Fifth International Conference on
Geometry, Integrability and Quantization
June 5–12, 2003, Varna, Bulgaria
Ivaïlo M. Mladenov and Allen C. Hirshfeld, Editors
SOFTEX, Sofia 2004, pp 144–148

POISSON STRUCTURES IN $\mathbb{R}^3$

METIN GÜRSES and KOSTYANTYN ZHELTUKHIN

Department of Mathematics, Faculty of Sciences,
Bilkent University, 06800 Ankara, Turkey

Abstract. The Poisson structures and Hamiltonian formulation of three dimensional systems is considered in general. A class of degenerate structures in higher dimensions is also briefly discussed.

1. Introduction

In a recent work we [1] have considered the Poisson structures in $\mathbb{R}^3$. We showed that, locally all such structures must have the form

$$J^{ij} = \mu \epsilon^{ijk} \partial_k \Psi \tag{1}$$

where μ and Ψ are arbitrary differentiable functions of x^i, $i = 1, 2, 3$ and ϵ^{ijk} is the Levi-Civita symbol. Here we use the summation convention. This has a very natural geometrical explanation. Let $\Psi = c_1$ and $H = c_2$ define two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, in $\mathbb{R}^3$, where c_1 and c_2 are some constants. Then the intersection of these surfaces define a curve C in $\mathbb{R}^3$. The velocity vector $\mathrm{d}x/\mathrm{d}t$ of this curve is parallel to the vector product of the normal vectors $\nabla\Psi$ and ∇H of the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, i.e.,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu \nabla\Psi \times \nabla H \tag{2}$$

where μ is any arbitrary function in $\mathbb{R}^3$. Equation (2) defines a Hamiltonian system in $\mathbb{R}^3$. In [1] we proved that all Hamiltonian systems in $\mathbb{R}^3$ are of the form (2).

In many examples the general form (1) of a Poisson structure is preserved globally, including the irregular points (points where the rank of the structure changes). That is a Poisson structure has the same form on different symplectic leaves [8].

The general form (1) allows to construct the compatible Poisson structures and the corresponding bi-Hamiltonian systems. The bi-Hamiltonian representation of a system is closely related to the notion of integrability. Given a bi-Hamiltonian system one can construct an infinite hierarchy of commuting first integrals, using