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WEITZENBÖCK FORMULAS ON POISSON PROBABILITY SPACES

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Abstract. This paper surveys and compares some recent approaches to stochastic infinite-dimensional geometry on the space Γ of configurations (i. e. locally finite subsets) of a Riemannian manifold M under Poisson measures. In particular, different approaches to Bochner-Weitzenböck formulas are considered. A unitary transform is also introduced by mapping functions of n configuration points to their multiple stochastic integral.

1. Weitzenböck Formula under a Measure

Let M be a Riemannian manifold with volume measure $\mathrm{d}x$, covariant derivative ∇, and exterior derivative d. Let ∇_μ^* and d_μ^* denote the adjoints of ∇ and d under a measure μ on M of the form $\mu(\mathrm{d}x) = \mathrm{e}^{\phi(x)}\,\mathrm{d}x$. The classical Weitzenböck formula under the measure μ states that

$$\mathrm{d}_\mu^*\,\mathrm{d} + \mathrm{d}\,\mathrm{d}_\mu^* = \nabla_\mu^*\nabla + R - \operatorname{Hess}\phi\,,$$

where R denotes the Ricci tensor on M. In terms of the de Rham Laplacian $H_R = \mathrm{d}_\mu^*\,\mathrm{d} + \mathrm{d}\,\mathrm{d}_\mu^*$ and of the Bochner Laplacian $H_B = \nabla_\mu^*\nabla$ we have

$$H_R = H_B + R - \operatorname{Hess}\phi\,.$$

In particular the term $\operatorname{Hess}\phi$ plays the role of a curvature under the measure μ.

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2. Probability: Poisson Space

In this section we recall some facts on random functionals on Poisson space. The Poisson probability measure on $\mathbb{N}$ can be introduced by considering N independent $\{0,1\}$-valued Bernoulli random variables $X_1, \dots, X_N$, with parameter λ/N, $\lambda > 0$. Then $X_1 + \cdots + X_N$ has a binomial law, and

$$P(X_1 + \cdots + X_N = k) = \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}$$

converges to $\frac{\lambda^k}{k!} \, \mathrm{e}^{-\lambda}$ as N goes to infinity. This defines a probability measure π_λ on $\mathbb{N}$ as

$$\pi_\lambda(\{k\}) = \frac{\lambda^k}{k!} \, \mathrm{e}^{-\lambda} .$$

Let X be a metric space with a σ-finite Borel measure σ. The measure π_λ has the convolution property $\pi_\lambda \star \pi_\mu = \pi_{\lambda+\mu}$, which allows to construct the Poisson measure π_σ with intensity σ on

$$\Gamma = \left\{ \gamma = \sum_{k=1}^{k=n} \delta_{x_k} ; \; x_1, \dots, x_n \in X, \; n \in \mathbb{N} \cup \{\infty\} \right\}$$

by letting

$$\begin{aligned} &\pi_\sigma(\{\gamma \in \Gamma ; \; \gamma(A_1) = k_1, \dots, \gamma(A_n) = k_n\}) \\ &\qquad = \frac{\sigma(A_1)^{k_1}}{k_1!} \, \mathrm{e}^{-\sigma(A_1)} \cdots \frac{\sigma(A_n)^{k_n}}{k_n!} \, \mathrm{e}^{-\sigma(A_n)} , \end{aligned}$$

where $A_1, \dots, A_n$ are disjoint compact subsets of X. This measure is characterized by its Fourier transform

$$\int_\Gamma \mathrm{e}^{\mathrm{i} \int_X f(x) \, \mathrm{d}\gamma(x)} \, \mathrm{d}\pi(\gamma) = \exp\left(\int_X (\mathrm{e}^{\mathrm{i} f(x)} - 1) \, \mathrm{d}\sigma(x) \right).$$

If $\gamma \in \Gamma$ is finite with cardinal $|\gamma| = n$ we write

$$\gamma = \sum_{i=1}^{i=n} \delta_{x_i} .$$

For a given compact subset Λ we consider $F \colon \Gamma \to \mathbb{R}$ such that $F(\gamma) = F(\gamma \cap \Lambda)$, and written as

$$F(\gamma) = F(\gamma \cap \Lambda) = \mathrm{e}^{\sigma(\Lambda)/2} \sum_{n=0}^{\infty} 1_{\{|\gamma \cap \Lambda| = n\}} n! f_n(x_1, \dots, x_n) = \sum_{n=0}^{\infty} J_n(f_n)$$

where f_n is a symmetric function with support in Λ^n, with

$$J_n(f_n)(\gamma) = J_n(f_n)(\gamma \cap \Lambda) = n! 1_{\{|\gamma \cap \Lambda| = n\}} \, \mathrm{e}^{\sigma(\Lambda)/2} f_n(x_1, \dots, x_n) , \quad n \geq 1 .$$

The multiple Poisson stochastic integral of f_n is defined as

$$I_n(f_n) = \int_{\substack{(x_1,\dots,x_n)\in X^n \\ x_i\neq x_j,\ i\neq j}} f_n(x_1,\dots,x_n)(\gamma-\sigma)(\,\mathrm{d}x_1)\cdots(\gamma-\sigma)(\,\mathrm{d}x_n)\,,$$

and extends to $f_n \in L^2_\sigma(X)^{\circ n}$ via the well-known isometry

$$\int_\Gamma I_n(f_n)I_m(g_m)d\pi = n!1_{\{n=m\}}\langle f_n, g_m\rangle_{L^2_\sigma(X)^{\circ n}}\,,$$
$$f_n \in L^2_\sigma(X)^{\circ n}\,, \quad g_m \in L^2_\sigma(X)^{\circ m}\,.$$

We introduce a combinatorial transform $\tilde{K}$ which has some similarities with the K-transform, cf. [6] and references therein. The transform $\tilde{K}$ identifies the functional $J_n(f_n)$, which makes sense only in finite volume, to $I_n(f_n)$ which is defined for all square-integrable f_n.

Proposition 2.1. *The operator $\tilde{K}$ defined by*

$$\tilde{K}J_n(f_n) = I_n(f_n)\,, \quad f_n \text{ symmetric in } \mathcal{C}_c(\Lambda^n)\,, n\in\mathbb{N}\,,$$

is unitary on $L^2_{\pi_\sigma}(\Gamma)$. Moreover, $\tilde{K}$ satisfies

$$\tilde{K}F(\gamma) = \sum_{\eta\subset\gamma}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\int_{X^k} F(\eta\cup\{y_1,\dots,y_k\})\sigma(\,\mathrm{d}y_1)\cdots\sigma(\,\mathrm{d}y_k).$$

Proof: We have

$$\begin{aligned}
&\int_\Gamma J_n(f_n)J_m(g_m)\,\mathrm{d}\pi \\
&\quad = n!^2 1_{\{n=m\}}\,\mathrm{e}^{\sigma(\Lambda)}\int_\Gamma 1_{\{|\gamma\cap\Lambda|=n\}}f_n(x_1,\dots,x_n)g_n(x_1,\dots,x_n)\,\mathrm{d}\pi(\gamma) \\
&\quad = \frac{n!^2}{\sigma(\Lambda)^n}\,\mathrm{e}^{\sigma(\Lambda)}\pi_\sigma(|\gamma\cap\Lambda|=n)\langle f_n, g_n\rangle_{L^2_\sigma(X)^{\otimes n}} \\
&\quad = n!1_{\{n=m\}}\langle f_n, g_n\rangle_{L^2_\sigma(X)^{\otimes n}}\,,
\end{aligned}$$

which shows the first statement. On the other hand we have

$$\begin{aligned}
&\tilde{K}J_n(f_n)(\gamma) \\
&\quad = \sum_{\eta\subset\gamma\cap\Lambda}\sum_{k=0}^\infty(-1)^k\frac{n!}{k!}\int_X 1_{\{|\eta\cup\{y_1,\dots,y_k\}|=n\}}f_n(\eta\cup\{y_1,\dots,y_k\}) \\
&\qquad\qquad \times\sigma(\,\mathrm{d}y_1)\cdots\sigma(\,\mathrm{d}y_k)
\end{aligned}$$

$$= \sum_{k=0}^{k=n} (-1)^k \frac{n!}{k!} \sum_{\substack{\eta \subset \gamma \cap \Lambda \\ |\eta| = n-k}} \int_X f_n(\eta \cup \{y_1, \ldots, y_k\}) \sigma(\mathrm{d}y_1) \cdots \sigma(\mathrm{d}y_k)$$

$$= \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \sum_{\substack{x_1, \ldots, x_{n-k} \in \gamma \cap \Lambda \\ x_i \neq x_j, \ i \neq j}} \int_X f_n(\{x_1, \ldots, x_{n-k}, y_1, \ldots, y_k\}) \times \sigma(\mathrm{d}y_1) \cdots \sigma(\mathrm{d}y_k)$$

$$= I_n(f_n)(\gamma),$$

the last relation follows e. g. from Prop. 4.1 of [9]. □

If Λ is compact and $F(\gamma) = F(\gamma \cap \Lambda)$ we have

$$\int_\Gamma F(\gamma)\, \mathrm{d}\pi(\gamma) = \mathrm{e}^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_X \cdots \int_X f_n(x_1, \ldots, x_n) \sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_n).$$

In the particular case $X = \mathbb{R}_+$ with σ the Lebesgue measure, the standard Poisson process is defined as

$$N_t(\gamma) = \gamma([0,t]) = \sum_{k=1}^{\infty} 1_{[T_k, \infty]}(t), \qquad t > 0,$$

i. e. every configuration $\gamma \in \Gamma$ can be viewed as the ordered sequence $\gamma = (T_k)_{k \geq 1}$ of jump times of $(N_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}_+$. Let $f_n \in \mathcal{C}_c([0,\lambda]^n)$ be symmetric. Then

$$\int_\Gamma f_n(T_1, \ldots, T_n) d\pi(\gamma) = \mathrm{e}^{-\lambda} \sum_{k=n}^{\infty} \frac{1}{k!} \int_0^\lambda \cdots \int_0^\lambda f_n(t_1, \ldots, t_n) \sigma(\mathrm{d}t_1) \cdots \sigma(\mathrm{d}t_k)$$

$$= \sum_{k=n}^{\infty} \mathrm{e}^{-\lambda} \int_0^\lambda \int_0^{t_k} \cdots \int_0^{t_1} f_n(t_1, \ldots, t_n) \sigma(\mathrm{d}t_1) \cdots \sigma(\mathrm{d}t_k)$$

$$= \sum_{k=n}^{\infty} \mathrm{e}^{-\lambda} \int_0^\lambda \frac{(\lambda - t_n)^{k-n}}{(k-n)!} \int_0^{t_n} \cdots \int_0^{t_1} f_n(t_1, \ldots, t_n) \sigma(\mathrm{d}t_1) \cdots \sigma(\mathrm{d}t_n)$$

$$= \int_0^\infty \mathrm{e}^{-t_n} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

This formula extends to f bounded and measurable.

3. Geometry

We recall the construction of [1, 2] in the case of 1-forms, see also [3] for the case of n-forms. We assume that X is a Riemannian manifold. The tangent space at $\gamma \in \Gamma$ is taken to be

$$L^2(X; TX, \gamma) = \oplus_{x\in\gamma} T_x X \, .$$

A differential form of order n maps $\gamma \in \Gamma$ into the antisymmetric tensor product

$$\wedge^n(T_\gamma \Gamma) = \wedge^n(\oplus_{x\in\gamma} T_x X) \, .$$

Bochner and de Rham Laplacians on differential forms over configuration spaces are then constructed from their counterparts at the level of the manifold X. Let d_x^X be the exterior differential on X, let ∇_x^X, Δ_x^X be the natural covariant derivative and Bochner Laplacian on the bundle $T_{\gamma\setminus\{x\}\cup\{y\}}\Gamma \to y \in \mathcal{O}_{\gamma,x}$, where $\mathcal{O}_{\gamma,x}$ is an open set in X such that $\bar{\mathcal{O}}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \emptyset$. The covariant derivative of the smooth differential 1-form W is defined as

$$(\nabla_x W_x(\gamma, x))_{x\in\gamma} \in T_\gamma\Gamma \otimes T_\gamma\Gamma \, ,$$

where $W_x(\gamma, y) = W((\gamma \setminus \{x\}) \cup \{y\})$, $x, y \in X$. The Bochner Laplacian H^B on Γ is defined as

$$H^B W(\gamma) = -\sum_{x\in\gamma} \Delta_x^X W_x(\gamma, x) \, .$$

The exterior derivative d^Γ is defined as

$$\mathrm{d}^\Gamma W = \sum_{x\in\gamma}\sum_{y\in\gamma} \mathrm{d}_x^X W_x(\gamma, x)_y \, ,$$

where $W_x(\gamma, x)_y$ is the component of $W_x(\gamma, x)$ of index $y \in \gamma$, with adjoint

$$\mathrm{d}^{\Gamma *} W = \sum_{x\in\gamma}\sum_{y\in\gamma} \mathrm{d}_x^{X*} W_x(\gamma, x)_{x,y} \, ,$$

where $W_x(\gamma, x)_{x,y}$ is the component of $W_x(\gamma, x)$ of index (x, y) and d_x^{X*} is the adjoint of d_x^X under the volume element σ on X. A Weitzenböck formula is stated in [1, 3] as

$$H^R = H^B + R \, , \tag{3.1}$$

where H^R is the de Rham Laplacian $H^R = \mathrm{d}^\Gamma \, \mathrm{d}^{\Gamma *} + \mathrm{d}^{\Gamma *} \, \mathrm{d}^\Gamma$ and the curvature term

$$R(\gamma) = \sum_{x\in\gamma} R(\gamma, x)$$

has the explicit expression

$$R(\gamma, x)(V(\gamma)_y) = 1_{\{x=y\}} \sum_{i,j=1}^{d} \mathrm{Ric}_{ij}(x) e_i \langle V(\gamma)_x, e_j \rangle_x \,,$$

where $(e_j)_{j=1}^{j=d}$ is an orthonormal basis of $T_x X$. Formula (3.1) can be viewed as the lifting to Γ of the Weitzenböck formula on X.

Note that in the above construction the curvature term in (3.1) is essentially due to the curvature of X, in particular it vanishes if $X = \mathbb{R}^d$ and no curvature term is induced from the Poisson measure itself.

In this paper we present a different geometry on the infinite-dimensional space Γ, in which the Ricci curvature tensor under the Poisson measure appears to be the identity operator when $X = \mathbb{R}_+$, see [8] when X is a more general Riemannian manifold.

Lifting of Differential Structure

Let $\mathcal{S}$ denote the space of cylindrical functionals of the form

$$F(\gamma) = f(T_1, \ldots, T_n)\,, \qquad f \in \mathcal{C}_b^\infty(\mathbb{R}^n)\,. \tag{3.2}$$

Let $\mathcal{U}$ denote the space of smooth processes of the form

$$u(\gamma, x) = \sum_{i=1}^{i=n} F_i(\gamma) h_i(x)\,, \qquad (\gamma, x) \in \Gamma \times \mathbb{R}_+\,,$$
$$h_i \in \mathcal{C}_c^\infty(\mathbb{R}_+)\,, \qquad F_i \in \mathcal{S}\,, \qquad i = 1, \ldots, n\,. \tag{3.3}$$

The differential geometric objects to be introduced below have finite dimensional counterparts, and each of them has a stochastic interpretation. The following table describes the correspondence between geometry and probability.

	Geometry	Probability
γ	element of Γ	point measure on $\mathbb{R}_+$
$\mathcal{C}_c^\infty(\mathbb{R}_+)$	tangent vectors to Γ	test functions on $\mathbb{R}_+$
σ	Riemannian metric on Γ	Lebesgue measure
d	gradient on Γ	stochastic gradient
$\mathcal{U}$	vector fields on Γ	stochastic processes
$\mathrm{d}u$	exterior derivative of $u \in \mathcal{U}$	two-parameter process
$\{\cdot, \cdot\}$	bracket of vector fields on Γ	bracket on $\mathcal{U} \times \mathcal{U}$
Ω	curvature tensor on Γ	trilinear mapping on $\mathcal{U}$
d^*	divergence on Γ	stochastic integral operator

Divergence Operator

The definition of the following gradient operator goes back to [4].

Definition 3.1. *Given $F \in \mathcal{S}$, $F = f(T_1, \ldots, T_d)$, let*

$$\mathrm{d}_t F(\gamma) = -\sum_{k=1}^{k=d} 1_{[0,T_k]}(t)\partial_k f(T_1, \ldots, T_d), \qquad t \geq 0.$$

The following is a finite-dimensional integration by parts formula for d.

Lemma 3.1. *We have for $F = f(T_1, \ldots, T_d)$ and $h \in \mathcal{C}_c(\mathbb{R}_+)$:*

$$\int_\Gamma \langle \mathrm{d}F, h\rangle_{L^2(\mathbb{R}_+)} \, \mathrm{d}\pi(\gamma) = \int_\Gamma F(\gamma)\left(\sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t)\,\mathrm{d}t\right)\mathrm{d}\pi(\gamma).$$

Proof: All $\mathcal{C}^\infty$ functions on $\Delta_d = \{(t_1, \ldots, t_d); \quad 0 \leq t_1 < \cdots < t_d\}$ are extended by continuity to the closure of Δ_d. We have

$$\int_\Gamma \langle \mathrm{d}F(\gamma), h\rangle_{L^2(\mathbb{R}_+)} \, \mathrm{d}\pi(\gamma)$$

$$= -\sum_{k=1}^{k=d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} \int_0^{t_k} h(s)\,\mathrm{d}s \partial_k f(t_1, \ldots, t_d)\,\mathrm{d}t_1 \ldots \mathrm{d}t_d$$

$$= \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_1) f(t_1, \ldots, t_d)\,\mathrm{d}t_1 \ldots \mathrm{d}t_d$$

$$- \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_3} \int_0^{t_2} h(s)\,\mathrm{d}s f(t_2, t_2, \ldots, t_d)\,\mathrm{d}t_2 \ldots \mathrm{d}t_d$$

$$+ \sum_{k=2}^{k=d} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_k) f(t_1, \ldots, t_d)\,\mathrm{d}t_1 \ldots \mathrm{d}t_d$$

$$- \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} h(s)\,\mathrm{d}s \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)\,\mathrm{d}t_1 \ldots \mathrm{d}t_d$$

$$- \sum_{k=2}^{k=d-1} \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdots \int_0^{t_2}$$

$$\int_0^{t_{k+1}} h(s)\,\mathrm{d}s f(t_1, ., t_{k-1}, t_{k+1}, t_{k+1}, ., t_d)\,\mathrm{d}t_1 \ldots \mathrm{d}t_{k+1}\,\mathrm{d}t_{k-1} \ldots \mathrm{d}t_d$$

$$+\sum_{k=2}^{k=d}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_k}\int_0^{t_{k-2}}\cdots\int_0^{t_2}\int_0^{t_k} h(s)\,\mathrm{d}s f(t_1,.,t_{k-2},t_k,t_k,.,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d$$

$$=\sum_{k=1}^{k=d}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_2} h(t_k)f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d$$

$$-\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d} h(s)\,\mathrm{d}s\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d$$

$$=\int_\Gamma F(\gamma)\left(\sum_{k=1}^{k=d} h(T_k)-\int_0^{T_d} h(t)\,\mathrm{d}t\right)\mathrm{d}\pi(\gamma)\,.$$

□

The following definition of the divergence coincides with the compensated Poisson stochastic integral with respect to $(N_t-t)_{t\in\mathbb{R}_+}$ on the adapted square-integrable processes.

Definition 3.2. *We define* d_π^* *on* $\mathcal{U}$ *by*

$$\mathrm{d}_\pi^*(hG)=\int_0^\infty h(t)(\gamma(\mathrm{d}t)-\mathrm{d}t)-\langle h,\mathrm{d}G\rangle_{L^2(\mathbb{R}_+)}\,,\qquad G\in\mathcal{S},\ h\in L^2(\mathbb{R}_+)\,.$$

Using this definition, an integration by parts formula can be obtained independently of the dimension.

Proposition 3.1. *The divergence operator* $\mathrm{d}_\pi^*\colon L^2(\Gamma\times\mathbb{R}_+)\to L^2(\Gamma)$ *is the closable adjoint of* d, *i. e.*

$$\int_\Gamma F\,\mathrm{d}_\pi^* u\,\mathrm{d}\pi(\gamma)=\int_\Gamma\langle\mathrm{d}F,u\rangle_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma)\,,\qquad F\in\mathcal{S}\,,\ u\in\mathcal{U}\,.\qquad(3.4)$$

Proof: Given Lemma 3.1 it suffices to notice that if $k>d$,

$$\int_\Gamma F(\gamma)h(T_k)d\pi(\gamma)=\int_0^\infty \mathrm{e}^{-t_k}h(t_k)\int_0^{t_k}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_k$$

$$=\int_0^\infty \mathrm{e}^{-t_k}\int_0^{t_k} h(s)\,\mathrm{d}s\int_0^{t_k}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_k$$

$$-\int_0^\infty \mathrm{e}^{-t_{k-1}}\int_0^{t_{k-1}} h(s)\,\mathrm{d}s\int_0^{t_{k-1}}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_{k-1}$$

$$= \int_\Gamma F(\gamma) \int_{x_{k-1}}^{T_k} h(t)\,\mathrm{d}t\,\mathrm{d}\pi(\gamma),$$

in other terms the discrete-time process

$$\left(\sum_{k=1}^{k=n} h(T_k) - \int_0^{T_k} h(t)\,\mathrm{d}t\right)_{k\geq 1} = \left(\int_0^{T_k} h(t)\,\mathrm{d}(N_t - t)\right)_{k\geq 1}$$

is a martingale. Hence relation (3.4) also implies that for $F, G \in \mathcal{S}$,

$$\begin{aligned}\int_\Gamma \langle\, \mathrm{d}F, hG\rangle_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) &= \int_\Gamma \langle\, \mathrm{d}(FG), h\rangle_{L^2(\mathbb{R}_+)} - F\langle\, \mathrm{d}G, h\rangle_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma)\\ &= \int_\Gamma F\left(G\int_0^\infty h(t)(\gamma(\,\mathrm{d}t) - \,\mathrm{d}t) - \langle h,\,\mathrm{d}G\rangle_{L^2(\mathbb{R}_+)}\right)\,\mathrm{d}\pi(\gamma)\\ &= \int_\Gamma F\,\mathrm{d}_\pi^*(hG)\,\mathrm{d}\pi(\gamma)\,.\end{aligned} \tag{3.5}$$

□

Covariant Derivative

Given $u \in \mathcal{U}$ we define the covariant derivative $\nabla_u v$ in the direction $u \in L^2(\mathbb{R}_+)$ of the vector field $v = \sum_{i=1}^{i=n} F_i h_i \in \mathcal{U}$ as

$$\nabla_u v(t) = \sum_{i=1}^{i=n} h_i(t)\,\mathrm{d}_u F_i - F_i \dot{h}_i(t)\int_0^t u(s)\,\mathrm{d}s\,, \qquad t \in \mathbb{R}_+\,, \tag{3.6}$$

where

$$\mathrm{d}_u F = \langle\, \mathrm{d}F, u\rangle_{L^2(\mathbb{R}_+)}\,, \qquad F \in \mathcal{S}.$$

We have

$$\nabla_{uF}(vG) = Fv\,\mathrm{d}_u G + FG\nabla_u v\,, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)\,, \quad F, G \in \mathcal{S}\,. \tag{3.7}$$

We also let

$$\nabla_s v(t) = \sum_{i=1}^{i=n} h_i(t)\,\mathrm{d}_s F_i - F_i \dot{h}_i(t) 1_{[0,t]}(s)\,, \qquad s, t \in \mathbb{R}_+\,,$$

in order to write

$$\nabla_u v(t) = \int_0^\infty u(s)\nabla_s v(t)\,\mathrm{d}s\,, \qquad t \in \mathbb{R}_+\,, \qquad u, v \in \mathcal{U}\,.$$

Lie-Poisson Bracket

Definition 3.3. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$*, is defined to be the unique element of* $\mathcal{C}_c^\infty(\mathbb{R}_+)$ *satisfying* $(\mathrm{d}_u \mathrm{d}_v - \mathrm{d}_v \mathrm{d}_u)F = \mathrm{d}_w F$*,* $F \in \mathcal{S}$*.*

The bracket $\{u, v\}$ is defined for $u, v \in \mathcal{U}$ with

$$\{Fu, Gv\}(x) = FG\{u, v\}(x) + v(x)F\,\mathrm{d}_u G - u(x)G\,\mathrm{d}_v F\,, \qquad x \in \mathbb{R}_+\,,$$

$u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F, G \in \mathcal{S}$.

Vanishing of Torsion

Proposition 3.2. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{U}$ *satisfies*

$$\{u, v\} = \nabla_u v - \nabla_v u\,,$$

i. e. the connection defined by ∇ *has a vanishing torsion.*

Proof: We have $F(\gamma) = T_n$. If $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ we have

$$\begin{aligned}(\mathrm{d}_u \mathrm{d}_v - \mathrm{d}_v \mathrm{d}_u)T_n &= -\mathrm{d}_u \int_0^{T_n} v(s)\,\mathrm{d}s + \mathrm{d}_v \int_0^{T_n} u(s)\,\mathrm{d}s \\ &= v(T_n)\int_0^{T_n} u(s)\,\mathrm{d}s - u(T_n)\int_0^{T_n} v(s)\,\mathrm{d}s \\ &= \int_0^{T_n}\left(\dot v(t)\int_0^t u(s)\,\mathrm{d}s - \dot u(t)\int_0^t v(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \mathrm{d}_{\nabla_u v - \nabla_v u} T_n\,.\end{aligned}$$

Since d is a derivation, this shows that

$$\mathrm{d}_u \mathrm{d}_v - \mathrm{d}_v \mathrm{d}_u = \mathrm{d}_{\nabla_u v - \nabla_v u}\,, \qquad u, v \in \mathcal{U}\,.$$

The extension to $u, v \in \mathcal{U}$ follows from (3.7). □

Vanishing of Curvature

Proposition 3.3. *The curvature tensor* Ω *of* ∇ *vanishes on* $\mathcal{U}$*, i. e.*

$$\Omega(u, v)h := [\nabla_u, \nabla_v]h - \nabla_{\{u,v\}}h = 0\,, \qquad u, v, h \in \mathcal{U}\,,$$

and $\mathcal{U}$ *is a Lie algebra under the bracket* $\{\cdot\,,\cdot\}$*.*

Proof: We have, letting $\tilde{u}(t) = -\int_0^t u(s)\,\mathrm{d}s$:

$$[\nabla_u, \nabla_v]h = \tilde{u}\dot{\overline{\nabla_v h}} - \tilde{v}\dot{\overline{\nabla_u h}} = \tilde{u}\,\dot{\overline{\tilde{v}\dot{h}}} - \tilde{v}\,\dot{\overline{\tilde{u}\dot{h}}} = -\tilde{u}v\dot{h} + \tilde{v}u\dot{h}\,,$$

and

$$\nabla_{\{u,v\}}h = \nabla_{\tilde{u}\dot{v}-\tilde{v}\dot{u}}h = (\widetilde{\tilde{u}\dot{v} - \tilde{v}\dot{u}})\dot{h} = (u\tilde{v} - v\tilde{u})\dot{h}\,,$$

hence $\Omega(u,v)h = 0$, $h, u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension of the result to $\mathcal{U}$ follows again from (3.7). The Lie algebra property follows from the vanishing of Ω. □

Exterior Derivative

The exterior derivative $\mathrm{d}u$ of a smooth vector field $u \in \mathcal{U}$ is defined from

$$\langle\, \mathrm{d}u, h_1 \wedge h_2 \rangle_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} = \langle \nabla_{h_1} u, h_2 \rangle_{L^2(\mathbb{R}_+)} - \langle \nabla_{h_2} u, h_1 \rangle_{L^2(\mathbb{R}_+)}\,,$$

$h_1, h_2 \in \mathcal{U}$. We have

$$\|\mathrm{d}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} = 2\int_0^\infty\int_0^\infty (\,\mathrm{d}u(s,t))^2\,\mathrm{d}s\,\mathrm{d}t\,, \tag{3.8}$$

where

$$\mathrm{d}u(s,t) = \frac{1}{2}(\nabla_s u(t) - \nabla_t u(s))\,, \qquad s,t \in \mathbb{R}_+\,,\ u \in \mathcal{U}\,.$$

Isometry Formula

Lemma 3.2. *We have for $u \in \mathcal{U}$:*

$$\int_\Gamma (\,\mathrm{d}^*_\pi u)^2\,\mathrm{d}\pi(\gamma) = \int_\Gamma \|u\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) + \int_\Gamma\int_0^\infty\int_0^\infty \nabla_s u(t)\nabla_t u(s)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\,. \tag{3.9}$$

Proof: (cf. [8, 7] and the proof of [5] for path spaces over Lie groups). Given $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ we have

$$\begin{aligned}\int_\Gamma \mathrm{d}^*_\pi(h_iF_i)\,\mathrm{d}^*_\pi(h_jF_j)\,\mathrm{d}\pi(\gamma) &= \int_\Gamma F_i\,\mathrm{d}_{h_i}\,\mathrm{d}^*_\pi(h_jF_j)\,\mathrm{d}\pi(\gamma)\\ &= \int_\Gamma F_i\,\mathrm{d}_{h_i}\Big(F_j\,\mathrm{d}^*_\pi(h_j) - \mathrm{d}_{h_j}F_j\Big)\,\mathrm{d}\pi(\gamma)\\ &= \int_\Gamma \Big(F_iF_j\,\mathrm{d}_{h_i}\,\mathrm{d}^*_\pi h_j + F_i\,\mathrm{d}^*_\pi(h_j)\,\mathrm{d}_{h_i}F_j - F_i\,\mathrm{d}_{h_i}\,\mathrm{d}_{h_j}F_j\Big)\,\mathrm{d}\pi(\gamma)\end{aligned}$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j \, \mathrm{d}_\pi^*(\nabla_{h_i} h_j) + F_i \, \mathrm{d}_\pi^*(h_j) \, \mathrm{d}_{h_i} F_j - F_i \, \mathrm{d}_{h_i} \, \mathrm{d}_{h_j} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + \mathrm{d}_{\nabla_{h_i} h_j}(F_i F_j) + \mathrm{d}_{h_j}(F_i \, \mathrm{d}_{h_i} F_j) - F_i \, \mathrm{d}_{h_i} \, \mathrm{d}_{h_j} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + \mathrm{d}_{\nabla_{h_i} h_j}(F_i F_j) + \mathrm{d}_{h_j} F_i \, \mathrm{d}_{h_i} F_j + F_i (\mathrm{d}_{h_j} \, \mathrm{d}_{h_i} F_j - \mathrm{d}_{h_i} \, \mathrm{d}_{h_j} F_j) \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + \mathrm{d}_{\nabla_{h_i} h_j}(F_i F_j) + \mathrm{d}_{h_j} F_i \, \mathrm{d}_{h_i} F_j + F_i \, \mathrm{d}_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \, \mathrm{d}_{\nabla_{h_i} h_j} F_i + F_i \, \mathrm{d}_{\nabla_{h_j} h_i} F_j + \mathrm{d}_{h_j} F_i \, \mathrm{d}_{h_i} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Bigg(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^\infty \mathrm{d}_s F_i \int_0^\infty \nabla_t h_j(s) h_i(t) \, \mathrm{d}t \, \mathrm{d}s + F_i \int_0^\infty \mathrm{d}_t F_j \int_0^\infty \nabla_s h_i(t) h_j(s) \, \mathrm{d}s \, \mathrm{d}t + \int_0^\infty h_i(t) \, \mathrm{d}_t F_j \int_0^\infty h_j(s) \, \mathrm{d}_s F_i \, \mathrm{d}s \, \mathrm{d}t \Bigg) \, \mathrm{d}\pi(\gamma) ,$$

where we used the commutation relation satisfied by the gradient d:

$$\mathrm{d}_u \, \mathrm{d}_\pi^* v = \mathrm{d}_\pi^* \nabla_u v + \langle u, v \rangle_{L^2(\mathbb{R}_+)} , \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+), \tag{3.10}$$

which can be proved as follows:

$$\mathrm{d}_u \, \mathrm{d}_\pi^* v = -\sum_{k=1}^\infty \dot{v}(T_k) \int_0^{T_k} u(s) \, \mathrm{d}s = - \mathrm{d}_\pi^* \left(v(\cdot) \int_0^{\cdot} u(s) \, \mathrm{d}s \right) - \int_0^\infty \dot{v}(t) \int_0^t u(s) \, \mathrm{d}s \, \mathrm{d}t = \mathrm{d}_\pi^* (\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)} .$$

□

Finally we state a Weitzenböck type identity on configuration space, that can be read as

$$\mathrm{d}\,\mathrm{d}_\pi^* + \mathrm{d}^*\,\mathrm{d} = \nabla^*\nabla + \mathrm{Id}_{L^2(\mathbb{R}_+)},$$

i. e. the Ricci tensor under the Poisson measure is the identity $\mathrm{Id}_{L^2(\mathbb{R}_+)}$ on $L^2(\mathbb{R}_+)$.

Theorem 3.1. *We have for* $u \in \mathcal{U}$:

$$\int_\Gamma (\mathrm{d}_\pi^* u)^2\,\mathrm{d}\pi(\gamma) + \int_\Gamma \|\,\mathrm{d}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) = \int_\Gamma \|u\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) + \int_\Gamma \|\nabla u\|^2_{L^2(\mathbb{R}_+)\otimes L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma)\,. \tag{3.11}$$

Proof: Relation (3.11) for $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ follows from (3.8) and Lemma 3.2. □

References

[1] Albeverio S., Daletskii A. and Lytvynov E., *Laplace Operators and Diffusions in Tangent Bundles over Poisson Spaces*, In:*Infinite Dimensional Stochastic Analysis*, R. Neth. Acad. Arts Sci., Amsterdam, 2000, pp 1–24.

[2] Albeverio S., Daletskii A. and Lytvynov E., *De Rham Cohomology of Configuration Spaces with Poisson Measure*, J. Funct. Anal. **185** (2001) 240–273.

[3] Albeverio S., Daletskii A. and Lytvynov E., *Laplace Operators on Differential Forms over Configuration Spaces*, J. Geom. Phys. **37** (2001) 15–46.

[4] Carlen E. and Pardoux E., *Differential Calculus and Integration by Parts on Poisson Space*, In: *Stochastics, Algebra and Analysis in Classical and Quantum Dynamics*, S. Albeverio, Ph. Blanchard and D. Testard (Eds), Kluwer Acad. Publ., Dordrecht 1990, pp 63–73.

[5] Fang S. and Franchi J., *Platitude de la structure riemannienne sur le groupe des chemins et identité d'énergie pour les intégrales stochastiques [Flatness of Riemannian Structure over the Path Group and Energy Identity for Stochastic Integrals]*, C.R. Acad. Sci. Paris Ser. I Math. **321** (1995) 1371–1376.

[6] Kondratiev Y. and Kuna T., *Harmonic Analysis on Configuration Space I. General Theory*, SFB 256 Preprint No 626, Bonn University, 1999.

[7] Privault N., *Connection, Parallel Transport, Curvature and Energy Identities on Spaces of Configurations*, C.R. Acad. Sci. Paris Ser. I Math. **330** (2000) 899–904.

[8] Privault N., *Connections and Curvature in the Riemannian Geometry of Configuration Spaces*, J. Funct. Anal. **185** (2001) 367–403.

[9] Surgailis D., *On Multiple Poisson Stochastic Integrals and Associated Markov Semi-groups*, Probab. Math. Statist. **3** (1984) 217–239.