomial of $\zeta(\mathcal{F})$ is

$$z^4 + (\alpha^2 - \sigma^2)z^2 - \alpha^2\sigma^2. \tag{24}$$

Because $\zeta$ is an inner automorphism, i.e., a conjugation, then the characteristic polynomial of $\mathcal{F}$ is the same and, therefore,

$$\alpha^2\sigma^2 = (\mathbf{E}.\mathbf{B})^2 \qquad \alpha^2 - \sigma^2 = \mathbf{B}^2 - \mathbf{E}^2. \tag{25}$$

From these relations we obtain

$$\begin{aligned} \sigma^2 &= \frac{\mathbf{E}^2 - \mathbf{B}^2 + \sqrt{(\mathbf{B}^2 - \mathbf{E}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}{2} \\ \alpha^2 &= \frac{\mathbf{B}^2 - \mathbf{E}^2 + \sqrt{(\mathbf{B}^2 - \mathbf{E}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}{2}. \end{aligned} \tag{26}$$

On the other hand

$$\mathrm{Exp}\left(\begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Exp}\left(\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}\right) & \mathbf{0} \\ \mathbf{0} & \mathrm{Exp}\left(\begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}\right) \end{bmatrix} \tag{27}$$

and, therefore,

$$\mathrm{Exp}(\widetilde{\mathcal{F}}) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \mathrm{ch}\sigma & \mathrm{sh}\sigma \\ 0 & 0 & \mathrm{sh}\sigma & \mathrm{ch}\sigma \end{bmatrix}. \tag{28}$$

Introducing

$$\mathrm{J}_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{J}_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

one can easily conclude, that (28) can be written in the form

$$\operatorname{Exp}(\widetilde{\mathcal{F}}) = (\cos\alpha\ \mathrm{J}_\alpha + \operatorname{ch}\sigma\ \mathrm{J}_\sigma) + \left(\frac{\sin\alpha}{\alpha}\mathrm{J}_\alpha + \frac{\operatorname{sh}\sigma}{\sigma}\mathrm{J}_\sigma\right)\widetilde{\mathcal{F}}. \tag{30}$$

**Remark 6.** *Taking into account that* $\lim\limits_{\alpha\to 0}\dfrac{\sin\alpha}{\alpha} = 1$ *and* $\lim\limits_{\alpha\to 0}\dfrac{\operatorname{sh}\sigma}{\sigma} = 1$, *the formula* (30) *is still valid even in the cases when either* $\alpha = 0$ *or* $\sigma = 0$.

Using

$$\widetilde{\mathcal{F}}^2 = \begin{bmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & -\alpha^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

it is possible to invert the above relation and to write down the expressions

$$\mathrm{J}_\alpha = \frac{1}{\alpha^2+\sigma^2}(\sigma^2\mathrm{I}_4 - \widetilde{\mathcal{F}}^2), \qquad \mathrm{J}_\sigma = \frac{1}{\alpha^2+\sigma^2}(\alpha^2\mathrm{I}_4 + \widetilde{\mathcal{F}}^2). \tag{31}$$

**Remark 7.** *These relations are also valid in the degenerate cases when either* $\alpha = 0$, $\sigma \neq 0$ *or* $\alpha \neq 0$, $\sigma = 0$.

Substituting (31) into (30), we obtain

$$\begin{aligned}\operatorname{Exp}(\widetilde{\mathcal{F}}) = {} & \frac{\sigma^2\cos\alpha + \alpha^2\operatorname{ch}\sigma}{\alpha^2+\sigma^2}\mathrm{I}_4 + \frac{\sigma^2\dfrac{\sin\alpha}{\alpha} + \alpha^2\dfrac{\operatorname{sh}\sigma}{\sigma}}{\alpha^2+\sigma^2}\widetilde{\mathcal{F}} \\ & + \frac{\operatorname{ch}\sigma - \cos\alpha}{\alpha^2+\sigma^2}\widetilde{\mathcal{F}}^2 + \frac{\dfrac{\operatorname{sh}\sigma}{\sigma} - \dfrac{\sin\alpha}{\alpha}}{\alpha^2+\sigma^2}\widetilde{\mathcal{F}}^3.\end{aligned} \tag{32}$$

Since $\zeta$ is an inner automorphism of the Lie algebra $\mathfrak{so}(3,1)$ and $\widetilde{\mathcal{F}} = \zeta(\mathcal{F})$, there exists such $g \in \mathrm{SO}(3,1)$ for which $\widetilde{\mathcal{F}} = g\mathcal{F}g^{-1}$ and consequently

$$g^{-1}\operatorname{Exp}(\widetilde{\mathcal{F}})g = g^{-1}\operatorname{Exp}(g\mathcal{F}g^{-1})g = \operatorname{Exp}(g^{-1}g\mathcal{F}g^{-1}g) = \operatorname{Exp}(\mathcal{F}).$$

Now (32) implies

$$\begin{aligned}\operatorname{Exp}(\mathcal{F}) = {} & \frac{\sigma^2\cos\alpha + \alpha^2\operatorname{ch}\sigma}{\alpha^2+\sigma^2}\mathrm{I}_4 + \frac{\sigma^2\dfrac{\sin\alpha}{\alpha} + \alpha^2\dfrac{\operatorname{sh}\sigma}{\sigma}}{\alpha^2+\sigma^2}\mathcal{F} \\ & + \frac{\operatorname{ch}\sigma - \cos\alpha}{\alpha^2+\sigma^2}\mathcal{F}^2 + \frac{\dfrac{\operatorname{sh}\sigma}{\sigma} - \dfrac{\sin\alpha}{\alpha}}{\alpha^2+\sigma^2}\mathcal{F}^3\end{aligned} \tag{33}$$

where $\alpha$ and $\sigma$ are determined from (26), and $\mathcal{F}$ is an arbitrary element in $\mathfrak{so}(3,1)$. In their turn Remark 6 and Remark 7 imply

**Remark 8.** *Formula* (33) *remains true in the cases*

$\mathcal{F} \neq 0$ *and* $\widetilde{\mathcal{F}}$ *is of type* I) *with* $\alpha = 0$

$\mathcal{F} \neq 0$ *and* $\widetilde{\mathcal{F}}$ *is of type* I) *with* $\sigma = 0$.

Further, we define a map

$$\begin{gathered} D : \mathfrak{so}(3,1) \to \mathfrak{so}(3,1) \\ D(X_i) = -X_{3+i}, \qquad D(X_{3+i}) = X_i, \qquad i = 1,2,3 \end{gathered} \tag{34}$$

acting on the matrices $X_i$, $i = 1, \dots, 6$ that form the basis of $\mathfrak{so}(3,1)$ fixed in (11). The result of the action of $D$ on an arbitrary matrix $\mathcal{F} \in \mathfrak{so}(3,1)$ is

$$D(\mathcal{F}) = D\left(\begin{bmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -E_3 & E_2 & B_1 \\ E_3 & 0 & -E_1 & B_2 \\ -E_2 & E_1 & 0 & B_3 \\ B_1 & B_2 & B_3 & 0 \end{bmatrix}.$$

This equality and direct calculations lead to the following identities

$$\begin{aligned} D(\mathcal{F})\mathcal{F} = \mathcal{F}D(\mathcal{F}) &= (\mathbf{E}.\mathbf{B})\mathrm{I}_4, \qquad \mathcal{F} \in \mathfrak{so}(3,1) \\ \mathcal{F}^2 &= D(\mathcal{F})^2 + (\mathbf{E}^2 - \mathbf{B}^2)\mathrm{I}_4. \end{aligned} \tag{35}$$

Following [16] we refer to the matrix $D(\mathcal{F})$ as the dual matrix of the matrix $\mathcal{F}$. The above identities ensure also another useful relationship

$$\begin{aligned} \mathcal{F}^3 = \mathcal{F}^2\mathcal{F} &= (D(\mathcal{F})^2 + (\mathbf{E}^2 - \mathbf{B}^2)\mathrm{I}_4)\mathcal{F} \\ &= D(\mathcal{F})^2\mathcal{F} + (\mathbf{E}^2 - \mathbf{B}^2)\mathcal{F} \\ &= (\mathbf{E}.\mathbf{B})D(\mathcal{F}) + (\mathbf{E}^2 - \mathbf{B}^2)\mathcal{F}. \end{aligned} \tag{36}$$

By means of the isomorphism $\theta : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(3,1)$, defined in (13), the map $D$ is conjugated to the following linear transformation

$$\begin{gathered} \theta^{-1} \circ D \circ \theta : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C}) \\ \theta^{-1} \circ D \circ \theta(e_i) = -e_{3+i}, \qquad \theta^{-1} \circ D \circ \theta(e_{3+i}) = e_i, \qquad i = 1,2,3 \end{gathered} \tag{37}$$

where $e_i$ are given in (9). Taken together (9) and (37) mean that we actually have

$$\theta^{-1} \circ D \circ \theta(e_i) = \mathrm{i}e_i, \qquad i = 1, \dots, 6. \tag{38}$$

Since $\zeta$ is an inner automorphism of $\mathfrak{so}(3,1)$, then $\theta^{-1} \circ \zeta \circ \theta$ is an inner automorphism of $\mathfrak{sl}(2,\mathbb{C})$ as well, and in particular it is a complex-linear map. Because of that, $\theta^{-1} \circ \zeta \circ \theta$ commutes with $\theta^{-1} \circ D \circ \theta$. Hence, $D$ commutes with $\zeta$ as well. Therefore

$$\begin{aligned} \zeta(\mathcal{F}\, D(\mathcal{F})) &= g(\mathcal{F}\, D(\mathcal{F}))g^{-1} = (g\mathcal{F}g^{-1})(gD(\mathcal{F})g^{-1}) \\ &= \zeta(\mathcal{F})\zeta(D(\mathcal{F})) \ = \zeta(\mathcal{F})\, D(\zeta(\mathcal{F})). \end{aligned} \tag{39}$$

On the other hand, using (35) and (23), it is easy to find that

$$\widetilde{\mathcal{F}}\, D(\widetilde{\mathcal{F}}) = \alpha\sigma \mathrm{I}_4. \tag{40}$$

Besides $\zeta(\mathcal{F}\ D(\mathcal{F})) = \zeta((\mathbf{E}.\mathbf{B})\mathrm{I}_4) = (\mathbf{E}.\mathbf{B})\,\mathrm{I}_4$ and for this reason

$$(\mathbf{E}.\mathbf{B})\,\mathrm{I}_4 = \alpha\sigma\mathrm{I}_4$$

and therefore

$$\mathbf{E}.\mathbf{B} = \alpha\sigma. \tag{41}$$

The equations (25), (36) and (41) finally imply the relation

$$\mathcal{F}^3 = \alpha\sigma D(\mathcal{F}) + (\sigma^2 - \alpha^2)\mathcal{F}. \tag{42}$$

Replacing (42) in (33), we get

$$\begin{aligned}\mathrm{Exp}(\mathcal{F}) = {} & \frac{\sigma^2\cos\alpha + \alpha^2 \operatorname{ch}\sigma}{\alpha^2+\sigma^2}\mathrm{I}_4 + \frac{\sigma\operatorname{sh}\sigma + \alpha\sin\alpha}{\alpha^2+\sigma^2}\mathcal{F} \\ & + \frac{\alpha\operatorname{sh}\sigma - \sigma\sin\alpha}{\alpha^2+\sigma^2}D(\mathcal{F}) + \frac{\operatorname{ch}\sigma - \cos\alpha}{\alpha^2+\sigma^2}\mathcal{F}^2\end{aligned} \tag{43}$$

where $\alpha$ and $\beta$ are defined in terms of the matrix elements $E_1$, $E_2$, $E_3$, $B_1$, $B_2$, $B_3$ of $\mathcal{F}$ in (26). A similar formula for the Minkowskian metric with signature $(1,3)$ can be found in [16]. Plebański has derived similar formulas equivalent to (33) and (43) in terms of projectors associated to $\mathcal{F}$ [12, 13]. These formulas within another context can be recognized in [11] too.

Second case: Let $\mathbf{E}.\mathbf{B} = 0$ and $\mathbf{B}^2 - \mathbf{E}^2 = 0$. This case is referred in the literature as the **null-field** (cf. [8], [12], [14]). Now we have

$$\zeta(\mathcal{F}) = \widetilde{\mathcal{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & -\nu & 0 & \nu \\ 0 & 0 & \nu & 0 \end{bmatrix}, \qquad \nu \in \mathbb{R}^+. \tag{44}$$

As we mentioned above, in this case the matrix $\mathcal{F}$ is nilpotent. More precisely: a direct calculation shows, that $\widetilde{\mathcal{F}}^3 = 0$. Since $\zeta(\mathcal{F})^3 = \zeta(\mathcal{F}^3)$, then $\mathcal{F}^3 = 0$. Because of that and according to the definition (18)

$$\mathrm{Exp}(\mathcal{F}) = \mathrm{I}_4 + \mathcal{F} + \frac{1}{2}\mathcal{F}^2. \tag{45}$$

## 4. Motion of a Charged Particle in a Constant Electromagnetic Field

In this section we will determine the trajectories of a particle with mass $m$ which carries an electric charge $e$ in a constant electromagnetic field $\mathcal{F}$ specified by a matrix of type (4). For that purpose we must solve the system of Lorentz equations (1) with respective initial data.

As Proposition 4 claims, this electromagnetic field can be represented in appropriate inertial system by a matrix $\widetilde{\mathcal{F}}$, which is either of type I) or type II). We will determine several types of trajectories starting from the “initial" position

$$\mathring{x} = x(0) \qquad \mathring{y} = y(0) \qquad \mathring{z} = z(0)$$

with a four-velocity $\mathring{U}$ which space part is a three dimensional vector chosen to be orthogonal to both vectors **E** and **B**, i.e.,

$$\mathring{U} = U(0) = {}^t\left(0, \mu, 0, \sqrt{1+\mu^2}\right) \qquad \mu \in \mathbb{R}. \tag{46}$$

### 4.1. The Case E.B $\neq 0$

According to the Proposition 4 and Proposition 5 in this case the electromagnetic field can be represented via a matrix of the type

$$\zeta(\mathcal{F}) = \widetilde{\mathcal{F}} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix}, \qquad \begin{matrix} \mathbf{E} = (0,0,\sigma) \\ \mathbf{B} = (0,0,\alpha) \end{matrix} \qquad \alpha, \sigma \in \mathbb{R} \setminus \{0\}. \tag{47}$$

Adopting (28) we obtain immediately

$$\operatorname{Exp}(a\widetilde{\mathcal{F}}\tau) = \begin{bmatrix} \cos(a\alpha\tau) & \sin(a\alpha\tau) & 0 & 0 \\ -\sin(a\alpha\tau) & \cos(a\alpha\tau) & 0 & 0 \\ 0 & 0 & \operatorname{ch}(a\sigma\tau) & \operatorname{sh}(a\sigma\tau) \\ 0 & 0 & \operatorname{sh}(a\sigma\tau) & \operatorname{ch}(a\sigma\tau) \end{bmatrix} \tag{48}$$

and taking into account (8) and (46)

$$U(\tau) = \begin{bmatrix} U^x(\tau) \\ U^y(\tau) \\ U^z(\tau) \\ U^t(\tau) \end{bmatrix} = \begin{bmatrix} \mu \sin(a\alpha\tau) \\ \mu \cos(a\alpha\tau) \\ \sqrt{1+\mu^2}\,\operatorname{sh}(a\sigma\tau) \\ \sqrt{1+\mu^2}\,\operatorname{ch}(a\sigma\tau) \end{bmatrix}. \tag{49}$$

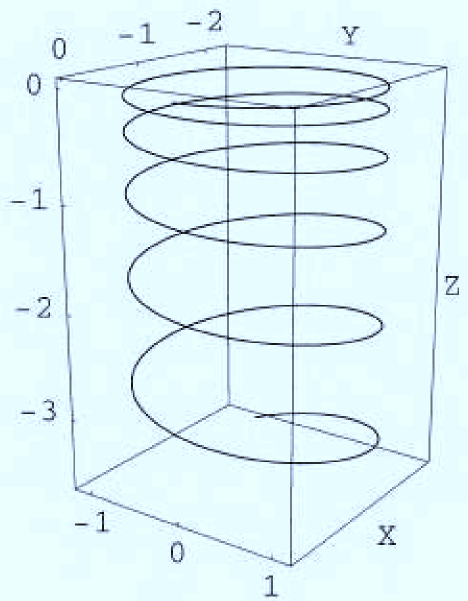



**Figure 1.** $\mathbf{E} = (0, 0, 41 \times 10^{-4})$, $\mathbf{B} = (0, 0, 6 \times 10^{2})$, $\tau \in [0,\ 10^{-4}]$

Space trajectories can be found after direct integration (cf. (3)) and this gives

$$\begin{aligned} x(\tau) &= \mathring{x} + \frac{\mu}{a\alpha}(1 - \cos(a\alpha\tau)) \\ y(\tau) &= \mathring{y} + \frac{\mu}{a\alpha}\sin(a\alpha\tau) \\ z(\tau) &= \mathring{z} + \frac{\sqrt{1+\mu^2}}{a\sigma}(\mathrm{ch}(a\sigma\tau) - 1). \end{aligned} \tag{50}$$

Such a trajectory, which is obviously a **helix** along the $z$-axis, is depicted in Fig. 1 for an electron starting with velocity $\mathring{\mathbf{U}}$ from the initial position $\mathring{\mathbf{x}}$ that are chosen to be respectively

$$\mathring{\mathbf{x}} = (\mathring{x}, \mathring{y}, \mathring{z}) = (0,\ 0,\ 0) \quad \text{and} \quad \mathring{\mathbf{U}} = (\mathring{U}^x, \mathring{U}^y, \mathring{U}^z) = (0,\ 4 \times 10^5,\ 0). \tag{51}$$

Actually, this is the general case in view of the considerations of Synge [15] (see also [2] and [5]) making a recourse to Frenet–Serret formalism and proving the surprising result that the world-line of a charged particle moving in a constant electromagnetic field is a generalized helix in the Minkowski spacetime.

Let us mention also that in the Figures of this paper the vectors **E** and **B** are represented respectively in *volt per meter* and *volt* $\times$ *second* $/(3 \times 10^8)$ *per squared* *meter*, the proper time is measured in *seconds*, the distances (with some exceptions) in *meters* and velocities in *meters per second*.

### 4.2. The Case E.B $= 0$ and $\mathbf{B}^2 - \mathbf{E}^2 < 0$

According to Proposition 4 and Proposition 5 the electromagnetic field can be represented again by a matrix of the type I) and taking into account (25) one easily concludes that this is possible only when

$$\alpha = 0. \tag{52}$$

Therefore we can write

$$\zeta(\mathcal{F}) = \widetilde{\mathcal{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix}, \qquad \begin{matrix} \mathbf{E} = (0, 0, \sigma) \\ \mathbf{B} = (0, 0, 0) \end{matrix} \qquad \sigma \in \mathbb{R} \setminus \{0\}. \tag{53}$$

Relying on (28) in this case we get

$$\operatorname{Exp}(a\widetilde{\mathcal{F}}\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \operatorname{ch}(a\sigma\tau) & \operatorname{sh}(a\sigma\tau) \\ 0 & 0 & \operatorname{sh}(a\sigma\tau) & \operatorname{ch}(a\sigma\tau) \end{bmatrix} \tag{54}$$

which combined with (8) and (46) produce

$$U(\tau) = \begin{bmatrix} U^x(\tau) \\ U^y(\tau) \\ U^z(\tau) \\ U^t(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \mu \\ \sqrt{1+\mu^2}\,\operatorname{sh}(a\sigma\tau) \\ \sqrt{1+\mu^2}\,\operatorname{ch}(a\sigma\tau) \end{bmatrix}. \tag{55}$$

Integration of (55) gives the trajectory

$$\begin{aligned} x(\tau) &= \mathring{x} \\ y(\tau) &= \mathring{y} + \mu\tau \\ z(\tau) &= \mathring{z} + \frac{\sqrt{1+\mu^2}}{a\sigma}(\operatorname{ch}(a\sigma\tau) - 1) \end{aligned} \tag{56}$$

and the curve which it traces is easily to be recognized as the **catenary** in the $Y-Z$ plane. More details about geometrical and mechanical properties of this curve can be found in the books by Oprea [9] and [10]. Using formulas (56) we have illustrated graphically the path of the electron in Fig. 2 with the initial data specified in (51) and the scale on the axes there is chosen so that one unit corresponds to $10^3$ meters.

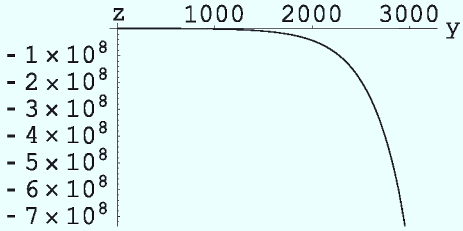



**Figure 2.** $\mathbf{E} = (0, 0, 2 \times 10^{-3})$, $\mathbf{B} = (0, 0, 0)$, $\tau \in [0, 8]$

### 4.3. The Case $\mathbf{E}.\mathbf{B} = 0$ and $\mathbf{B}^2 - \mathbf{E}^2 > 0$

This time Proposition 4 and Proposition 5 taken together with (25) say that the electromagnetic field can be represented via a matrix of the type **I)** in which

$$\sigma = 0 \tag{57}$$

and therefore

$$\zeta(\mathcal{F}) = \widetilde{\mathcal{F}} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} \mathbf{E} = (0, 0, 0) \\ \mathbf{B} = (0, 0, \alpha) \end{array} \qquad \alpha \in \mathbb{R} \setminus \{0\}. \tag{58}$$

Following the same route as in the previous subsection we obtain

$$\operatorname{Exp}(a\widetilde{\mathcal{F}}\tau) = \begin{bmatrix} \cos(a\alpha\tau) & \sin(a\alpha\tau) & 0 & 0 \\ -\sin(a\alpha\tau) & \cos(a\alpha\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{59}$$

and

$$U(\tau) = \begin{bmatrix} U^x(\tau) \\ U^y(\tau) \\ U^z(\tau) \\ U^t(\tau) \end{bmatrix} = \begin{bmatrix} \mu \sin(a\alpha\tau) \\ \mu \cos(a\alpha\tau) \\ 0 \\ \sqrt{1 + \mu^2 \tau)} \end{bmatrix}. \tag{60}$$

Integration of (60) gives the trajectory

$$\begin{aligned} x(\tau) &= \mathring{x} + \frac{\mu}{a\alpha}(1 - \cos(a\alpha\tau)) \\ y(\tau) &= \mathring{y} + \frac{\mu}{a\alpha}\sin(a\alpha\tau) \\ z(\tau) &= \mathring{z}. \end{aligned} \tag{61}$$

which is obviously a circle in the $X - Y$ plane and this case is of great importance for the theory of cyclotrons and mass spectrometers.

### 4.4. The Case $\mathbf{E}.\mathbf{B} = 0$ and $\mathbf{B}^2 - \mathbf{E}^2 = 0$

As a direct consequence of Proposition 4 and Proposition 5 in this case the electromagnetic field can be represented via a matrix of the type II), i.e.,

$$\widetilde{\mathcal{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & -\nu & 0 & \nu \\ 0 & 0 & \nu & 0 \end{bmatrix}, \qquad \begin{matrix} \mathbf{E} = (0, 0, \nu) \\ \mathbf{B} = (\nu, 0, 0) \end{matrix} \qquad \nu \in \mathbb{R}. \tag{62}$$

Relying on (45) we get

$$\operatorname{Exp}(a\widetilde{\mathcal{F}}\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{1}{2}a^2\nu^2\tau^2 & a\nu\tau & \frac{1}{2}a^2\nu^2\tau^2 \\ 0 & -a\nu\tau & 1 & a\nu\tau \\ 0 & -\frac{1}{2}a^2\nu^2\tau^2 & a\nu\tau & 1 + \frac{1}{2}a^2\nu^2\tau^2 \end{bmatrix} \tag{63}$$

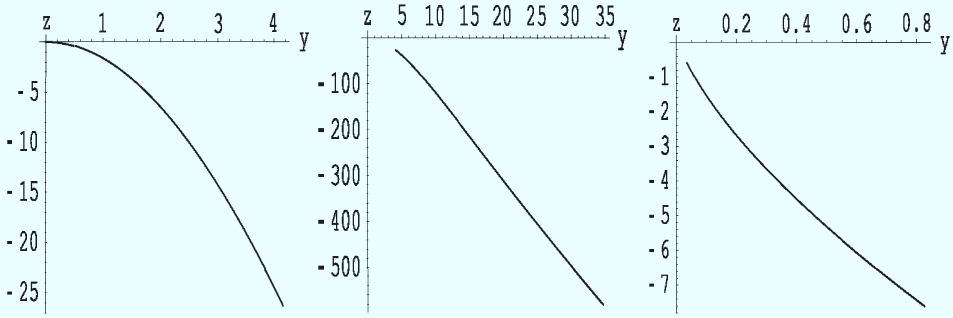
which combined with (8) and (46) produce

$$U(\tau) = \begin{bmatrix} U^x(\tau) \\ U^y(\tau) \\ U^z(\tau) \\ U^t(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \mu + \frac{1}{2}a^2\nu^2(\sqrt{1+\mu^2} - \mu)\tau^2 \\ a\nu(\sqrt{1+\mu^2} - \mu)\tau \\ \sqrt{1+\mu^2} + \frac{1}{2}a^2\nu^2(\sqrt{1+\mu^2} - \mu)\tau^2 \end{bmatrix}. \tag{64}$$

Integrating (64) we find that the trajectory is the generalized **semi-cubic parabola** in the $Y - Z$ plane (see [4])

$$\begin{aligned} x(\tau) &= \mathring{x} \\ y(\tau) &= \mathring{y} + \mu\tau + \frac{1}{6}a^2\nu^2(\sqrt{1+\mu^2} - \mu)\tau^3 \\ z(\tau) &= \mathring{z} + \frac{1}{2}a\nu(\sqrt{1+\mu^2} - \mu)\tau^2. \end{aligned} \tag{65}$$

Three different patches of such a trajectory generated with the initial data specified in (51) are presented in Fig. 2 in order to stress its character within different intervals of the proper time. In the third picture the axes are scaled so that one unit there corresponds to $10^3$ meters.




$\tau \in [0, 10^{-5}]$ $\tau \in [10^{-5}, 4.7 \times 10^{-5}]$ $\tau \in [4.7 \times 10^{-5}, 17 \times 10^{-5}]$

**Figure 3.** $\mathbf{E} = (0, 0, 3)$, $\mathbf{B} = (3, 0, 0)$

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