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HARMONIC FORMS ON COMPACT SYMPLECTIC 2-STEP NILMANIFOLDS

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Abstract. In this paper we study harmonic forms on compact symplectic nilmanifolds. We consider harmonic cohomology groups of dimension 3 and of codimension 2 for 2-step nilmanifolds and give examples of compact 2-step symplectic nilmanifolds G/Γ such that the dimension of harmonic cohomology groups varies.

1. Introduction

Let $(M, \mathbf{G})$ be a Poisson manifold with a Poisson structure $\mathbf{G}$, that is, a skew-symmetric contravariant 2-tensor $\mathbf{G}$ on M satisfying $[\mathbf{G}, \mathbf{G}] = 0$, where $[\,,\,]$ denotes the Schouten-Nijenhuis bracket. For a Poisson manifold $(M, \mathbf{G})$, Koszul [5] introduced a differential operator $\mathrm{d}^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ by $\mathrm{d}^* = [\mathrm{d}, i(\mathbf{G})]$, where $\Omega^k(M)$ denotes the space of all k-forms on M. The operator d^* is called the **Koszul differential**. For a symplectic manifold (M^{2m}, ω), let $\mathbf{G}$ be the skew-symmetric bivector field dual to ω. Then $\mathbf{G}$ is a Poisson structure on M. Brylinski [1] defined the star operator $*\colon \Omega^k(M) \to \Omega^{2m-k}(M)$ for the symplectic structure ω as an analogue of the star operator for an oriented Riemannian manifold and proved that the Koszul differential d^* satisfies $\mathrm{d}^* = (-1)^k * \mathrm{d} *$ on $\Omega^k(M)$ and the identity $*^2 = \mathrm{id}$. A form α on M is called **harmonic form** if it satisfies $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$. Let $\mathcal{H}^k_\omega(M) = \mathcal{H}^k(M)$ denote the space of all harmonic k-form on M. Brylinski [1] defined symplectic harmonic k-cohomology group $H^k_{\omega\text{-}hr}(M) = H^k_{hr}(M)$

by $\mathcal{H}^k_\omega(M)/(B^k(M)\cap\mathcal{H}^k_\omega(M))$, as a subspace of de Rham cohomology group $H^k_{DR}(M)$. We denote by $L_\omega\colon \Omega^k(M)\to\Omega^{k+2}(M)$ the linear operator defined by ω and the induced homomorphism in de Rham cohomology groups by $L_{[\omega]}\colon H^k_{DR}(M)\to H^{k+2}_{DR}(M)$.

Mathieu [6] proved the following theorem.

Theorem 1. (Mathieu) *Let (M^{2m},ω) be a symplectic manifold of dimension $2m$. Then the following two assertions are equivalent:*

a) *For any $k\leq m$, the homomorphism $L_{[\omega]^k}\colon H^{m-k}_{DR}(M)\to H^{m+k}_{DR}(M)$ is surjective.*

b) *For any k, $H^k_{DR}(M)=H^k_{\omega\text{-}hr}(M)$.*

Mathieu's theorem is a generalization of Hard Lefschetz Theorem for compact Kähler manifolds. Mathieu [6] proved also that, for $i=0,1,2$, $H^i_{hr}(M)=H^i_{DR}(M)$. Yan [10] gave a simpler, elegant proof of Mathieu's Theorem by using a special type of infinite dimensional $\mathfrak{sl}(2)$-representation theory.

In connection with the study of harmonic forms, we are interested in the following question raised by Khesin and McDuff (see Yan [10]).

Question: On which compact manifold M, there exists a family ω_t of symplectic forms such that the dimension of $H^k_{\omega_t\text{-}hr}(M)$ varies for some k?

For 6-dimensional compact nilmanifolds, the above question is considered independently by one of the present authors [9] and Ibáñez *et al* [3]. Actially Ibáñez *et al* [3] have proved that there exist at least five 6-dimensional nilmanifolds M with a family ω_t of symplectic forms such that the dimension of $H^k_{\omega_t\text{-}hr}(M)$ varies by computing $H^4_{\omega_t\text{-}hr}(M)$ and $H^5_{\omega_t\text{-}hr}(M)$. Note that, in [9], it is proved that the dimension of $H^{2m-1}_{\omega\text{-}hr}(M)$ for compact 2-step nilmanifold M^{2m} is independent of symplectic forms ω (cf. Theorem 5).

In this paper we study symplectic harmonic cohomology groups $H^3_{\omega\text{-}hr}(M)$ of dimension 3 and $H^{2m-2}_{\omega\text{-}hr}(M)$ of codimension 2 for compact nilmanifolds and give examples of higher dimensional compact 2-step symplectic nilmanifolds G/Γ such that the dimension of harmonic cohomology group varies.

2. Harmonic Cohomology Groups of Nimanifolds

For a $2m$-dimensional symplectic manifold (M,ω) let $\mathbf{G}$ be the skew-symmetric bivector field dual to ω. Then $\mathbf{G}$ is a Poisson structure on M. By the Darboux's theorem, going to the canonical coordinates $\{p_1,q_1,\ldots,p_m,q_m\}$, we can write symplectic structure ω as $\omega=\mathrm{d}p_1\wedge\mathrm{d}q_1+\cdots+\mathrm{d}p_m\wedge\mathrm{d}q_m$ and respectively the Poisson structure $\mathbf{G}$ as $\mathbf{G}=\dfrac{\partial}{\partial q_1}\wedge\dfrac{\partial}{\partial p_1}+\cdots+\dfrac{\partial}{\partial q_m}\wedge\dfrac{\partial}{\partial p_m}$.

Brylinski [1] defined the star operator $*\colon \Omega^k(M) \to \Omega^{2m-k}(M)$ by requiring

$$\alpha \wedge *\beta = (\wedge^k(\mathbf{G}))(\alpha, \beta) v_M$$

for k-forms α, β, where $v_M = \omega^m/m!$. The star operator $*$ satisfies the identities

$$*^2 = \mathrm{id}, \quad \mathrm{d}^* = (-1)^k * \mathrm{d} *$$

and consequently, the Koszul differential d^* is a symplectic codifferential of the exterior differential d with respect to the star operator $*$. We denote by $L_\omega = L\colon \Omega^k(M) \to \Omega^{k+2}(M)$ the operator defined by $L(\alpha) = \alpha \wedge \omega$.
The following Propositions are due to Yan [10]:

Proposition 1. (Duality on forms) *The linear mapping* $L^k\colon \Omega^{m-k}(M) \to \Omega^{m+k}(M)$ *is an isomorphism for any* k.

Proposition 2. (Duality on harmonic forms) *The linear mapping* $L^k : \mathcal{H}^{m-k}(M) \to \mathcal{H}^{m+k}(M)$ *is an isomorphism for any* k. *In particular, we have* $H^{m+k}_{hr}(M) = \mathrm{Im}\{L^k : H^{m-k}_{hr}(M) \to H^{m+k}_{DR}(M)\}$.

Note also that we have $H^i_{hr}(M) = H^i_{DR}(M)$ for $i = 0, 1, 2$. Thus we have the following corollary from Proposition 2.

Corollary 1. *We have*

$$H^{2m-1}_{hr}(M) = \mathrm{Im}\{L^{m-1}\colon H^1_{DR}(M) \to H^{2m-1}_{DR}(M)\}$$

and

$$H^{2m-2}_{hr}(M) = \mathrm{Im}\{L^{m-2}\colon H^2_{DR}(M) \to H^{2m-2}_{DR}(M)\}\,.$$

Let $\mathfrak{g}$ be a Lie algebra and put $\mathfrak{g}^{(0)} = \mathfrak{g}$ and let $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$ for $i \geq 0$. A Lie algebra $\mathfrak{g}$ is said to be $(r+1)$**-step nilpotent** if $\mathfrak{g}^{(r)} \neq (0)$ and $\mathfrak{g}^{(r+1)} = (0)$ and a Lie group G is said to be $(r+1)$-step nilpotent if the Lie algebra $\mathfrak{g}$ is $(r+1)$-step nilpotent. If G is a simply- connected $(r+1)$-step nilpotent Lie group and Γ is a lattice of G, that is, a discrete subgroup of G such that G/Γ is compact, then G/Γ is called an $(r+1)$**-step compact nilmanifold**. We denote by $\bigwedge^k \mathfrak{g}^*$ the space of all left G- invariant k-forms on G and regard it as a subspace of $\Omega^k(G/\Gamma)$. Then we have a subcomplex $(\bigwedge^* \mathfrak{g}^*, \mathrm{d})$ of the de Rham complex $(\Omega^*(G/\Gamma), \mathrm{d})$ of compact nilmanifold G/Γ and denote by $H^k(\mathfrak{g})$ the k-th cohomology groups of the complex $(\bigwedge^* \mathfrak{g}^*, \mathrm{d})$.

Theorem 2. (Nomizu) *For a compact nilmanifold* G/Γ, *the inclusion* $\iota\colon (\bigwedge^* \mathfrak{g}^*, \mathrm{d}) \to (\Omega^*(G/\Gamma), \mathrm{d})$ *induces an isomorphism on cohomology groups*

$$\iota^* : H^k(\mathfrak{g}) \cong H^k_{DR}(G/\Gamma)\,.$$

For a symplectic form ω on a compact nilmanifold G/Γ, there exists a G-invariant closed 2-form ω_0 on G/Γ such that $\omega - \omega_0 = \mathrm{d}\gamma$. Note that ω_0 is also a symplectic form on G/Γ. For a G-invariant symplectic form ω_0, we denote by $\mathcal{H}^k(\mathfrak{g})$ the space of all G-invariant harmonic k-forms on G/Γ and by $H^k_{\omega_0\text{-}hr}(\mathfrak{g}) = \mathcal{H}^k(\mathfrak{g})/(B^k(\mathfrak{g}) \cap \mathcal{H}^k(\mathfrak{g}))$ a subspace of Lie algebra cohomology group $H^k(\mathfrak{g})$.

In [9] we have proved the following propositions:

Proposition 3. *Let $(G/\Gamma, \omega)$ be a compact symplectic nilmanifold and let ω_0 be a G-invariant symplectic form such that $\omega - \omega_0 = \mathrm{d}\gamma$ as above. Then we have*

$$H^k_{\omega\text{-}hr}(G/\Gamma) = H^k_{\omega_0\text{-}hr}(G/\Gamma) = H^k_{\omega_0\text{-}hr}(\mathfrak{g}) \tag{1}$$

for any k.

Proposition 4. *Let $(G/\Gamma, \omega)$ be a $2m$-dimensional compact symplectic nilmanifold with a G- invariant symplectic form $\omega \in \bigwedge^2(\mathfrak{g}^*)$. Then the linear mapping*

$$L^k : \mathcal{H}^{m-k}_{\omega\text{-}hr}(\mathfrak{g}) \to \mathcal{H}^{m+k}_{\omega\text{-}hr}(\mathfrak{g})$$

is an isomorphism for any k.

Now we may assume that symplectic structures on G/Γ are G-invariant in order to study harmonic cohomology groups on compact nilmanifolds M. A nilpotent Lie algebra $\mathfrak{g}$ with a non-degenerate invariant closed 2-form is called a symplectic nilpotent Lie algebra. For an $(r+1)$-step nilpotent Lie algebra $\mathfrak{g}$, let $\mathfrak{a}^{(i)}$ denote a complementary vector subspace of $\mathfrak{g}^{(i+1)}$ in $\mathfrak{g}^{(i)}$: $\mathfrak{g}^{(i)} = \mathfrak{g}^{(i+1)} + \mathfrak{a}^{(i)}$ for $i = 0, 1, \ldots, r$. For simplicity, put $\bigwedge^{i_0,\ldots,i_r} = \bigwedge^{i_0} \mathfrak{a}^{(0)^*} \wedge \cdots \wedge \bigwedge^{i_r} \mathfrak{a}^{(r)^*}$. Then we have $\bigwedge^s \mathfrak{g}^* = \sum_{i_0+\cdots+i_r=s} \bigwedge^{i_0,\ldots,i_r}$.

The following lemma is due to Benson and Gordon [2].

Lemma 1. *Each closed 2-form $\theta \in \bigwedge^2 \mathfrak{g}^*$ belongs to $\bigwedge^{1,0,\ldots,0,1} + \sum \bigwedge^{i_0,\ldots,i_{r-1},0}$.*

Let $\{\lambda_1, \ldots, \lambda_{n_r}\}$ be a basis of $\bigwedge^{0,\ldots,0,1}$. By Lemma 1, a G-invariant symplectic form ω can be written in the form

$$\omega = \beta_1 \wedge \lambda_1 + \cdots + \beta_{n_r} \wedge \lambda_{n_r} \quad \text{modulo} \quad \sum \bigwedge^{i_0,\ldots,i_{r-1},0} \tag{2}$$

where $\beta_1, \ldots, \beta_{n_r}$ are elements of $\bigwedge^{1,0,\ldots,0}$. Note that $\beta_1, \ldots, \beta_{n_r}$ are linearly independent, since ω is non-degenerate. We extend these elements to a basis $\{\beta_1, \ldots, \beta_{n_r}, \ldots, \beta_{n_0}\}$ for $\bigwedge^{1,0,\ldots,0}$. Put $\mathfrak{a}_1^{(0)^*} = \mathrm{span}\{\beta_1, \ldots, \beta_{n_r}\}$ and $\mathfrak{a}_2^{(0)^*} = \mathrm{span}\{\beta_{n_r+1}, \ldots, \beta_{n_0}\}$. Then $\mathfrak{a}^{(0)^*} = \mathfrak{a}_1^{(0)^*} + \mathfrak{a}_2^{(0)^*}$. Put $n_0^1 = \dim \mathfrak{a}_1^{(0)^*} = n_r$ and $n_0^2 = \dim \mathfrak{a}_2^{(0)^*} = n_0 - n_r$. For simplicity, we also put $\bigwedge^{(i_0^1, i_0^2), i_1, \ldots, i_r} =$

$\bigwedge^{i_0^1} \mathfrak{a}_1^{(0)^*} \wedge \bigwedge^{i_0^2} \mathfrak{a}_2^{(0)^*} \wedge \bigwedge^{i_1} \mathfrak{a}^{(1)^*} \wedge \cdots \wedge \bigwedge^{i_r} \mathfrak{a}^{(r)^*}$. Moreover, let $\{\beta_1^{(k)}, \ldots, \beta_{n_k}^{(k)}\}$ be a basis of $\mathfrak{a}^{(k)^*}$ and put

$$\{\omega_1, \ldots, \omega_{2m}\} = \{\beta_1, \ldots, \beta_{n_0}, \ldots, \beta_1^{(k)}, \ldots, \beta_{n_k}^{(k)}, \ldots, \lambda_1, \ldots, \lambda_{n_r}\}.$$

Let $\{X_1, \ldots, X_{2m}\}$ be a basis of $\mathfrak{g}$ which is dual to the basis $\{\omega_1, \ldots, \omega_{2m}\}$. If we write the symplectic form ω as

$$\omega = \sum a_{ij}\omega_i \wedge \omega_j \qquad a_{ij} = -a_{ji} \in \mathbb{R}$$

then it is easy to see that the Poisson structure $\mathbf{G}$ which is dual to ω is given by

$$\mathbf{G} = -\sum c_{ij} X_i \wedge X_j \tag{3}$$

where c_{ij} is the inverse of the transposed matrix of (a_{ij}).

Lemma 2. *With respect to the basis $\{X_1, \ldots, X_{2m}\}$ above, $\mathbf{G}$ is given in the form*

$$(c_{ij}) = \left(\begin{array}{c|c|c|c} 0_{n_r,n_r} & 0_{n_r,n_0-n_r} & 0_{n_r,2m-n_0-n_r} & E_{n_r} \\ \hline 0_{n_0-n_r,n_r} & * & * & * \\ \hline 0_{2m-n_0-n_r,n_r} & * & * & * \\ \hline -E_{n_r} & * & * & * \end{array}\right). \tag{4}$$

Proof: Note that (c_{ij}) is an alternating matrix. We have $a_{j\,n_0+\cdots+n_{r-1}+i} = \delta_{ji}$ for $i, j = 1, \ldots, n_r$ and $a_{jk} = 0$ for $j = 1, \ldots, n_r$; $k = 1, \ldots, n_0 + \cdots + n_{r-1}$ from (2). Then we have

$$c_{ik} = c_{ik} a_{i\,n_0+\cdots+n_{r-1}+i} = \sum_{l=1}^{n_0+\cdots+n_r} c_{lk} a_{l\,n_0+\cdots+n_{r-1}+i} = \delta_{k\,n_0+\cdots+n_{r-1}+i}$$

for $i = 1, \ldots, n_r$ and $k = 1, \ldots, n_0 + \cdots + n_r$. □

Lemma 3. *Let G/Γ be a 2-step compact nilmanifold with a G-invariant symplectic form $\omega = \beta_1 \wedge \lambda_1 + \cdots + \beta_{n_1} \wedge \lambda_{n_1}$ modulo $\bigwedge^{2,0}$. Then we have*

$$\mathrm{d}\lambda_k = \sum_{i<j\leq n_1} b_{ij}^k \beta_i \wedge \beta_j + \sum_{i\leq n_1<j} b_{ij}^k \beta_i \wedge \beta_j \in \bigwedge^{(2,0),0} + \bigwedge^{(1,1),0}$$

for $k = 1, \ldots, n_1$.

Proof: Put $\mathrm{d}\lambda_k = \sum_{i<j\le n_1} b^k_{i,j}\beta_i\wedge\beta_j + \sum_{i\le n_1<j} b^k_{ij}\beta_i\wedge\beta_j + \sum_{n_1<i<j} b^k_{ij}\beta_i\wedge\beta_j$. Since ω is closed, we have

$$\beta_1 \wedge \sum_{n_1\le i<j} b^1_{ij}\beta_i\wedge\beta_j + \cdots + \beta_{n_1}\wedge \sum_{n_1\le i<j} b^{n_1}_{ij}\beta_i\wedge\beta_j = 0\,.$$

Therefore, since $\beta_1\wedge\beta_{i_1}\wedge\beta_{j_1},\ldots,\beta_{n_1}\wedge\beta_{i_{n_1}}\wedge\beta_{j_{n_1}}$ $(n_1<i_k<j_k)$ are linearly independent, we conclude that $b^k_{ij}=0$ for $n_1<i<j$. □

Note that, from Lemma 2, we have $i(\mathbf{G})(\bigwedge^{(2,0)}) = i(\mathbf{G})(\bigwedge^{(1,1)}) = 0$.

Theorem 3. *Let G/Γ be a 2-step compact nilmanifold with a G-invariant symplectic form ω. Then we have $B^3(\mathfrak{g})\subset\mathcal{H}^3(\mathfrak{g})$.*

Proof: Since $\mathrm{d}\bigwedge^{2,0}=0$ and $\mathrm{d}\bigwedge^{1,1}\subset\bigwedge^{3,0}$, we have to consider only the case of $\bigwedge^{0,2}$. From Lemma 3, we end with

$$\overset{0,2}{\bigwedge} \xrightarrow{\ \mathrm{d}\ } \overset{(2,0),1}{\bigwedge} + \overset{(1,1),1}{\bigwedge}\,.$$

Thus, from (4) in Lemma 2, we get

$$i(\mathbf{G})(\mathrm{d}\overset{0,2}{\bigwedge}) \subset \overset{1,0}{\bigwedge} \xrightarrow{\ \mathrm{d}\ } 0,$$

which implies that $\mathrm{d}\bigwedge^{0,2}\subset\mathcal{H}^3(\mathfrak{g})$. □

By a straightforward computation, we have also that

$$\begin{aligned} &*(\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}) \\ &= \sum_{j_1<\cdots<j_s} (-1)^s a\det(c_{kh})^{k=j_1,\ldots,j_s}_{h=i_1,\ldots,i_s}\omega_1\wedge\cdots\wedge\hat{\omega}_{j_1}\wedge\cdots\wedge\hat{\omega}_{j_s}\wedge\cdots\wedge\omega_{2m} \end{aligned} \tag{5}$$

where $\omega^m/m! = a\cdot(\omega_{j_1}\wedge\cdots\wedge\omega_{j_s})\wedge(\omega_1\wedge\cdots\wedge\hat{\omega}_{j_1}\wedge\cdots\wedge\hat{\omega}_{j_s}\wedge\cdots\wedge\omega_{2m})$. In fact, let

$$*(\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}) = \sum_{j_1<\cdots<j_s} a^{j_1\ldots j_s}_{i_1\ldots i_s}\omega_1\wedge\cdots\wedge\hat{\omega}_{j_1}\wedge\cdots\wedge\hat{\omega}_{j_s}\wedge\cdots\wedge\omega_{2m}.$$

Then, we get that

$$\begin{aligned} &(\omega_{j_1}\wedge\cdots\wedge\omega_{j_s})\wedge*(\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}) \\ &\quad = (\overset{s}{\bigwedge}(\mathbf{G}))(\omega_{j_1}\wedge\cdots\wedge\omega_{j_s},\omega_{i_1}\wedge\cdots\wedge\omega_{i_s})\omega^m/m! \\ &\quad = \det(i(\mathbf{G})(\omega_k,\omega_h))^{k=j_1,\ldots,j_s}_{h=i_1,\ldots,i_s}\omega^m/m! = (-1)^s\det(c_{kh})^{k=j_1,\ldots,j_s}_{h=i_1,\ldots,i_s}\omega^m/m!\,. \end{aligned}$$

Thus we have finally

$$a^{j_1\dots j_s}_{i_1\dots i_s} = (-1)^s a \cdot \det(c_{kh})^{k=j_1,\dots,j_s}_{h=i_1,\dots,i_s}.$$

Theorem 4. *Let $\mathfrak{g}$ is an $(r+1)$-step symplectic nilpotent Lie algebra. Then, for $q = 0, \dots, n_r$, we have*

$$\bigwedge^{n_0,\dots,n_{r-1},n_r-q} \subset \mathcal{H}^{2m-q}(\mathfrak{g}).$$

Proof: Note that the star operator $*\colon \mathcal{H}^{m-k}(\mathfrak{g}) \longrightarrow \mathcal{H}^{m+k}(\mathfrak{g})$ is an isomorphism for each k and we have $\bigwedge^{(q,0),0,\dots,0} \subset \mathcal{H}^q(\mathfrak{g})$ from (4) in Lemma 2. Now, from (5) and (4) in Lemma 2, we see that the star operator

$$*\colon \bigwedge^{(q,0),0,\dots,0} \longrightarrow \bigwedge^{n_0,\dots,n_{r-1},n_r-q}$$

is an isomorphism. Thus we have $\bigwedge^{n_0,\dots,n_{r-1},n_r-q} \subset \mathcal{H}^{2m-q}(\mathfrak{g})$ for $q = 0, \dots, n_r$. □

Corollary 2. *Let G/Γ be a 2-step compact nilmanifold with a G-invariant symplectic form ω. Then we have*

$$\bigwedge^{n_0,n_1-q} \subset \mathcal{H}^{2m-q}(\mathfrak{g}).$$

In particular, we have that

$$\begin{aligned} &\dim H^2_{DR}(G/\Gamma) - \dim H^{2m-2}_{hr}(G/\Gamma) \\ &\quad = \dim(\mathrm{d}\bigwedge^{n_0-3,n_1} \cap \mathcal{H}(\mathfrak{g})) + \dim \mathrm{d}\bigwedge^{n_0-2,n_1-1} - n_1 . \end{aligned} \tag{6}$$

Proof: Since $\mathrm{d}\bigwedge^{n_0-2,n_1-1} \subset \bigwedge^{n_0,n_1-2}$, $\mathrm{d}\bigwedge^{n_0-2,n_1-1}$ is a subspace of $\mathcal{H}^{2m-2}(\mathfrak{g})$. Thus we have that

$$\begin{aligned} &\dim H^2(\mathfrak{g}) - \dim H^{2m-2}_{hr}(\mathfrak{g}) \\ &\quad = \dim \mathcal{H}^2(\mathfrak{g}) - \dim(B^2(\mathfrak{g}) \cap \mathcal{H}^2(\mathfrak{g})) - \dim \mathcal{H}^{2m-2}(\mathfrak{g}) \\ &\qquad + \dim(B^{2m-2}(\mathfrak{g}) \cap \mathcal{H}^{2m-2}(\mathfrak{g})) \\ &\quad = \dim(B^{2m-2}(\mathfrak{g}) \cap \mathcal{H}^{2m-2}(\mathfrak{g})) - \dim B^2(\mathfrak{g}) \\ &\quad = \dim(\mathrm{d}\bigwedge^{n_0-3,n_1} \cap \mathcal{H}^{2m-2}(\mathfrak{g})) + \dim \mathrm{d}\bigwedge^{n_0-2,n_1-1} - \dim B^2(\mathfrak{g}) \\ &\quad = \dim(\mathrm{d}\bigwedge^{n_0-3,n_1} \cap \mathcal{H}^{2m-2}(\mathfrak{g})) + \dim \mathrm{d}\bigwedge^{n_0-2,n_1-1} - n_1 . \end{aligned}$$

□

Theorem 5. *Let G/Γ be a 2-step compact nilmanifold with a G-invariant symplectic form ω. Then we have*

$$\bigwedge^{(n_0^1, n_0^2 - p), n_1 - q} \subset \mathcal{H}^{2m-p-q}(\mathfrak{g})\,. \tag{7}$$

In particular,

$$\bigwedge^{(n_0^1, n_0^2 - 1), n_1} + \bigwedge^{(n_0^1, n_0^2), n_1 - 1} = \mathcal{H}^{2m-1}(\mathfrak{g})$$

which implies $\dim H_{hr}^{2m-1}(\mathfrak{g}) = n_0^2 = \dim \mathfrak{g} - 2\dim[\mathfrak{g}, \mathfrak{g}]$.

Proof: From Lemma 3, it is obvious that $\bigwedge^{(n_0^1, n_0^2-p), n_1-q} \subset Z^{2m-p-q}(\mathfrak{g})$. Since $\mathrm{d}^* = (-1)^k * \mathrm{d} *$ on $\bigwedge^k(\mathfrak{g}^*)$, it is enough to prove that

$$*\colon \bigwedge^{(n_0^1, n_0^2 - p), n_1 - q} \longrightarrow \bigwedge^{p+q, 0}\,.$$

Note also that

$$\begin{aligned} &* \left(\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}\right) \\ &= \sum_{j_1 < \cdots < j_s} (-1)^s a \det \left(c_{kh}\right)_{h=i_1,\ldots,i_s}^{k=j_1,\ldots,j_s} \omega_1 \wedge \cdots \wedge \hat{\omega}_{j_1} \wedge \cdots \wedge \hat{\omega}_{j_s} \wedge \cdots \wedge \omega_{2m} \end{aligned} \tag{8}$$

where $s = n_0^1 + n_0^2 + n_1 - p - q$. Thus if $\{j_1, \ldots, j_s\} \not\ni n_0 + j$, then we get that $\det(c_{kh})_{h=i_1,\ldots,i_s}^{k=j_1,\ldots,j_s} = 0$ from Lemma 2. In fact, noting that $n_0^1 = n_1$, we have

$$\begin{aligned} \det(c_{kh})_{h=i_1,\ldots,i_s}^{k=j_1,\ldots,j_s} &= \begin{vmatrix} c^{j_1}{}_1 & \ldots & c^{j_1}{}_{n_0{}^1} & \ldots & c^{j_1}{}_{i_s} \\ \vdots & & \vdots & & \vdots \\ c^{j_r}{}_1 & \ldots & c^{j_r}{}_{n_0{}^1} & \ldots & c^{i_s}{}_{j_s} \end{vmatrix} \\ &= \begin{vmatrix} c^{j_1}{}_1 & \ldots & 0 & \ldots & c^{j_1}{}_{n_0{}^1} & \ldots & c^{j_1}{}_{i_s} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c^{j_s}{}_1 & \ldots & 0 & \ldots & c^{j_s}{}_{n_0{}^1} & \ldots & c^{i_s}{}_{j_s} \end{vmatrix} = 0\,. \end{aligned}$$

Thus we get that if $\det(c_h^k)_{h=i_1,\ldots,i_s}^{k=j_1,\ldots,j_s} \neq 0$, $\{j_1, \ldots, j_s\} \supset \{n_0 + 1, \ldots, n_0 + n_1\}$. Therefore we have $*(\bigwedge^{(n_0^1, n_0^2-p), n_1-q}) \subset \bigwedge^{p+q,0}$. □

In particular, we have $\bigwedge^{(n_0^1, n_0^2-1), n_1} + \bigwedge^{(n_0^1, n_0^2), n_1-1} \subset \mathcal{H}^{2m-1}(\mathfrak{g})$. Since $L^{m-1}\colon \mathcal{H}^1(\mathfrak{g}) \to \mathcal{H}^{2m-1}(\mathfrak{g})$ is an isomorphism by Proposition 2, we obtain that $\dim \mathcal{H}^{2m-1}(\mathfrak{g}) = n_0$. On the other hand, we have that $\dim(\bigwedge^{(n_0{}^1, n_0{}^2-1), n_1} + \bigwedge^{(n_0{}^1, n_0{}^2), n_1-1}) = (n_0 - n_1) + n_1 = \dim \mathcal{H}^{2m-1}(\mathfrak{g})$. Thus we have proved our second claim. The last claim follows from the fact that $B^{2m-1}(\mathfrak{g}) = \bigwedge^{(n_0^1, n_0^2), n_1-1}$ which is due to Benson and Gordon [2].

3. Examples

Example 1. We consider a 2-step nilpotent Lie algebra of dimension $2m$ given by $\mathfrak{g} = \mathrm{span}\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$ with $[X_i, X_{i+1}] = Y_i$ $i = 1, \dots, m$, where we set $X_{m+1} = X_1$. Then a simply connected nilpotent Lie group G with the Lie algebra $\mathfrak{g}$ has a lattice Γ. Let $\{\beta_1, \dots, \beta_m, \lambda_1, \dots, \lambda_m\}$ be the dual basis of $\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$. Note that $\mathrm{d}\bigwedge^{k,0} = (0)$ and $\mathrm{d}\bigwedge^{k,j} \subset \bigwedge^{k+2,j-1}$. It is easy to see that $\mathrm{d}\colon \bigwedge^{0,j} \to \bigwedge^{2,j-1}$ is injective.
The space of invariant closed 2-forms $Z^2(\mathfrak{g})$ is given by

$$Z^2(\mathfrak{g}) = \left\{ \sum_{i=1}^{m} a_{ii+1}(\beta_i \wedge \lambda_{i+1} - \beta_{i+2} \wedge \lambda_i) + \sum_{i=1}^{m} a_{ii}\, \beta_i \wedge \lambda_i \right.$$
$$\left. + \sum_{i=1}^{m} a_{i+1i}\, \beta_{i+1} \wedge \lambda_i \, ; \, a_{ii+1}, a_{ii}, a_{i+1i} \in \mathbb{R} \right\} + \bigwedge^{2,0}$$

where we have put $\beta_{m+1} = \beta_1$, $\beta_{m+2} = \beta_2$, $\lambda_{m+1} = \lambda_1$.
Now we consider a symplectic form ω given by an element of $\bigwedge^{1,1}$. Then the Poisson structure $\mathbf{G}$ is of the form

$$\mathbf{G} = -\sum_{i,j=1}^{m} c_{ij}\, X_i \wedge Y_j$$

with respect to the basis $\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$. We see that

$$i(\mathbf{G})\colon \bigwedge^{3,0} \to (0), \quad i(\mathbf{G})\colon \bigwedge^{2,1} \to \bigwedge^{1,0}, \quad i(\mathbf{G})\colon \bigwedge^{1,2} \to \bigwedge^{0,1}$$

and the space of harmonic 3-forms $\mathcal{H}^3(\mathfrak{g})$ is given by

$$\mathcal{H}^3(\mathfrak{g}) = \bigwedge^{3,0} + Z^3(\mathfrak{g}) \cap \bigwedge^{2,1} + \mathcal{H}^3(\mathfrak{g}) \cap \bigwedge^{1,2}.$$

For $m \geq 6$, we see that

$$Z^3(\mathfrak{g}) \cap \bigwedge^{1,2} = \left\{ \sum_{j=1}^{m} b_j\, \beta_{j+1} \wedge \lambda_j \wedge \lambda_{j+1} \, ; \, b_j \in \mathbb{R}, \ j = 1, \dots, m \right\}$$

where we have put $\beta_{m+1} = \beta_1$, $\lambda_{m+1} = \lambda_1$.
Case 1. The case when $\omega = \sum_{j=1}^{m} a_{jj} \beta_j \wedge \lambda_j$ where $a_{jj} \neq 0$.
Note that $\mathbf{G} = -\sum_{j=1}^{m} \dfrac{1}{a_{jj}} X_j \wedge Y_j$ and hence we have $\mathcal{H}^3(\mathfrak{g}) \cap \bigwedge^{1,2} = (0)$ for $m \geq 6$.
Case 2. The case when $\omega = \sum_{j=1}^{m-1} a_{jj+1}(\beta_j \wedge \lambda_{j+1} - \beta_{j+2} \wedge \lambda_j)$.

Note that for the above basis ω can be written in the form $\begin{pmatrix} 0 & A \\ -{}^tA & 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 0 & a_{12} & 0 & 0 & 0 & \dots & 0 & -a_{m-1\,m} & 0 \\ 0 & 0 & a_{23} & 0 & 0 & \dots & 0 & 0 & 0 \\ -a_{12} & 0 & 0 & a_{34} & 0 & \dots & 0 & 0 & 0 \\ 0 & -a_{23} & 0 & 0 & a_{45} & \dots & 0 & 0 & 0 \\ 0 & 0 & -a_{34} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{45} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & a_{m-3\,m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{m-2\,m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{m-1\,m} \\ 0 & 0 & 0 & 0 & 0 & \dots & -a_{m-2\,m-1} & 0 & 0 \end{pmatrix}$$

We can prove also that, for $m \geq 3$, the matrix A is non-degenerate if and only if $m = 3\ell$ for $\ell \in \mathbb{N}$. Put $A^{-1} = (c_{ij})$ for $m = 3\ell$. Then we have also that the components of the matrix $A^{-1} = (c_{ij})$ satisfy the conditions

$$c_{jj} = 0 \ \text{ for } j = 1, \dots, m, \quad c_{j+1j} = 0 \ \text{ for } \ j = 1, \dots, m-1 \ \text{ and } \ c_{1m} = 0.$$

Since $\mathbf{G}$ is given by $\begin{pmatrix} 0 & -A^{-1} \\ {}^tA^{-1} & 0 \end{pmatrix}$, we see that for $\alpha = \sum_{j=1}^m b_j \beta_{j+1} \wedge \lambda_j \wedge \lambda_{j+1}$, $i(\mathbf{G})\alpha = 0$ and α therefore is harmonic. This implies that $\mathcal{H}^3(\mathfrak{g}) \cap \bigwedge^{1,2} = Z^3(\mathfrak{g}) \cap \bigwedge^{1,2}$ and $\dim(\mathcal{H}^3(\mathfrak{g}) \cap \bigwedge^{1,2}) = 3\ell$ for $\ell \geq 2$. Thus, from Theorem 3, we see that compact 2-step nilmanifolds G/Γ admit such symplectic structures that the dimension of harmonic cohomology group $H^3_{\omega\text{-}hr}(G/\Gamma)$ varies.

Example 2. For $p \geq 2$ let $\mathfrak{h}(1,p)$ be a 2-step nilpotent Lie algebra of dimension $2p+1$ spanned by $\{X_1, \dots, X_{2p+1}\}$ with

$$[X_1, X_i] = X_{p+i} \quad i = 2, \dots, p+1$$

We consider the Lie algebra $\mathfrak{g} = \mathfrak{h}(1,p) \oplus \mathbb{R}$ of dimension $2p+2$. Then a simply connected nilpotent Lie group G with the Lie algebra $\mathfrak{g}$ has a lattice Γ. Let X_{2p+2} denote a generator of the Lie algebra $\mathbb{R}$ and let $\{\omega_1, \dots, \omega_{p+1}, \omega_{p+2}, \dots, \omega_{2p+1}, \omega_{2p+2}\}$ be the dual basis of the basis $\{X_1, \dots, X_{p+1}, X_{p+2}, \dots, X_{2p+1}, X_{2p+2}\}$. Then we have

$$\bigwedge^{(0,1),0} \supset \operatorname{span}\{\omega_{2p+2}\}, \qquad \bigwedge^{(0,0),1} = \operatorname{span}\{\omega_{p+2}, \dots, \omega_{2p+1}\}.$$

Consider a G-invariant symplectic structure ω on G/Γ. We write the Poisson structure $\mathbf{G}$ dual to ω as $\mathbf{G} = -\sum c_{ij} X_i \wedge X_j$ with respect to the basis $\{X_1, \dots, X_{2p+1}, X_{2p+2}\}$ above.

We prove that

$$\dim H^2(\mathfrak{g}) - \dim H^{2p}_{\omega\text{-}hr}(\mathfrak{g}) = \begin{cases} {}_pC_2, & \text{if } c_{1\,2p+2} \neq 0 \\ {}_pC_2 + (p-1) & \text{if } c_{1\,2p+2} = 0\,. \end{cases}$$

In this example we use the following notations. For $i < j$, we put

$$\hat{\omega}_i\hat{\omega}_j = \omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_{2p+2}.$$

Similarly, for $i_1 < \cdots < i_k$ we put

$$\hat{\omega}_{i_1} \ldots \hat{\omega}_{i_k} = \omega_1 \wedge \cdots \wedge \hat{\omega}_{i_1} \wedge \cdots \wedge \hat{\omega}_{i_k} \wedge \cdots \wedge \omega_{2p+2}.$$

For $2 \leq i < j \leq p+1$, we put

$$\alpha_{ij} = \omega_2 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_{p+1} \wedge \omega_{2p+2} \wedge \mathrm{d}(\omega_{p+2} \wedge \cdots \wedge \omega_{2p+1})\,.$$

Then we have that

$$\begin{array}{ll} \text{for } 2 \leq i < j < k, & \mathrm{d}(\hat{\omega}_i\hat{\omega}_j\hat{\omega}_k) = 0 \\ \text{for } 2 \leq i < j \leq p+1, & \mathrm{d}(\hat{\omega}_1\hat{\omega}_i\hat{\omega}_j) = -\alpha_{ij} \\ \text{for } 2 \leq i \leq p+1, & \mathrm{d}(\hat{\omega}_1\hat{\omega}_i\hat{\omega}_{2p+2}) = (-1)^p\hat{\omega}_{p+i}\hat{\omega}_{2p+2} \\ \text{for } 2 \leq i, j \leq p+1, & \mathrm{d}(\hat{\omega}_1\hat{\omega}_i\hat{\omega}_{p+j}) = \begin{cases} \pm\hat{\omega}_{p+i}\hat{\omega}_{p+j} & \text{for } i < j \\ \pm\hat{\omega}_{p+j}\hat{\omega}_{p+i} & \text{for } j < i \\ 0 & \text{for } i = j \end{cases} \\ \text{for } 2 \leq i < j \leq p+1, & \mathrm{d}(\hat{\omega}_1\hat{\omega}_{p+i}\hat{\omega}_{p+j}) = 0 \\ \text{for } 2 \leq i \leq p+1, & \mathrm{d}(\hat{\omega}_1\hat{\omega}_{p+i}\hat{\omega}_{2p+2}) = 0. \end{array} \tag{9}$$

From (6) of Corollary 2, we have

$$\dim H^2(\mathfrak{g}) - \dim H^{2p}_{\omega\text{-}hr}(\mathfrak{g}) = \dim(\mathrm{d}\bigwedge^{p-1,p} \cap \mathcal{H}^{2p}_{\omega}(\mathfrak{g})) + \dim \mathrm{d}\bigwedge^{p,p-1} - p\,. \tag{10}$$

Note that $\mathrm{d}(\hat{\omega}_1\hat{\omega}_i\hat{\omega}_{2p+2}) = (-1)^p\hat{\omega}_{p+i}\hat{\omega}_{2p+2} \in \bigwedge^{(p,1),p-1}$ for $2 \leq i \leq p+1$. Thus, from Theorem 5, we see that $\mathrm{d}(\hat{\omega}_1\hat{\omega}_i\hat{\omega}_{2p+2}) \in \mathcal{H}^{2p}_{\omega}(\mathfrak{g})$. From (9), we see that $\dim \mathrm{d}\bigwedge^{p,p-1} = {}_pC_2$. We put

$$V = \mathrm{span}\{\alpha_{ij}\,;\, 2 \leq i < j \leq p+1\}\,.$$

Then, from (10), we have

$$\dim H^2(\mathfrak{g}) - \dim H^{2p}_{\omega\text{-}hr}(\mathfrak{g}) = \dim(V \cap \mathcal{H}^{2p}_{\omega}(\mathfrak{g})) + {}_pC_2\,. \tag{11}$$

Note that $\mathfrak{a}^{(0)} = \mathrm{span}\{X_1, \ldots, X_{p+1}, X_{2p+2}\}$ and $\mathfrak{a}^{(1)} = \mathrm{span}\{X_{p+2}, \ldots, X_{2p+1}\}$. Put $\bigwedge_{i_0,i_1} = \bigwedge^{i_0}\mathfrak{a}^{(0)} \wedge \bigwedge^{i_1}\mathfrak{a}^{(1)}$ and write the Poisson structure $\mathbf{G}$ dual to ω as

$$\mathbf{G} = \mathbf{G}_{2,0} + \mathbf{G}_{1,1} + \mathbf{G}_{0,2} \qquad \mathbf{G}_{i,j} \in \bigwedge_{i,j}\,.$$

Note that

$$\begin{aligned} \mathrm{d}i(\mathbf{G}_{2,0})\Big(\bigwedge^{p+1,p-1}\Big) &= \mathrm{d}\bigwedge^{p-1,p-1} \subset \bigwedge^{p+1,p-2} \\ \mathrm{d}i(\mathbf{G}_{1,1})\Big(\bigwedge^{p+1,p-1}\Big) &= \mathrm{d}\bigwedge^{p,p-2} \subset \bigwedge^{p+2,p-3} \\ \mathrm{d}i(\mathbf{G}_{0,2})\Big(\bigwedge^{p+1,p-1}\Big) &= \mathrm{d}\bigwedge^{p+1,p-3} = (0)\,. \end{aligned} \tag{12}$$

Note that, from Lemma 1, the G-invariant symplectic structure ω on G/Γ is of the form

$$\omega = \sum_{i<j\le p+1} a_{ij}\omega_i\wedge\omega_j + \sum_{j\le 2p+1} a_{j\,2p+2}\omega_j\wedge\omega_{2p+2} + \sum_{i\le p+1<j\le 2p+1} a_{ij}\omega_i\wedge\omega_j\,.$$

Since $\mathrm{d}\omega_{p+i} = -\omega_1\wedge\omega_i$, we see that $a_{p+i\,2p+2}=0$ for $2\le i\le p+1$. Thus the matrix form of $\mathbf{G}$ with respect to the basis $\{X_1,\dots,X_{2p+1},X_{2p+2}\}$ is given by

$$-\begin{pmatrix} & & & c_{1\,p+2} & \cdots & c_{1\,2p+1} & c_{1\,2p+2} \\ & 0 & & \vdots & & \vdots & \vdots \\ & & & c_{p+1\,p+2} & \cdots & c_{p+1\,2p+1} & c_{p+1\,2p+2} \\ -c_{1\,p+2} & \cdots & -c_{p+1\,p+2} & & & & \\ \vdots & & \vdots & & & & \\ -c_{1\,2p+1} & \cdots & -c_{p+1\,2p+1} & & & * & \\ -c_{1\,2p+2} & \cdots & -c_{p+1\,2p+2} & & & & \end{pmatrix}.$$

Thus $\mathbf{G}_{2,0} = -c_{1\,2p+2}X_1\wedge X_{2p+2}$. Moreover, we have

$$\begin{aligned} \mathrm{d}i(\mathbf{G}_{2,0})(\alpha_{ij}) &= -\,\mathrm{d}i(\mathbf{G}_{2,0})(\omega_1\wedge\cdots\wedge\hat{\omega}_i\wedge\cdots\wedge\omega_{p+1}\wedge\omega_{2p+2} \\ &\qquad \wedge\omega_{p+2}\wedge\cdots\wedge\hat{\omega}_{p+j}\wedge\cdots\wedge\omega_{2p+1}) \\ &\quad + \mathrm{d}i(\mathbf{G}_{2,0})(\omega_1\wedge\cdots\wedge\hat{\omega}_j\wedge\cdots\wedge\omega_{p+1}\wedge\omega_{2p+2} \\ &\qquad \wedge\omega_{p+2}\wedge\cdots\wedge\hat{\omega}_{p+i}\wedge\cdots\wedge\omega_{2p+1}) \\ &= -2c_{1\,2p+2}(\omega_2\wedge\cdots\wedge\omega_{p+1}\wedge\omega_{p+2}\wedge\dots \\ &\qquad \wedge\hat{\omega}_{p+i}\wedge\cdots\wedge\hat{\omega}_{p+j}\wedge\cdots\wedge\omega_{2p+1})\,. \end{aligned} \tag{13}$$

i) The case of $c_{12p+2}\neq 0$.
From (12) and (13), we see that $V\cap\mathcal{H}^{2p}(\mathfrak{g})=(0)$, and hence we get

$$\dim H^2(\mathfrak{g}) - \dim H^{2p}_{\omega\text{-}hr}(\mathfrak{g}) = {}_pC_2\,.$$

ii) The case of $c_{12p+2}=0$.

Since $\mathbf{G}$ is non-degenerate, $c_{1\,p+k} \neq 0$ for some $k \in \{2, \ldots, p+1\}$. For simplicity, we may assume that $c_{1\,p+2} \neq 0$.
Now we put

$$W = \mathrm{span}\{\hat{\omega}_{p+i}\hat{\omega}_{p+j}\hat{\omega}_{p+k}\,;\, 2 \leq i < j < k \leq p+1\}\,.$$

Note that $\dim V = {}_pC_2$ and $\dim W = {}_pC_3$.
We consider the linear mapping $\mathrm{d}i(\mathbf{G}_{1,1})\colon V \longrightarrow W$. We claim that

$$\dim \mathrm{d}i(\mathbf{G}_{1,1})(V) = {}_{p-1}C_2\,.$$

Then $\dim \mathrm{Ker}(\mathrm{d}i(\mathbf{G}_{1,1})) = {}_pC_2 - {}_{p-1}C_2 = p-1$ and hence

$$\dim H^2(\mathfrak{g}) - \dim H^{2p}_{\omega\text{-}hr}(\mathfrak{g}) = {}_pC_2 + p - 1\,.$$

By a straightforward computation, we see

$$\begin{aligned} \mathrm{d}i(\mathbf{G}_{1,1})(\alpha_{ij}) = {} & 2\sum_{k<i}(-1)^{p+k}c_{1\,p+k}\hat{\omega}_{p+k}\hat{\omega}_{p+i}\hat{\omega}_{p+j} \\ & - 2\sum_{i<k<j}(-1)^{p+k}c_{1\,p+k}\hat{\omega}_{p+i}\hat{\omega}_{p+k}\hat{\omega}_{p+j} \\ & + 2\sum_{j<k}(-1)^{p+k}c_{1\,p+k}\hat{\omega}_{p+i}\hat{\omega}_{p+j}\hat{\omega}_{p+k}\,. \end{aligned} \tag{14}$$

Consider the basis

$$\{\alpha_{34}, \ldots, \alpha_{3\,p+1}, \alpha_{45}, \ldots, \alpha_{p\,p+1}, \alpha_{23}, \ldots, \alpha_{2\,p+1}\}$$

of V and the basis

$$\{\hat{\omega}_{p+2}\hat{\omega}_{p+3}\hat{\omega}_{p+4}, \hat{\omega}_{p+2}\hat{\omega}_{p+3}\hat{\omega}_{p+5}, \ldots, \hat{\omega}_{p+2}\hat{\omega}_{2p}\hat{\omega}_{2p+1}, \hat{\omega}_{p+3}\hat{\omega}_{p+4}\hat{\omega}_{p+5}, \ldots, \hat{\omega}_{2p-1}\hat{\omega}_{2p}\hat{\omega}_{2p+1}\}$$

of W. Then, from (14), we see that, with respect to the bases above, the matrix form of $\mathrm{d}i(\mathbf{G}_{1,1})$ is of the form

$$\mathrm{d}i(\mathbf{G}_{1,1}) = \left(\begin{array}{ccc|c} 2(-1)^p c_{1\,p+2} & & 0 & \\ & \ddots & & C \\ 0 & & 2(-1)^p c_{1\,p+2} & \\ \hline & D & & 0 \end{array}\right)$$

where D is a matrix of ${}_{p-1}C_3 \times {}_{p-1}C_2$ and C is a matrix of ${}_{p-1}C_2 \times (p-1)$. Then we see that the $\mathrm{rank}(\mathrm{d}i(\mathbf{G}_{1,1})) \geq {}_{p-1}C_2$ and hence $\dim \mathrm{Ker}(\mathrm{d}i(\mathbf{G}_{1,1})) \leq {}_pC_2 - {}_{p-1}C_2 = p-1$.

For $j = 3, \dots, p+1$, we put

$$\gamma_j = (-1)^{p+2} c_{1\,p+2}\alpha_{2j} + \sum_{2<\ell<j} (-1)^{p+\ell} c_{1\,p+\ell}\alpha_{\ell j} + \sum_{j<\ell\leq p+1} (-1)^{p+\ell} c_{1\,p+\ell}\alpha_{j\ell}\,.$$

From (14), it is easy to conclude that, for $j = 3, \dots, p+1$,

$$\mathrm{d}i(\mathbf{G}_{1,1})(\gamma_j) = 0\,.$$

Since $\{\gamma_j\,;\, j = 3, \dots, p+1\}$ are linearly independent, we have proved our claim.

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