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EQUIVARIANT LOCALIZATION AND STATIONARY PHASE

GREGORY L. NABER

Department of Mathematics, California State University
Chico, CA 95929-0525, USA

Abstract. Equivariant cohomology in general and the equivariant localization theorems in particular have taken on a role of increasing significance in theoretical physics of late (see e. g. [3], [4] and [10]). These lectures are an attempt to provide a self-contained and elementary introduction to the Cartan model of equivariant cohomology, a complete proof of the simplest of the localization theorems, and, as an application, a proof of the famous Duistermaat–Heckman theorem on exact stationary phase approximations.

1. Stationary Phase Approximation

We consider a compact, oriented, smooth manifold M of dimension $n = 2k$ and denote by ν a volume form on M. Suppose $H\colon M \to \mathbb{R}$ is a Morse function on M, i. e., a smooth function whose critical points p $(\mathrm{d}H(p) = 0)$ are all nondegenerate (this means that the Hessian $\mathcal{H}_p\colon T_p(M) \times T_p(M) \to \mathbb{R}$, defined by $\mathcal{H}_p(V_p, W_p) = V_p(W(H))$, where $V_p, W_p \in T_p(M)$ and W is a vector field on M with $W(p) = W_p$, is a nondegenerate bilinear form). Finally, let T denote some real parameter. We consider the integral

$$\int\limits_M \mathrm{e}^{\mathrm{i}TH}\nu \tag{1.1}$$

and are especially interested in its asymptotic behavior as $T \to \infty$. The Stationary Phase Theorem (Chapter I of [6]) asserts roughly that, for large T, the dominant contributions to such an integral come from the critical points of H.