ON THE MULTI-COMPONENT NLS TYPE EQUATIONS ON SYMMETRIC SPACES: REDUCTIONS AND SOLITON SOLUTIONS

GEORGI G. GRAHOVSKI[†], VLADIMIR S. GERDJIKOV[†]
and NIKOLAY A. KOSTOV[‡]

[†]*Institute for Nuclear Research and Nuclear Energy*
 Bulgarian Academy of Sciences
 72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria
[‡]*Institute of Electronics, Bulgarian Academy of Sciences*
 72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria

Abstract. The fundamental properties of the multi-component nonlinear Schrödinger (MNLS) type models related to symmetric spaces are analyzed. New types of reductions of these systems are constructed. The Lax operators L and the corresponding recursion operators Λ are used to formulate some of the fundamental properties of the MNLS-type equations. The results are illustrated by specific examples of MNLS-type systems related to the **D.III** symmetric space for the $\mathfrak{so}(8)$-algebra. The effect of the reductions on their soliton solutions is outlined.

1. Introduction

The (scalar) **nonlinear Schrödinger (NLS) equation** [26]

$$iu_t + u_{xx} + 2|u|^2 u = 0, \quad u = u(x,t) \tag{1}$$

has numerous applications in a wide variety of physical problems [8, 24]. Its complete integrability has been proven at the very early stage of the inverse scattering method (ISM). Indeed, equation (1) allows Lax representation using as Lax operator the 2×2 **Zakharov–Shabat system** related to $\mathfrak{sl}(2)$-algebra

$$L(\lambda)\psi(x,t,\lambda) = (i\partial_x + q(x,t) - \lambda\sigma_3)\psi(x,t,\lambda) = 0 \tag{2}$$

$$q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ -u^*(x,t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and that it is a Hamiltonian system described by the Hamiltonian

$$H = \int_{-\infty}^{\infty} \mathrm{d}x \, (-|q_x|^2 + |q|^4).$$

(3)

The first multi-component NLS type model with applications to physics is the so-called vector NLS equation (Manakov model) [2, 22]

$$\mathrm{i}\mathbf{v}_t + \mathbf{v}_{xx} + 2\|\mathbf{v}\|^2\mathbf{v} = 0, \qquad \mathbf{v} = \begin{pmatrix} v_1(x, t) \\ v_2(x, t) \end{pmatrix}.$$

(4)

It obviously allows generalization to n-component vectors.

The applications of the differential geometric and Lie algebraic methods to soliton type equations lead to the discovery of close relationship between the multi-component (matrix) NLS equations and the homogeneous and **symmetric spaces** [9]. It was shown that the integrable MNLS systems have Lax representation with the generalized Zakharov–Shabat system as the **Lax operator**

$$L\psi(x, t, \lambda) \equiv \mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}x} + (Q(x, t) - \lambda J)\psi(x, t, \lambda) = 0$$

(5)

where J is a constant element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$, $Q(x, t) \equiv [J, \widetilde{Q}(x, t)] \in \mathfrak{g}/\mathfrak{g}_0$ and $\mathfrak{g}_0$ is the subalgebra of the elements commuting with J. In other words $Q(x, t)$ belongs to the co-adjoint orbit $\mathcal{M}_J$ of $\mathfrak{g}$ passing through J.

The choice of J determines the dimension of $\mathcal{M}_J$ which can be viewed as the phase space of the relevant nonlinear evolution equations (NLEE). It is is equal to the number of roots of $\mathfrak{g}$ such that $\alpha(J) \neq 0$. Taking into account that if α is a root, then $-\alpha$ is also a root of $\mathfrak{g}$ then $\dim \mathcal{M}_J$ is always even.

We concentrate on those most degenerate choices of J for which ad $_J$ has just two non-vanishing eigenvalues $\pm 2a$; in this case $J^2 = a^2 \mathbb{1}$. Such choices of J are compatible with several types of symmetric spaces [9, 18]: **A.III** $\simeq$ SU($p + q$)/S(U(p) $\otimes$ U(q)), **C.I** $\simeq$ Sp($2p$)/U(p) and **D.III** $\simeq$ SO($2p$)/U(p). The construction of soliton solutions proposed in [2, 22] is valid for **A.III** symmetric spaces; its extension to **C.I** and **D.III** is proposed in [25], see also [15].

The interpretation of the ISM as a generalized Fourier transforms and the expansion over the so-called **squared solutions** (see [17] for regular and [11] for non-regular J) allow one to study all the fundamental properties of the corresponding NLEE's. These include: i) the description of the class of NLEE related to a given Lax operator $L(\lambda)$ and solvable by the ISM; ii) derivation of the infinite family of integrals of motion; and iii) their hierarchy of Hamiltonian structures.

The present article is organized as follows. In Section 2 we give some preliminaries about the simple Lie algebras. In Section 3 we describe the general form of the MNLS models and the relevant recursion operators. In Section 4 we briefly discuss

the scattering data and the Hamiltonian properties of the considered systems. The soliton solutions are discussed in Section 5. In Section 6 are derived few examples of NLS equations by applying the reduction group [23] action. In the same section the reductions of the soliton solutions are analyzed.

2. Preliminaries. Simple Lie Algebras

Here we fix the notations and the normalization conditions for the Cartan–Weyl generators $\{h_k, E_\alpha\}$ of $\mathfrak{g}$ ($r = \operatorname{rank}\mathfrak{g}$) with root system Δ. We introduce $h_k \in \mathfrak{h}$, $k = 1, \dots, r$ as the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$ and the Weyl generators E_α, $\alpha \in \Delta$. Their commutation relations

$$[h_k, E_\alpha] = (\alpha, e_k)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \frac{2}{(\alpha, \alpha)} \sum_{k=1}^{r} (\alpha, e_k)h_k$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha + \beta \in \Delta \\ 0 & \text{for } \alpha + \beta \notin \Delta \cup \{0\}. \end{cases} \tag{6}$$

Here $\vec{a} = \sum_{k=1}^{r} a_k e_k$ is a r-dimensional vector dual to $J \in \mathfrak{h}$ and $(\cdot, \cdot)$ is the scalar product in $\mathbb{E}^r$. The normalization of the basis is determined by

$$E_{-\alpha} = E_\alpha^T, \qquad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha, \alpha)}, \qquad N_{-\alpha,-\beta} = -N_{\alpha,\beta} \tag{7}$$

where $N_{\alpha,\beta} = \pm(p + 1)$ and the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \dots, p$, $\alpha + (p + 1)\beta \notin \Delta$ and $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}$, see [18]. The root system Δ of $\mathfrak{g}$ is invariant with respect to the group $W_\mathfrak{g}$ of Weyl reflections S_α

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)}\alpha, \qquad \alpha \in \Delta. \tag{8}$$

With each reflection S_α one can relate an internal automorphism of the algebra $\operatorname{Ad}_{A_\alpha} \in \operatorname{Aut}_0 \mathfrak{g}$ which act in a natural way on the Cartan–Weyl basis, namely

$$S_\alpha(H_\beta) \equiv A_\alpha H_\beta A_\alpha^{-1} = H_{\beta'}, \qquad\qquad \beta' = S_\alpha\beta$$

$$S_\alpha(E_\beta) \equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta}E_{\beta'}, \qquad\qquad n_{\alpha,\beta} = \pm 1. \tag{9}$$

Since $S_\alpha^2 = \mathbb{1}$ we must have $A_\alpha^2 = \pm\mathbb{1}$.

As we already mentioned in the Introduction the MNLS equations correspond to Lax operator (5) with non-regular (constant) Cartan elements $J \in \mathfrak{h}$. If J is a regular element of the Cartan subalgebra of $\mathfrak{g}$ then ad_J has as many different eigenvalues as is the number of the roots of the algebra and they are given by $a_j = \alpha_j(J)$, $\alpha_j \in \Delta$. Such J's can be used to introduce an ordering in the root system by assuming that $\alpha > 0$ if $\alpha(J) > 0$. In what follows we will assume that all roots for which $\alpha(J) > 0$ are positive.

Obviously we can consider the eigensubspaces of ad_J as grading of the algebra $\mathfrak{g}$. In what follows we will consider symmetric spaces related to maximally degenerated J, i.e. ad_J has only two non-vanishing eigenvalues $\pm 2a$. Then $\mathfrak{g}$ is split into a direct sum of the subalgebra $\mathfrak{g}_0$ and the linear subspaces $\mathfrak{g}_\pm$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_- \qquad \mathfrak{g}_\pm = \mathrm{l.c.}\left\{ X_{\pm j}\,;\, [J, X_\pm] = \pm 2a X_\pm \right\}.$$

The subalgebra $\mathfrak{g}_0$ contains the Cartan subalgebra $\mathfrak{h}$ and also all root vectors $E_{\pm\alpha} \in \mathfrak{g}$ corresponding to the roots α such that $(\vec{a}, \alpha) = 0$. The root system Δ is split into subsets of roots $\Delta = \theta_0 \cup \theta_+ \cup (-\theta_+)$, where

$$\theta_0 = \left\{ \alpha \in \Delta\,;\, \alpha(J) = 0 \right\} \qquad \theta_+ = \left\{ \alpha \in \Delta\,;\, \alpha(J) = a > 0 \right\}. \tag{10}$$

We can use the gauge transformation commuting with J to simplify Q; in particular we can remove all components of Q in $\mathfrak{g}_0$; effectively this means that our $Q(x, t) = Q_+(x, t) + Q_-(x, t) \in \mathfrak{g}_+ \cup \mathfrak{g}_-$ can be viewed as a local coordinate in the co-adjoint orbit $\mathcal{M}_J \simeq \mathfrak{g} \setminus \mathfrak{g}_0$

$$Q_+(x, t) = \sum_{\alpha \in \theta_+} q_\alpha(x, t) E_\alpha, \qquad Q_-(x, t) = \sum_{\alpha \in \theta_+} p_\alpha(x, t) E_{-\alpha}. \tag{11}$$

Obviously $Q_\pm \in \mathfrak{g}_\pm$ and

$$\mathrm{ad}_J Q \equiv [J, Q] = 2a(Q_+ - Q_-), \qquad (\mathrm{ad}_J)^{-1} Q = \frac{1}{2a}(Q_+ - Q_-) \tag{12}$$

besides $[E_\alpha, E_\beta] = 0$ for any pair of roots $\alpha, \beta \in \theta_+$. This simplifies solving the recursion relations and the explicit calculation of the recursion operator Λ.

3. The MNLS Type Models

3.1. Lax Representation and General Form of the NLEE

The operator (5) together with the corresponding operator $M(\lambda)$

$$M(\lambda)\psi \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - [Q, \mathrm{ad}_J^{-1} Q] + 2\mathrm{i}\,\mathrm{ad}_J^{-1} Q_x + 2\lambda Q - 2\lambda^2 J \right) \psi(x, t, \lambda) = 0 \tag{13}$$

where $Q = Q(x, t)$, provide the Lax representation for the MNLS type systems. The compatibility condition $[L(\lambda), M(\lambda)] = 0$ of (5) and (13) gives the general form of the MNLS equations on symmetric spaces

$$\frac{\mathrm{i}}{2}\left[J, \frac{\mathrm{d}Q}{\mathrm{d}t} \right] + \frac{\mathrm{d}^2 Q}{\mathrm{d}x^2} - 2a^2 [\mathrm{ad}_J^{-1} Q, [\mathrm{ad}_J^{-1} Q, Q]] = 0. \tag{14}$$

Following [1] one can consider more general M-operators of the form

$$M(\lambda)\Psi \equiv \mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}t} + \left(\sum_{k=1}^{N} V_k(x, t)\lambda^k \right) \Psi(x, t, \lambda) = 0, \quad f(\lambda) = \lim_{x \to \pm\infty} V(x, t, \lambda).$$

The Lax representation $[L(\lambda), M(\lambda)] = 0$ leads to a recurrent relations between $V_k(x,t) = V_k^{\mathrm{f}} + V_k^{\mathrm{d}}$

$$V_{k+1}^{\mathrm{f}}(x,t) \equiv \pi_J(V_{k+1}) = \Lambda_{\pm} V_k^{\mathrm{f}}(x,t) - \mathrm{ad}\,_J^{-1}[C_k, Q(x,t)], \quad k = 1, \ldots, N$$

$$V_k^{\mathrm{d}}(x,t) \equiv (\mathbb{1} - \pi_J)(V_k) = C_k + \mathrm{i} \int_{\pm\infty}^{x} \mathrm{d}y\, [Q(y,t), V_k^{\mathrm{f}}(y,t)] \tag{15}$$

where $\pi_J = \mathrm{ad}\,_J^{-1} \circ \mathrm{ad}\,_J$ and $C_k = (\mathbb{1} - \pi_J)C_k$ are block-diagonal integration constants, for details see, e.g. [1,9]. These relations are resolved by the recursion operators (16)

$$\Lambda_{\pm} Z = \frac{\mathrm{ad}\,_J}{4a^2} \left(\mathrm{i}\frac{\mathrm{d}Z}{\mathrm{d}x} + \mathrm{i}\left[Q(x), \int_{\pm\infty}^{x} \mathrm{d}y\, [Q(y), Z(y)] \right] \right) \tag{16}$$

where we assume that $Z \equiv \pi_J Z \in \mathcal{M}_J$. As a result we obtain that the class of (generically nonlocal) NLEE solvable by the ISM have the form

$$\mathrm{i}\mathrm{ad}\,_J^{-1}\frac{\mathrm{d}Q}{\mathrm{d}t} + \sum_{k=0}^{N} \Lambda_{\pm}^{N-k} \left[C_k, \mathrm{ad}\,_J^{-1} Q(x,t) \right] = 0, \quad f(\lambda) = \begin{pmatrix} f^+(\lambda) & 0 \\ 0 & f^-(\lambda) \end{pmatrix} \tag{17}$$

where $f(\lambda) = \sum_{k=0}^{N} C_k \lambda^{N-k}$ determines their dispersion law. The NLEE (17) become local if $f(\lambda) = f_0(\lambda)J$, where $f_0(\lambda)$ is a scalar function. In particular, if $f(\lambda) = -2\lambda^2 J$ we get the MNLS equation (14). If the dispersion law $f(\lambda)$ is non-degenerate then the corresponding NLEE allows solutions of "boomeron" and "trappon" types [4–6] whose velocities are time-dependent.

3.2. A Basic Example: The Symmetric Space D.III

We choose $\mathfrak{g} = \mathbf{D}_4 \simeq \mathfrak{so}(8)$. It has 4 simple roots, namely $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$ and $\alpha_4 = e_3 + e_4$. We fix up the Catran element

$$J = \mathrm{diag}(a,a,a,a,-a,-a,-a,-a), \qquad J^2 = a^2\mathbb{1} \tag{18}$$

which means that the subset $\theta_+ = \{e_i + e_j\}$ with $1 \leq i < j \leq 4$. Then the corresponding potential $Q \in \mathbf{D.III}$ (11) takes the form

$$Q = \begin{pmatrix}
0 & 0 & 0 & 0 & q_{14} & q_{13} & q_{12} & 0 \\
0 & 0 & 0 & 0 & q_{24} & q_{23} & 0 & q_{12} \\
0 & 0 & 0 & 0 & q_{34} & 0 & q_{23} & -q_{13} \\
0 & 0 & 0 & 0 & 0 & q_{34} & -q_{24} & q_{14} \\
p_{14} & p_{24} & p_{34} & 0 & 0 & 0 & 0 & 0 \\
p_{13} & p_{23} & 0 & p_{34} & 0 & 0 & 0 & 0 \\
p_{12} & 0 & p_{23} & -p_{24} & 0 & 0 & 0 & 0 \\
0 & p_{12} & -p_{13} & p_{14} & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{19}$$

Here by $q_{ij}(x,t)$ and $p_{ij}(x,t)$ where i,j belong to the set of indices $\mathcal{J} = \{(ij); 1 \leq i < j \leq 4\}$ denote the coefficients of the generators E_α and $E_{-\alpha}$ with

$\alpha = e_i + e_j$. Then the NLEE (14) becomes a system of 12 equations described by following Hamiltonian and symplectic form

$$H = H_{\text{kin}} + H_{\text{int}}, \qquad H_{\text{kin}} = -\frac{1}{a} \int_{-\infty}^{\infty} \mathrm{d}x \sum_{(ij)\in\mathcal{J}} \frac{\partial q_{ij}}{\partial x} \frac{\partial p_{ij}}{\partial x} \tag{20}$$

$$H_{\text{int}} = \frac{1}{a} \int_{-\infty}^{\infty} \mathrm{d}x \left(\sum_{(ij)\in\mathcal{J}} p_{ij} q_{ij} \right)^2$$
$$- \frac{2}{a} \int_{-\infty}^{\infty} \mathrm{d}x \, (p_{12}p_{34} + p_{13}p_{24} - p_{14}p_{23})(q_{13}q_{24} + q_{12}q_{34} - q_{14}q_{23}) \tag{21}$$

$$\Omega^{(0)} = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}x \sum_{(ij)\in\mathcal{J}} \delta q_{ij}(x,t) \wedge \delta p_{ij}(x,t).$$

4. Scattering Data and Hamiltonian Properties of the MNLS Models

Here we will start with a brief sketch of the direct scattering problem for (5), (18), (19). It is based on the **Jost solutions** [8, 24] defined by their asymptotics

$$\lim_{x\to\infty} \psi(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda J x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda J x} = \mathbb{1}$$

and the scattering matrix $T(t,\lambda) = (\psi(x,t,\lambda))^{-1}\phi(x,t,\lambda)$ and its inverse $\hat{T}(\lambda,t)$

$$T(t,\lambda) = \begin{pmatrix} a^+(t,\lambda) & -b^-(t,\lambda) \\ b^+(t,\lambda) & a^-(t,\lambda) \end{pmatrix}, \quad \hat{T}(t,\lambda) = \begin{pmatrix} c^-(t,\lambda) & d^-(t,\lambda) \\ -d^+(t,\lambda) & c^+(t,\lambda) \end{pmatrix} \tag{22}$$

where $a^\pm(t,\lambda)$ and $b^\pm(t,\lambda)$ are 4×4 block matrices. The **fundamental analytic solutions** (FAS) $\chi^\pm(x,t,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\text{Im}\,\lambda \gtrless 0$ and are related to the Jost solutions by

$$\chi^\pm(x,t,\lambda) = \phi(x,t,\lambda)S_J^\pm(t,\lambda) = \psi(x,t,\lambda)T_J^\mp(t,\lambda). \tag{23}$$

Here $S_J^\pm, T_J^\pm$ denote upper- and lower- block-triangular matrices

$$S_J^+(t,\lambda) = \begin{pmatrix} \mathbb{1} & d^-(t,\lambda) \\ 0 & c^+(t,\lambda) \end{pmatrix}, \qquad S_J^-(t,\lambda) = \begin{pmatrix} c^-(t,\lambda) & 0 \\ -d^+(t,\lambda) & \mathbb{1} \end{pmatrix} \tag{24}$$

$$T_J^+(t,\lambda) = \begin{pmatrix} \mathbb{1} & -b^-(t,\lambda) \\ 0 & a^-(t,\lambda) \end{pmatrix}, \qquad T_J^-(t,\lambda) = \begin{pmatrix} a^+(t,\lambda) & 0 \\ b^+(t,\lambda) & \mathbb{1} \end{pmatrix} \tag{25}$$

satisfy $T_J^\pm(t,\lambda)\hat{S}_J^\pm(t,\lambda) = T(t,\lambda)$ and can be viewed as the factors of a generalized Gauss decompositions of $T(t,\lambda)$ [11]. If $Q(x,t)$ evolves according to (17)

then

$$i\frac{db^{\pm}}{dt} + f^{\mp}(\lambda)b^{\pm}(t,\lambda) - b^{\pm}(t,\lambda)f^{\pm}(\lambda) = 0$$

$$i\frac{da^{\pm}}{dt} + [f^{\pm}(\lambda), a^{\pm}(t,\lambda)] = 0. \tag{26}$$

On the real axis in the complex λ-plane both FAS $\chi^{\pm}(x,t,\lambda)$ are linearly dependent

$$\chi^{+}(x,t,\lambda) = \chi^{-}(x,t,\lambda)G_0(t,\lambda) \tag{27}$$

and $G_0(t,\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues for the Lax operator (5), see [13].

The MNLS equations possess hierarchies of Hamiltonian structures. The phase space $\mathcal{M}_J$ of the MNLS equations is the co-adjoint orbit of the $\mathfrak{g} \simeq \mathbf{D}_r$ determined by J. In addition we assume that the matrix elements of $Q(x,t)$ are smooth functions tending to zero fast enough for $|x| \rightarrow \infty$.

On the $\mathbf{D.III}$-type symmetric spaces the Hamiltonian of (14) is given by

$$H_{\mathrm{MNLS}}^{(0),\mathrm{symm}} = a \int_{-\infty}^{\infty} dx \left\{ 2\langle \operatorname{ad}{}_J^{-1}Q_x, \operatorname{ad}{}_J^{-1}Q_x \rangle + \frac{1}{2}\langle [\operatorname{ad}{}_J^{-1}Q, Q], [\operatorname{ad}{}_J^{-1}Q, Q]\rangle \right\}.$$

The hierarchies of symplectic structures defined on $\mathcal{M}_J$ are generated by the corresponding recursion operators and are given by the following families of compatible two-forms

$$\Omega_{\mathrm{MNLS}}^{(k)} = -ia \int_{-\infty}^{\infty} dx \left\langle \delta Q(x,t) \wedge \Lambda^k \operatorname{ad}{}_J^{-1}\delta Q(x,t)\right\rangle.$$

For $f(\lambda) = -2\lambda^2 J$ equation (26) gives $da^{\pm}/dt = 0$ and $a^{\pm}(\lambda)$ can be viewed as generating functionals of integrals of motion whose number $2r^2$ is larger than the rank r of $\mathfrak{g}$. This is obviously due to the degeneracy of the dispersion law. For generic $f(\lambda)$ from (26) there follows that only the eigenvalues of $a^{\pm}(\lambda)$ will be conserved. So for MNLS we have extra integrals of motion but one can check that they are not all in involution. Indeed, it follows from the **classical R-matrix** approach [8]

$$\{T(\lambda,t) \underset{,}{\otimes} T(\mu,t)\} = [R(\lambda - \mu), T(\lambda,t) \otimes T(\mu,t)] \tag{28}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket and the R-matrix equals [9]

$$R(\lambda - \mu) = \frac{1}{\lambda - \mu}\left(\sum_{k=1}^{r} h_k \otimes h_k + \sum_{\alpha \in \Delta} \frac{E_\alpha \otimes E_{-\alpha}}{\langle E_\alpha, E_{-\alpha}\rangle}\right) \tag{29}$$

where h_k are introduced in (6) and $\langle h_i, h_k\rangle = \delta_{ik}$.

The principal series of integrals of motion $C^{(k)}$ is generated by

$$\log \det a^{\pm}(t, \lambda) = \sum_{k=1}^{\infty} C^{(k)} \lambda^k.$$

From (28) it follows that the first integrals $C^{(k)}$ from this series are in involution. Due to the special degenerate choice of the dispersion law $f(\lambda) = -2\lambda^2 J$, any matrix elements of the blocks $a^{\pm}(\lambda)$ will generate integrals of motion, which however will not be in involution, see (28). The Hamiltonian for the MNLS models is proportional to $C^{(3)}$, i.e. belongs to the principal series. If we choose a generic (i.e., non-degenerate) dispersion law then the Hamiltonian of the corresponding NLEE will not be in involution with $C^{(k)}$. Such are the dispersion laws for the NLEE's that allow "boomeron" and "trappon" type solutions [4–6]. This is the reason why their velocities become time-dependent.

5. Dressing Factors and Soliton Solutions

The main idea of the dressing method is starting from a FAS $\chi_{(0)}^{\pm}(x, \lambda)$ of $L(\lambda)$ with potential $q_{(0)}$ to construct a new singular solution $\chi_{(1)}^{\pm}(x, \lambda)$ of the Riemann–Hilbert Problem (27) with singularities located at prescribed positions $\lambda_1^{\pm}$. Then the new solutions $\chi_{(1)}^{\pm}(x, \lambda)$ will correspond to a potential $q_{(1)}$ of $L(\lambda)$ with two discrete eigenvalues $\lambda_1^{\pm}$. It is related to the regular one by the dressing factors $u(x, \lambda)$

$$\chi_{(1)}^{\pm}(x, \lambda) = u(x, \lambda)\chi_{(0)}^{\pm}(x, \lambda)u_{-}^{-1}(\lambda), \qquad u_{-}(\lambda) = \lim_{x \to -\infty} u(x, \lambda). \tag{30}$$

If $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$ the **dressing factors** take the form [15]

$$u(x, \lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1(x) + (c_1^{-1}(\lambda) - 1)P_{-1}(x), \qquad P_{-1} = SP_1^T S^{-1} \tag{31}$$

where the rank 1 projector $P_1(x)$ and the function $c_1(\lambda)$ are given by

$$P_1(x) = \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle} \qquad c_1(\lambda) = \frac{\lambda - \lambda^+}{\lambda - \lambda^-}$$

$$|n(x)\rangle = \chi_0^+(x, \lambda_1^+)|n_0\rangle \qquad \langle m(x)| = \langle m_0|\hat{\chi}_0^-(x, \lambda_1^-). \tag{32}$$

$|n_0\rangle$ and $\langle m_0|$ are constant vectors and

$$S = \sum_{k=1}^{4}(-1)^{k+1}(E_{k\bar{k}} + E_{\bar{k}k}), \qquad \bar{k} = 9 - k.$$

Here E_{kn} is an 8×8 matrix whose matrix elements are $(E_{kn})_{ij} = \delta_{ik}\delta_{nj}$. Then the "dressed" potential have the form

$$Q_{(1)}(x, t) = Q_{(0)}(x, t) - (\lambda_1^+ - \lambda_1^-)[J, p(x, t)], \qquad p(x, t) = P_1(x, t) - P_{-1}(x, t) \tag{33}$$

where

$$p(x,t) = \frac{2}{\langle m|n \rangle} \left(\sum_{k=1}^{4} h_k(x,t) H_{e_k} + \sum_{\alpha \in \Delta_+} (P_\alpha(x,t) E_\alpha + P_{-\alpha}(x,t) E_{-\alpha}) \right)$$

$$\langle m|n \rangle = \sum_{k=1}^{4} \left(n_{0k} m_{0k} e^{2a\nu_1 x + 16a\mu_1 \nu_1 t} + n_{0\overline{k}} m_{0\overline{k}} e^{-2a\nu_1 x - 16a\mu_1 \nu_1 t} \right)$$

$$h_k(x,t) = n_{0k} m_{0k} e^{2a\nu_1 x + 16a\mu_1 \nu_1 t} - n_{0\overline{k}} m_{0\overline{k}} e^{-2a\nu_1 x - 16a\mu_1 \nu_1 t}$$

and

$$P_\alpha(x,t) = \left(n_{0k} m_{0\overline{s}} - (-1)^{s+k} n_{0s} m_{0\overline{k}} \right) e^{-2ia\mu_1 x - 8ia(\mu_1^2 - \nu_1^2)t}$$

$$P_{-\alpha}(x,t) = \left(n_{0\overline{s}} m_{0k} - (-1)^{s+k} n_{0\overline{k}} m_{0s} \right) e^{2ia\mu_1 x + 8ia(\mu_1^2 - \nu_1^2)t}$$

for $\alpha = e_k + e_s$, and

$$P_\alpha(x,t) = n_{0k} m_{0s} e^{2a\nu_1 x + 16a\mu_1 \nu_1 t} - (-1)^{s+k} n_{0\overline{s}} m_{0\overline{k}} e^{-2a\nu_1 x - 16a\mu_1 \nu_1 t}$$

$$P_{-\alpha}(x,t) = n_{0\overline{k}} m_{0\overline{s}} e^{-2a\nu_1 x - 16a\mu_1 \nu_1 t} - (-1)^{s+k} n_{0s} m_{0k} e^{2a\nu_1 x + 16a\mu_1 \nu_1 t} \tag{34}$$

for $\alpha = e_k - e_s$, $k < s$.

Let us consider now the purely solitonic case, i.e. $Q_{(0)}(x) = 0$ and $\chi_{(0)}^+(x,t,\lambda_1^+)$ $= \exp(-i\lambda_1^+ Jx - 4i\lambda_1^{+2} Jt)$. Thus the 1-soliton solution is

$$q_{jk} = -\frac{2ia\nu_1 \left(n_{0j} m_{0\overline{k}} - (-1)^{j+k} n_{0k} m_{0\overline{j}} \right) e^{-2ia\mu_1 x - 8ia(\mu_1^2 - \nu_1^2)t}}{\sqrt{\varphi_1 \varphi_2} \, \text{ch} \left(2a\nu_1 x + 16a\nu_1 \mu_1 t + \frac{1}{2} \ln \frac{\varphi_1}{\varphi_2} \right)} \tag{35}$$

where $(ij) \in \mathcal{J}$ and

$$\varphi_1 = \sum_{j=1}^{4} n_{0j} m_{0j}, \qquad \varphi_2 = \sum_{j=1}^{4} n_{0\overline{j}} m_{0\overline{j}}, \qquad \lambda_1^\pm = \mu_1 \pm i\nu_1. \tag{36}$$

6. New Reductions of MNLS Equations

6.1. The Reduction Group

The **reduction group** G_R introduced by Mikhailov [23] provides a powerful tool for constructing new integrable equations [7,15,16,20,21,25] starting from known ones. It is a finite group which preserves the Lax representation, i.e. it ensures that the reduction constrains are automatically compatible with the evolution. The main idea of the reduction group is to impose an invariance condition on the Lax operators (5) and (13). In particular this means that the dispersion law $f_{\text{MNLS}}(\lambda) = -2\lambda^2 J$ must also be compatible with the reduction group action.

Here we consider two types of $G_R \simeq \mathbb{Z}_2$ reductions and we will embed them as subgroup of $W_\mathfrak{g}$

Type I:
$$B^{-1}U^\dagger(x,t,\lambda^*)B = U(x,t,\lambda), \qquad B^{-1}JB = J$$
$$B^{-1}V^\dagger(x,t,\lambda^*)B = V(x,t,\lambda) \tag{37}$$

Type II:
$$C^{-1}U^*(x,t,\lambda^*)C = -U(x,t,\lambda), \qquad C^{-1}JC = -J$$
$$C^{-1}V^*(x,t,\lambda^*)C = -V(x,t,\lambda) \tag{38}$$

where

$$U(x,t,\lambda) = Q(x,t) - \lambda J$$
$$V(x,t,\lambda) = -[Q, \mathrm{ad}_J^{-1}Q] + 2\mathrm{iad}_J^{-1}Q_x(x,t) + 2\lambda Q(x,t) - 2\lambda^2 J$$

and the automorphisms C and B must be of even order.

6.2. Examples of $\mathbb{Z}_2$-Reductions.

Type B1: Let us impose on the potential $U(x,t,\lambda)$ of the Lax operator the following $\mathbb{Z}_2$-reduction

$$B_1^{-1}(U^\dagger(x,t,\lambda^*))B_1 = U(x,t,\lambda), \qquad U(x,t,\lambda) = Q(x,t) - \lambda J$$
$$B_1 = w_{e_1-e_2}, \qquad B_1^2 = \mathbb{1}$$

where $w_{e_i-e_j}$ are the **Weyl reflection** with respect to the roots $e_i - e_j$, $i < j$ of the $\mathfrak{so}(8)$-algebra. The corresponding constraints on the potential $U(x,t,\lambda)$ are given by $B_1(J) = J$ and

$$p_{12}(x,t) = -q_{12}^*(x,t), \quad p_{13}(x,t) = q_{23}^*(x,t), \quad p_{14}(x,t) = -q_{24}^*(x,t)$$
$$p_{23}(x,t) = q_{13}^*(x,t), \quad p_{24}(x,t) = -q_{14}^*(x,t), \quad p_{34}(x,t) = -q_{34}^*(x,t). \tag{39}$$

Thus one gets a six component NLS type system described by the following Hamiltonian and symplectic form in the terms of the independent fields $q_{ij}(x,t)$

$$H_{\mathrm{kin}}^{\mathrm{red}} = \frac{1}{a}\int_{-\infty}^\infty dx \left(|q_{12,x}|^2 + |q_{34,x}|^2 + q_{14,x}q_{24,x}^* + q_{14,x}^*q_{24,x} \right.$$
$$\left. - q_{13,x}q_{23,x}^* - q_{13,x}^*q_{23,x} \right)$$

$$H_{\mathrm{int}}^{\mathrm{red}} = \frac{1}{a}\int_{-\infty}^\infty dx \{(-|q_{12}|^2 - |q_{34}|^2 - q_{14}q_{24}^* - q_{14}^*q_{24} + q_{13}q_{23}^* + q_{13}^*q_{23})^2$$
$$- 2|q_{14}q_{23} - q_{13}q_{24} - q_{12}q_{34}|^2\} \tag{40}$$

$$\Omega^{(0),\mathrm{red}} = i\int_{-\infty}^\infty dx \left(\delta q_{12} \wedge \delta q_{12}^* + \delta q_{34} \wedge \delta q_{34}^* - \delta q_{14} \wedge \delta q_{24}^* \right.$$
$$\left. - \delta q_{24} \wedge \delta q_{14}^* + \delta q_{13} \wedge \delta q_{23}^* + \delta q_{23} \wedge \delta q_{13}^* \right).$$

Type B2: Let us impose another $\mathbb{Z}_2$-reduction

$$B_2^{-1}(U^\dagger(x,t,\lambda^*))B_2 = U(x,t,\lambda), \qquad U(x,t,\lambda) = Q(x,t) - \lambda J$$

$$B_2 = w_{e_1-e_2} \cdot w_{e_3-e_4}, \qquad B_2^2 = \mathbb{1}.$$

Then again $B_2(J) = J$ and for the matrix elements of $Q(x,t)$ we have

$$
\begin{aligned}
p_{12}(x,t) &= \quad q_{12}^*(x,t), & p_{13}(x,t) &= q_{24}^*(x,t), & p_{14}(x,t) &= -q_{23}^*(x,t) \\
p_{23}(x,t) &= -q_{14}^*(x,t), & p_{24}(x,t) &= q_{13}^*(x,t), & p_{34}(x,t) &= q_{34}^*(x,t).
\end{aligned}
\tag{41}
$$

and one derives the following 6 component MNLS system related to $\mathbf{D}_4$-algebra with the Hamiltonian and symplectic form

$$
\begin{aligned}
H_{\text{kin}}^{\text{red}} &= \frac{1}{a}\int_{-\infty}^{\infty} dx \left(|q_{12,x}|^2 + |q_{34,x}|^2 - q_{14,x}q_{23,x}^* - q_{14,x}^*q_{23,x} \right. \\
&\qquad\qquad \left. + q_{13,x}q_{24,x}^* + q_{13,x}^*q_{24,x} \right) \\
H_{\text{int}}^{\text{red}} &= \frac{1}{a}\int_{-\infty}^{\infty} dx \left\{ (|q_{12}|^2 + |q_{34}|^2 + q_{13}q_{24}^* + q_{13}^*q_{24} - q_{14}q_{23}^* - q_{14}^*q_{23})^2 \right. \\
&\qquad\qquad \left. - 2|q_{14}q_{23} - q_{13}q_{24} - q_{12}q_{34}|^2 \right\} \\
\Omega^{(0),\text{red}} &= i\int_{-\infty}^{\infty} dx \left(\delta q_{12} \wedge \delta q_{12}^* + \delta q_{34} \wedge \delta q_{34}^* + \delta q_{13} \wedge \delta q_{24}^* \right. \\
&\qquad\qquad \left. + \delta q_{24} \wedge \delta q_{13}^* - \delta q_{14} \wedge \delta q_{23}^* - \delta q_{23} \wedge \delta q_{14}^* \right).
\end{aligned}
\tag{42}
$$

Type C: We impose one more reduction but of type (38)

$$C(U^*(x,t,\lambda^*)) = -U(x,t,\lambda), \quad U(x,t,\lambda) = Q(x,t) - \lambda J$$

realized with

$$C = S_{e_1+e_2}S_{e_1-e_2}S_{e_3+e_4}S_{e_3-e_4}$$

then we will get the following reduction constraints

$$p_{ij} = q_{ij}^*, \qquad C(J) = -J.$$

This leads to the 6 component NLS system with Hamiltonian and symplectic form

$$
\begin{aligned}
H_{\text{kin}}^{\text{red}} &= -\frac{1}{a}\int_{-\infty}^{\infty} dx \sum_{(ij)\in\mathcal{J}} \frac{\partial q_{ij}}{\partial x}\frac{\partial q_{ij}^*}{\partial x} \\
H_{\text{int}}^{\text{red}} &= \frac{1}{a}\int_{-\infty}^{\infty} dx \left\{ \left(\sum_{(ij)\in\mathcal{J}} |q_{ij}|^2 \right)^2 - 2|q_{13}q_{24} + q_{12}q_{34} - q_{14}q_{23}|^2 \right\} \\
\Omega^{(0),\text{red}} &= i\int_{-\infty}^{\infty} dx \sum_{(ij)\in\mathcal{J}} \delta q_{ij}(x,t) \wedge \delta q_{ij}^*(x,t).
\end{aligned}
\tag{43}
$$

This reduction of MNLS systems on symmetric spaces is among the first nontrivial examples studied by Fordy and Kulish [9].

6.3. $\mathbb{Z}_2$-Reductions and the Soliton Solutions

The reduction conditions (37) and (38) imposed on the potential of the Lax operator (5) induce an invariance conditions for the corresponding FAS

$$\textbf{Type I:}\quad B^{-1}(\chi^+(x,t,\lambda^*))^\dagger B = \begin{pmatrix} c^-(\lambda) & 0 \\ 0 & a^-(\lambda) \end{pmatrix} (\chi^-(x,t,\lambda))^{-1} \qquad (44)$$

$$\textbf{Type II:}\quad C^{-1}(\chi^\pm(x,t,\lambda^*))^* C = \chi^\pm(x,t,\lambda) \qquad (45)$$

and for the scattering matrix (22)

$$\textbf{Type I:}\quad B^{-1}T^\dagger(t,\lambda^*)B = (T(t,\lambda))^{-1} \qquad (46)$$

$$\textbf{Type II:}\quad C^{-1}T^*(t,\lambda^*)C = T(t,\lambda). \qquad (47)$$

As a consequence for the dressing factors $u(x,t,\lambda)$ one gets

$$\textbf{Type I:}\quad B^{-1}u^\dagger(x,t,\lambda^*)B = (u(x,t,\lambda))^{-1} \qquad (48)$$

$$\textbf{Type II:}\quad C^{-1}u^*(x,t,\lambda^*)C = u(x,t,\lambda). \qquad (49)$$

The concrete restrictions on the corresponding eigenvalues λ_1^+ and λ_1^- and on the projectors $P_{\pm 1}$ will depend on the explicit choices of the automorphisms B and C. Skipping the details we will present here the final results

$$\textbf{Type I:}\quad \lambda_1^+ = (\lambda_1^-)^*, \quad B^{-1}P_{\pm 1}^\dagger(x,t)B = P_{\pm 1}(x,t)$$
$$|m(x,t)\rangle = B|n^*(x,t)\rangle \qquad (50)$$

$$\textbf{Type II:}\quad \lambda_1^+ = (\lambda_1^-)^*, \quad C^{-1}P_{+1}^*(x,t)C = P_{-1}(x,t)$$
$$|m(x,t)\rangle = SC^{-1}|n^*(x,t)\rangle. \qquad (51)$$

Applying the above restrictions to (31) we get dressing factors satisfying automatically these reduction conditions.

If we parameterize the constant vectors $|n_0\rangle$ and $\langle m_0|$ as follows

$$|n_0\rangle = (n_{01}, n_{02}, n_{03}, n_{04}, n_{0\bar 4}, n_{0\bar 3}, n_{0\bar 2}, n_{0\bar 1})^T$$
$$\langle m_0| = (m_{01}, m_{02}, m_{03}, m_{04}, m_{0\bar 4}, m_{0\bar 3}, m_{0\bar 2}, m_{0\bar 1})$$

and apply the reductions of **type B1** from Section 6.2, then the above formulas for the reduction constrains on the polarization vectors $|m\rangle$ and $|n\rangle$ read

$$
\begin{aligned}
&m_{01} = -n_{02}^*, & &m_{02} = -n_{01}^*, & &m_{03} = -n_{03}^*, & &m_{04} = -n_{04}^* \\
&m_{0\bar 4} = n_{0\bar 4}^*, & &m_{0\bar 3} = -n_{0\bar 3}^*, & &m_{0\bar 2} = n_{0\bar 1}^*, & &m_{0\bar 1} = n_{0\bar 2}^*.
\end{aligned}
\qquad (52)
$$

For the reductions on these vectors in the example of **type B2** one gets

$$
\begin{aligned}
&m_{01} = n_{02}^*, & &m_{02} = -n_{01}^*, & &m_{03} = -n_{04}^*, & &m_{04} = n_{03}^* \\
&m_{0\bar 4} = -n_{0\bar 3}^*, & &m_{0\bar 3} = n_{0\bar 4}^*, & &m_{0\bar 2} = n_{0\bar 1}^*, & &m_{0\bar 1} = -n_{0\bar 2}^*.
\end{aligned}
\qquad (53)
$$

Finally for the last example of **type C** we have

$$
\begin{aligned}
&m_{01} = -n_{01}^*, \qquad m_{02} = -n_{02}^*, \qquad m_{03} = -n_{03}^*, \qquad m_{04} = -n_{04}^* \\
&m_{0\bar{4}} = -n_{0\bar{4}}^*, \qquad m_{\overline{03}} = -n_{\overline{03}}^*, \qquad m_{\overline{02}} = -n_{\overline{02}}^*, \qquad m_{\overline{01}} = -n_{\overline{01}}^*.
\end{aligned}
\tag{54}
$$

Finally the soliton solutions of the reduced systems (40), (42) and (43) could be obtained directly from (35) by inserting there (52), (53) and (54), respectively.

7. Conclusions

We have described two new systems of MNLS type obtained as $\mathbb{Z}_2$-reductions of the MNLS related to a **D.III** symmetric space. The Hamiltonian formalism and the theory of Λ-operators for MNLS related to simple Lie algebras are briefly discussed. The corresponding soliton solutions for these systems are derived. These results can be extended to study the reductions of MNLS-type equations related to other symmetric and homogeneous spaces. We show how the method, presented in [15] for the N-wave equations and their gauge equivalent systems can be extended to MNLS type systems [16]. The method is explicitly gauge covariant and can also be applied to their gauge equivalent systems of Heisenberg ferromagnet type.

Thus one can systematically obtain and classify new integrable systems of MNLS type. Such research would entail a voluminous calculations and will be continued in subsequent publications. Other interesting reductions of MNLS type equations were reported in [14] and a systematic study of the problem is on the way.

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