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TOPOLOGICAL PROPERTIES OF SOME COHOMOGENEITY ON RIEMANNIAN MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract. In this paper we study some non-positively curved Riemannian manifolds acted on by a Lie group of isometries with principal orbits of codimension one. Among other results it is proved that if the universal covering manifold satisfies some conditions then every non-exceptional singular orbit is a totally geodesic submanifold. When M is flat and is not toruslike, it is proved that either each orbit is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ or there is a singular orbit. If the singular orbit is unique and non-exceptional, then it is isometric to $\mathbb{R}^k \times \mathbb{T}^m$.

1. Introduction

Recently, cohomogeneity one Riemannian manifolds have been studied from different points of view. A. Alekseevsky and D. Alekseevsky in [1] and [2] gave a description of such manifolds in terms of Lie subgroups of a Lie group G, Podesta and Spiro in [13] got some nice results in negatively curved case, Searle in [14] provided a complete classification of such manifolds in dimensions less than 6 when they are compact and of positive curvature. The aim of this paper is to deal with some non-positively curved cohomogeneity one Riemannian manifolds. We generalize some of the theorems of [13] to the case where M is a product of negatively curved manifolds. Also in Section 4 we study some cohomogeneity one flat Riemannian manifolds. Our main results are Theorems 3.5, 3.7, 3.10, and 4.4.

2. Preliminaries

Definition 2.0. Let M be a complete Riemannian manifold and G a Lie group of isometries which is closed in the full group of isometries of M. We say

that M is of cohomogeneity one under the action of G if G has an orbit of codimension one.

It is known (see [1] and [4, 11]) that the orbit space $\Omega = M/G$ is a topological Hausdorff space homeomorphic to one of the following spaces: \mathbb{R} , S^1 , $\mathbb{R}^+ = [0, +\infty)$ and [0, 1]. In the following we will indicate by $k: M \to \Omega$ the projection to the orbit space. Given a point $x \in M$, the orbit D = Gx is called **principal** (resp. **singular**) if the corresponding image in the orbit space is an **internal** (resp. **boundary**) point of Ω , and the point x is called a **regular** (resp. **singular**) point. We say that a singular orbit is exceptional if it has codimension one. Also note that the principal orbits are diffeomorphic to each other and M is diffeomorphic to $\Omega \times D$ if M/G = R.

If G_p is the isotropy subgroup of G at $p, (p \in M)$, then G_x and G_y are conjugate if both x, y are regular, while G_x is conjugate to a subgroup of G_y if x is regular and y is singular.

Definition 2.1. A (complete) geodesic γ on a Riemannian manifold of cohomogeneity one is called a normal geodesic if it crosses each orbit orthogonally.

We know (see [2]) that a geodesic γ is a normal geodesic if and only if it is orthogonal to each orbit Gx at one point $x \in \gamma$, and that each regular point belongs to a unique normal geodesic.

Definition 2.2. A differentiable real valued function F on a complete Riemannian manifold M is said to be convex (resp. strictly convex) if for each geodesic $\gamma \colon \mathbb{R} \to M$ the composed function $F \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is convex (resp. strictly convex), that is $(F \circ \gamma)'' \geq 0$ (resp. $(F \circ \gamma)'' > 0$).

Let φ be an isometry of a simply connected Riemannian manifold M, the squared **displacement function** of φ is the function defined by $d_{\varphi}^2(p) = d^2(p, \varphi(p)), p \in M$, where d denotes the distance on M.

In the next proposition we list some known properties of cohomogeneity one Riemannian manifolds, which we will use in the sequel.

Proposition 2.3. ([4], [8] and [13]) Let M be a cohomogeneity one Riemannian manifold under the action of a connected Lie group G which is closed in the full isometry goup of M, then

- a) If M is simply connected with nonpositive curvature, there is at most one singular orbit;
- b) If M has nonpositive curvature and B is the unique singular orbit of M, $\pi_1(M) = \pi_1(B)$;
- c) If M is simply connected no exceptional orbit may exist;

- d) If M is simply connected and without singular orbit then $\Omega \neq S^1$, i. e. $\Omega = \mathbb{R}$;
- e) No exceptional orbit is simply connected;
- f) If γ is a normal geodesic then the map $k \colon \gamma \to \Omega$ is surjective and it defines a covering over the set Ω^0 of internal points of Ω . When $\Omega = \mathbb{R}^+$ or \mathbb{R} , we can endow Ω with the metric given by the covering k.

The following proposition and theorems will be needed later.

Proposition 2.4. (see [3]) Let M be a simply connected and complete Riemannian manifold of nonpositive curvature, then

- a) If the minimum point set C of a real valued convex function F defined on M is a submanifold of M then C is totally geodesic in M, and each critical point of F belongs to C;
- b) d_{φ}^2 is a convex function for each isometry φ of M and if M has negative curvature it is strictly convex except at the minimum point set C which is at most the image of a geodesic.

Theorem 2.5. ([15]) Let M be a connected homogeneous Riemannian manifold with nonpositive curvature, then M is diffeomorphic to the product of a torus and a Euclidean space.

Theorem 2.6. ([9]) Let M be a homogeneous Riemannian manifold with non-positive curvature and negative definite Ricci tensor then M is simply connected.

3. Cohomogeneity on UND Manifolds

Throught the following M will denote a complete Riemannian manifold of dimension n with nonpositive curvature and of cohomogeneity one under the action of G, a connected Lie group which is closed in the full group of isometries of M. If M is not simply connected then \tilde{M} will denote the universal Riemannian covering manifold of M endowed with the pulled back metric and $\pi\colon \tilde{M}\to M$ will be the covering projection, with the symbol Δ we will denote the deck transformation group of the universal covering of M. We know (see [4] page 63) that the group G always admits a connected covering group G which acts on G0 by isometries and of cohomogeneity one, the projection G1. We such that G2 is such that G3 is a constant it maps G4-orbits onto G5-orbits, so for each G6 constant along orbits.

Definition 3.0. We say that a Riemannian manifold M is universally and negatively decomposable (UND) when its universal covering manifold \tilde{M} decomposes as $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$ and for each i, \tilde{M}_i has negative curvature and each $\varphi \in \Delta$ decomposes as $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_k$ where φ_i is an isometry of \tilde{M}_i .

Lemma 3.1. If $M=M_1\times M_2$ is a complete simply connected Riemannian manifold of nonpositive curvature such that for a geodesic $\gamma(t)=(\gamma_1(t),\gamma_2(t))$ and for an isometry $\varphi=\varphi_1\times\varphi_2,\ d_{1\varphi_1}^2\circ\gamma_1\colon\mathbb{R}\to\mathbb{R}$ is strictly convex, then $d_{\varphi}^2\circ\gamma\colon\mathbb{R}\to\mathbb{R}$ is a strictly convex function.

Lemma 3.2. If $\varphi \in \Delta$ is nontrivial and for a normal geodesic γ , $d_{\varphi}^2 \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ does not have any minimum point then, φ maps each orbit \tilde{D} onto itself.

Lemma 3.3. Let M be a UND cohomogeneity one Riemannian manifold and let $\varphi \in \Delta$ be nontrivial, then there exists a normal geodesic γ on \tilde{M} such that $d^2_{\omega} \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function.

Lemma 3.4. Let γ be a normal geodesic in \tilde{M} and $\varphi \in \Delta$ be such that $d_{\varphi}^2 \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is strictly convex and $t_1 \in \mathbb{R}$ is not a minimum point of the function $F(t) = d_{\varphi}^2 \circ \gamma(t)$, then the orbit $\tilde{B} = \tilde{G}\gamma(t_1)$ is a hypersurface in \tilde{M} .

Theorem 3.5. If M is a non-simply connected UND cohomogeneity one Riemannian manifold with only one singular orbit B, and B is not exceptional, then it is a totally geodesic submanifold of M diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and $\pi_1(M) = \mathbb{Z}^m$.

Proof: First note that since $\dim \pi^{-1}(B) = \dim B < n-1$, each component of $\pi^{-1}(B)$ must be a non-exceptional singular orbit in \tilde{M} . Therefore by 2.3(a), $\pi^{-1}(B)$ has only one component \tilde{B} . Now let $\varphi \in \Delta$ be a nontrivial deck transformation and γ a normal geodesic in \tilde{M} such that $F = \mathrm{d}_{\varphi}^2 \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function (see 3.3), then we have two cases.

Case 1: F has only one minimum point $t_0 \in \mathbb{R}$.

In this case since d_{φ}^2 is constant along orbits, we get that $\tilde{G}\gamma(t_0)$ is the minimum point set of d_{φ}^2 , so by 2.4(a) it is a totally geodesic submanifold of \tilde{M} . We show that $\tilde{B} = \tilde{G}.\gamma(t_0)$. If not, then $\tilde{B} = \tilde{G}\gamma(t_1)$, $t_1 \neq t_0$, so by 3.4 \tilde{B} must be a hypersurface in \tilde{M} , since $\dim \tilde{B} < n-1$ this is a contradiction, therefore $\tilde{B} = \tilde{G}\gamma(t_0)$ and \tilde{B} is a totally geodesic submanifold of \tilde{M} . Consequently $B = \pi(\tilde{B})$ is totally geodesic in M, so is of nonpositive curvature. Since B is homogeneous we get by 2.5 that B is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = \mathbb{Z}^m$.

Case 2: F has not any minimum point.

This case can not occur because by 3.4 each orbit of \tilde{M} must be a hypersurface, so \tilde{B} is a hypersurface, which is in contrast with the fact $\dim \tilde{B} < n-1$. \square

Lemma 3.6. If for each deck transformation $\varphi \in \Delta$ and each orbit \tilde{D} in \tilde{M} , φ maps \tilde{D} onto itself and if there is no singular orbit in M, then each orbit D in M is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

Proof: The proof of this lemma in given in a portion of the proof of Theorem 3.7 in [4] and the sketch of the proof is as follows: for an orbit D in M, $\pi^{-1}(D)$ has only one component \tilde{D} and $\tilde{D} = \tilde{G}/\tilde{K}$ with \tilde{K} maximal compact in \tilde{G} . So there is a solvable subgroup H acting transitively on \tilde{D} . Since $D = \tilde{D}/\Delta$ and Δ centeralizes \tilde{G} (and hence H too). we obtain that H acts transitively on D, so D is a solvmanifold and diffeomorphic to a product $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$ (see [13], p. 76 and [16]). \square

Theorem 3.7. If M is a non-simply connected UND cohomogeneity one Riemannian manifold without any singular orbit, then each orbit is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$. In this case if $M/G = \mathbb{R}$, then M is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$, $k = k_1 + 1$.

Proof: By 3.3 for each nontrivial $\varphi \in \Delta$, there is a normal geodesic γ (related to φ) such that $d_{\varphi}^2 \circ \gamma$ is a strictly convex function. We have two cases.

Case 1: There exists a $\varphi \in \Delta$ such that $d_{\varphi}^2 \circ \gamma$ has a minimum point $t_0 \in \mathbb{R}$. In this case the orbit $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of the function d_{φ}^2 . Therefore by 2.4(a) it is totally geodesic and so $B = \pi(\tilde{B})$ is totally geodesic in M, hence is of nonpositive curvature. Since B is homogeneous, it is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$ by 2.5. From the fact that the (principal) orbits are diffeomorphic we get that each orbit is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

Case 2: For each nontrivial $\varphi \in \Delta$, $d_{\varphi}^2 \circ \gamma$ does not have any minimum point. In this case by 3.2, φ maps each orbit \tilde{D} onto itself. Therefore by 3.6 each orbit D in M is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

If $M/G = \mathbb{R}$, from the fact that M is diffeomorphic to $M/G \times D$ we get that M is diffeomorphic to $\mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{T}^{m_1} = \mathbb{R}^k \times \mathbb{T}^{m_1}$. \square

Lemma 3.8. Let $M = M_1 \times M_2$ and $X = X_1 + X_2$, Z be two vectors at the point $p = (p_1, p_2)$ such that X_1 , Z are tangent to M_1 and X_2 is tangent to M_2 , then $K_M(X, Z) = K_{M_1}(X_1, Z)$.

Lemma 3.9. If $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$, where for each i, \tilde{M}_i is negatively curved with dim $\tilde{M}_i \geq 3$, then each totally geodesic hypersurface S of \tilde{M} has negative definite Ricci tensor.

Theorem 3.10. If M is a nonsimply connected UND cohomogeneity one Riemannian manifold and $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$, where for each i, dim $\tilde{M}_i \geq 3$, then

- a) There is at most one singular orbit;
- b) If there is a singular orbit B, it is non-exceptional and diffeomorphic to $\mathbb{R}^{K_1} \times \mathbb{T}^{m_1}$ and $\pi_1(M) = \mathbb{Z}^{m_1}$.

Proof: We prove the theorem in two steps.

Step 1: M does not have two exceptional singular orbits.

If M has two exceptional singualr orbits, then the dimension of each orbit of M (and so the dimension of each orbit of \tilde{M}) is n-1, so by 2.3(c, e), \tilde{M} does not have any singual orbit and $\tilde{M}/\tilde{G}=\mathbb{R}$. Therefore each normal geodesic γ in \tilde{M} intersects an orbit \tilde{D} exactly once. But since M/G = [0, 1], the normal geodesic $\pi \circ \gamma$ intersects a principal orbit D in M infinitely many times, so $\pi^{-1}(D)$ has more than one connected component. Therefore if \tilde{D} is a component of $\pi^{-1}(D)$, there exist a nontrivial $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$ thus by Lemmas 3.2, 3.3, for a normal geodesic γ , $d_{\omega}^{2} \circ \gamma$ is strictly convex with a minimum point $t_0 \in \mathbb{R}$, and since d_{φ}^2 is constant along orbits, $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of d_{φ}^2 . So it is totally geodesic by 2.4(a). Now since each factor of the decomposition of \tilde{M} is negatively curved with dim $\tilde{M}_i \geq 3$, we get by 3.9 that every totally geodesic hypersurface of \tilde{M} has negative definite Ricci tensor, so \tilde{B} (hence $B = \pi(\tilde{B})$) has negative definite Ricci tensor, thus by 2.6, B is simply connected. Since dim B = n - 1, we get by 2.3(d) that B is not a singular orbit. As B is simply connected, B = G/K ($K = G_x$) $x \in B$), where K is maximal compact subgroup of G (see [10], Vol II, p. 112), which is in contrast with the fact that there exists singular orbit.

Step 2: M does not have two singular orbits, at least one orbit non-exceptional. Let B_1 be a non-exceptional singular orbit of M then $\tilde{B} = \pi^{-1}(B_1)$ is the unique singular orbit of \tilde{M} . Because of dimensional reasons for each $\varphi \in \Delta$ we have $\varphi(\tilde{B}) = \tilde{B}$. The isometry φ induces an isometry φ^* on the orbis pace \mathbb{R}^+ of \tilde{M} such that for each orbit \tilde{D} we have $\varphi^*(k(\tilde{D}) = k(\varphi(\tilde{D}))$. Since $\varphi(\tilde{B}) = \tilde{B}$, we get that $\varphi^*(0) = \varphi^*(k(\tilde{B})) = k\varphi(\tilde{B}) = k(\tilde{B}) = 0$, so for each $t \in \mathbb{R}^+$ we have $\varphi^*(t) = t$. Thus $\varphi(\tilde{D}) = \tilde{D}$. Now we have a contradiction because a normal geodesic γ in \tilde{M} intersects each principal orbit in two points $(\tilde{M}/\tilde{G} = \mathbb{R}^+)$ while $\pi \circ \gamma$ intersects a principal orbit infinitely many times (M/G = [0,1]). So there exists $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$.

We need only to show that B can not be an exceptional orbit, the other parts of the claim is a simple consequence of Theorem 3.5. To prove the claim observe that if it were the case, \tilde{M} would admit only principal orbits and a normal geodesic intersects each orbit in \tilde{M} exactly in one point while since

 $M/G = \mathbb{R}^+$, a normal geodesic in M intersects each principal orbit in two points, and a contradiction arises as in the Step 1. \square

4. Cohomogeneity One Flat Manifolds

In this section we study cohomogeneity one flat Riemannian manifolds which are not toruslike.

It is known that every isometry $\varphi \in \operatorname{Iso}(\mathbb{R}^n)$ is of the form $\varphi = (A, b)$, $A \in O(n)$, $b \in \mathbb{R}^n$ that is, $\varphi(x) = Ax + b$, $x \in \mathbb{R}^n$. We say that φ is an ordinary translation when $A = \operatorname{Id}$ (Id is the identity map on \mathbb{R}^n).

Note that \mathbb{R}^n is the universal Riemannian covering manifold of each flat manifold M of dimension n.

Definition 4.1. We say that a flat Riemannian manifold M is "toruslike" if each deck transformation of the universal covering manifold of M is an ordinary translation.

In the following V.W denotes the inner product of the vectors V and W in \mathbb{R}^n and |V| is the length of V.

Lemma 4.2. Let \mathbb{R}^n be of cohomogeneity one under the action of a closed Lie subgroup $G \subset \operatorname{Iso}(\mathbb{R}^n)$ and let $\varphi = (A,b) \in G$, $A \neq \operatorname{Id}$. Then there is a normal geodesic γ on \mathbb{R}^n such that the function $F(t) = \operatorname{d}_{\varphi}^2 \circ \gamma(t)$ is a strictly convex function with the minimum point $t_0 \in R$.

Lemma 4.3. If \mathbb{R}^n is of cohomogeneity one under the action of a closed Lie subgroup G of $Iso(\mathbb{R}^n)$ and if all the orbits are regular and one orbit is isometric to \mathbb{R}^{n-1} , then other orbits are isometric to \mathbb{R}^{n-1} .

Theorem 4.4. If M is a flat cohomogeneity one Riemannian manifold under the action of a closed Lie group $G \subset \text{Iso}(M)$ and M is not torus-like, then

- a) Either each orbit D of M is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ for some m, k, m + k = n 1, or there is a singular orbit B in M;
- b) If there is a unique singular orbit B which is non-exceptional, then B is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ for some m, k and $\pi_1(M) = \mathbb{Z}^m$.

Proof: Let $\tilde{M} = \mathbb{R}^n$ be the universal covering manifold of M and let \tilde{G} be the corresponding covering Lie group of G, which acts on $\tilde{M} = \mathbb{R}^n$ by cohomogeneity one.

(a): Since M is not toruslike there is a deck transformation φ such that $\varphi = (A,b), A \neq \mathrm{Id}$. By Lemma 4.2 there is a normal geodesic γ in \tilde{M} such that the function $F(t) = \mathrm{d}_{\varphi}^2 \circ \gamma(t)$ is a strictly convex function with a minimum point t_0 . Since d_{φ}^2 is constant along orbits we get that the orbit $\tilde{D}_0 = \tilde{G}\gamma(t_0)$ is the

minimum point set of d_{φ}^2 Thus by 2.4(a) it is totally geodesic in $\tilde{M}=\mathbb{R}^n$, so it is flat and therefore isometric to \mathbb{R}^r for some r. Now let there is not any singular orbit in M. So $\tilde{M}=\mathbb{R}^n$ does not have any singular orbit, therefore r=n-1 and \tilde{D}_0 is isometric to \mathbb{R}^{n-1} , so by Lemma 4.3 we get that each orbit \tilde{D} of \tilde{M} is isometric to \mathbb{R}^{n-1} , therefore each orbit D (= $\pi(\tilde{D})$) of M is flat, and since it is homogeneous we get by Theorem 2.5 that D is isometric to $\mathbb{R}^k \times \mathbb{T}^m$, for some m, k, m+k=n-1. This proves the part (a).

(b): Let B be the unique non-exceptional singular orbit of M and $B = \pi^{-1}(B)$ and let F(t) be the function obtained in the proof of part (a) with the minimum point t_0 . For each $t \in R$ we have $\tilde{G}\gamma(t) = g^{-1}(F(t))$, where $g = \mathrm{d}_{\varphi}^2$. If c and b are regular values of g then $g^{-1}(c)$ and $g^{-1}(b)$ are diffeomorphic (see [3], p. 10, Corollary 3.11), from these facts we get that $\tilde{B} = g^{-1}(F(t_0))$ (because if not, then $\tilde{B} = g^{-1}(b)$ where b is a regular value of g, and so \tilde{B} must be diffeomorphic to principal orbits which is a contradiction). So \tilde{B} is the minimum point set of g and therefore by 2.4(a) it is totally geodesic in M and is flat, thus g is flat. Since it is homogeneous we get by 2.5 that g is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = \mathbb{Z}^m$. \square

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