Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 328–336

ONE-PARAMETER SYSTEMS OF DEVELOPABLE SURFACES OF CODIMENSION TWO IN EUCLIDEAN SPACE

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Abstract. In the present paper we consider a class of hypersurfaces of conullity two in Euclidean space, which are one-parameter systems of developable surfaces of codimension two and we prove a characterization theorem for them in terms of their second fundamental tensor.

1. Preliminaries

It is well-known that the curvature tensor R of a locally symmetric Riemannian manifold $(M^n_{\mathbb{C}}, g)$ satisfies the identity

$$R(X,Y) \cdot R = 0 . \tag{1.1}$$

for all tangent vector fields X and Y. This is the reason why the spaces satisfying the identity (1) are called semi-symmetric spaces.

In 1968 Nomizu [6] conjectured that in all dimensions $n \geq 3$ every irreducible complete Riemannian semi-symmetric space is locally symmetric.

In 1972 Takagi [9] constructed a complete irreducible hypersurface in $\mathbb{E}^4$ which satisfies the condition (1.1) that is not locally symmetric. In 1972 Sekigawa [7] proved that in $\mathbb{E}^{m+1}$ $(m \geq 3)$ there exist complete irreducible hypersurfaces satisfying the condition (1.1) but are not locally symmetric. In 1982 Szabo [8] gave a local classification of Riemannian semi-symmetric spaces. According to his classification there are three types of classes, namely:

1) trivial class, consisting of all locally symmetric Riemannian spaces and all 2-dimensional Riemannian spaces;

2) exceptional class of all elliptic, hyperbolic, Euclidean and Kählerian cones;
3) typical class of all Riemannian manifolds foliated by Euclidean leaves of codimension two.

Riemannian manifolds of the typical class were studied under the name Riemannian manifolds of conullity two in [2] with respect to their metrics.

Let $(M^n, g, \bar{\Delta})$ be a Riemannian manifold endowed with a two-dimensional distribution $\bar{\Delta}$. Since our considerations are local, we can assume that there is an orthonormal frame field $\{W, \xi\}$ on M^n, which spans $\bar{\Delta}$, i. e. $\bar{\Delta}_p = \mathrm{span}\{W, \xi\}, p \in M^n$. We denote by T_pM^n the tangent space to M^n at a point $p \in M^n$ and by $\mathfrak{X}M^n$ — the Lie algebra of all C^∞ vector fields on M^n.

A Riemannian manifold $(M^n, g, \bar{\Delta})$ with curvature tensor R is of **conullity two** [2], if at every point $p \in M^n$ there exists an orthonormal frame $\{e_1 = W, e_2 = \xi, e_3, \dots, e_n\}$ of the tangent space T_pM^n such that

i) $R(e_1, e_2, e_2, e_1) = -R(e_2, e_1, e_2, e_1) = -R(e_1, e_2, e_1, e_2) = R(e_2, e_1, e_1, e_2) = k(p) \neq 0$;

ii) $R(e_i, e_j, e_k, e_l) = 0$, otherwise.

Let M^n be a hypersurface in Euclidean space $\mathbb{E}^{n+1}$. We denote the standard metric in $\mathbb{E}^{n+1}$ by g and its Levi-Civita connection by ∇'. Further, let ∇ be the induced connection on M^n and

$$h(X, Y) = g(AX, Y), \quad X, Y \in \mathfrak{X}M^n$$

be the second fundamental tensor of the hypersurface M^n. Hypersurfaces of conullity two are characterized in terms of the second fundamental tensor as follows [5]:

Proposition 1.1. *A hypersurface $(M^n, g, \bar{\Delta})$ in $\mathbb{E}^{n+1}$ is of conullity two iff its second fundamental tensor h has the form*

$$h = \lambda\omega \otimes \omega + \mu(\omega \otimes \eta + \eta \otimes \omega) + \nu\eta \otimes \eta, \quad \lambda\nu - \mu^2 \neq 0\,, \qquad (1.2)$$

where ω and η are one-forms; λ, μ and ν are functions on M^n.

Let Δ_0 be the distribution on M^n, orthogonal to W and ξ, and Δ — the distribution orthogonal to ξ.

The integrability conditions for a hypersurface of conullity two, obtained in [5], are

1) $\nabla_{x_0}\xi = \gamma(x_0)W$;
2) $\nabla_{x_0}W = -\gamma(x_0)\xi$;
3) $g(\nabla_W W, x_0) = \dfrac{\nu\,\mathrm{d}\lambda(x_0) - \mu\,\mathrm{d}\mu(x_0)}{\lambda\nu - \mu^2} + \dfrac{\mu(\lambda + \nu)}{\lambda\nu - \mu^2}\gamma(x_0)$;

4) $g(\nabla_W \xi, x_0) = \dfrac{\lambda\,\mathrm{d}\mu(x_0) - \mu\,\mathrm{d}\lambda(x_0)}{\lambda\nu - \mu^2} - \dfrac{\lambda^2 + 2\mu^2 - \lambda\nu}{\lambda\nu - \mu^2}\gamma(x_0);$

5) $g(\nabla_\xi W, x_0) = \dfrac{\nu\,\mathrm{d}\mu(x_0) - \mu\,\mathrm{d}\nu(x_0)}{\lambda\nu - \mu^2} + \dfrac{2\mu^2 + \nu^2 - \lambda\nu}{\lambda\nu - \mu^2}\gamma(x_0);$

6) $g(\nabla_\xi \xi, x_0) = \dfrac{\lambda\,\mathrm{d}\nu(x_0) - \mu\,\mathrm{d}\mu(x_0)}{\lambda\nu - \mu^2} - \dfrac{\mu(\lambda + \nu)}{\lambda\nu - \mu^2}\gamma(x_0);$

7) $\{(\lambda - \nu)^2 + 4\mu^2\} g(\nabla_W W, \xi) = 2\mu\,\mathrm{d}\mu(\xi) - (\lambda - \nu)\,\mathrm{d}\mu(W) - 2\mu\,\mathrm{d}\nu(W) + (\lambda - \nu)\,\mathrm{d}\lambda(\xi);$

8) $\{(\lambda-\nu)^2+4\mu^2\} g(\nabla_\xi \xi, W) = (\lambda-\nu)\,\mathrm{d}\mu(\xi)+2\mu\,\mathrm{d}\mu(W)-(\lambda-\nu)\,\mathrm{d}\nu(W)-2\mu\,\mathrm{d}\lambda(\xi),$

where $x_0 \in \Delta_0$ and γ is a one-form on Δ_0, defined by the eduallity $\gamma(x_0) = g(\nabla_{x_0}\xi, W)$.

With the help of these integrability conditions two interesting classes of hypersurfaces of conullity two are characterized in [5] — the class of ruled hypersurfaces and the class of one-parameter systems of torses of codimension two. Ruled hyper surfaces in $\mathbb{E}^{n+1}$ are characterized by the following

Theorem 1.1. *Let* (M^n, g, W, ξ) *be a hypersurface in* $\mathbb{E}^{n+1}$ *with second fundamental tensor* h*. Then* M^n *is locally a ruled hypersurface iff*

i) $h = \mu(\omega \otimes \eta + \eta \otimes \omega) + \nu\eta \otimes \eta$*;*

ii) $\gamma = 0$*;*

iii) $\operatorname{div}\xi = 0$.

Hypersurfaces of conullity two, which are one-parameter systems of torses of codimension two, are characterized as follows

Theorem 1.2. *Let* (M^n, g, W, ξ) *be a hypersurface of conullity two in* $\mathbb{E}^{n+1}$ *with second fundamental tensor (1.1). Then* M^n *is locally a one-parameter system of torses iff*

1. $\lambda \neq 0$*;*
2. *the distribution* Δ *is involutive;*
3. $\gamma = 0$*;*
4. $\mu = -W\left(\arctan \dfrac{\operatorname{div}\xi}{\lambda}\right)$.

In this paper we prove a characterization theorem for hypersurfaces of conullity two in $\mathbb{E}^{n+1}$, which are one-parameter systems of developable $(n-1)$-surfaces in terms of their second fundamental tensor (Theorem 3.1).

2. Developable Surfaces in Euclidean Space

A $(k+1)$-dimensional surface M^{k+1} in Euclidean space $\mathbb{E}^{n+1}$, which is a one-parameter system $\{\mathbb{E}^k(s)\}$, $s \in J$ of k-dimensional linear subspaces of $\mathbb{E}^{n+1}$, defined in an interval $J \subset \mathbb{R}$, is said to be a **ruled $(k+1)$-surface** [3, 1]. The planes $\mathbb{E}^k(s)$ are called *generators* of M^{k+1}. A ruled surface M^{k+1} is said to be **developable** [1], if the tangent space T_pM^{k+1} at all regular points p of an arbitrary fixed generator $\mathbb{E}^k(s)$ is one and the same.

We call a developable ruled hypersurface $M^n = \{\mathbb{E}^{n-1}(s)\}$, $s \in J$ in $\mathbb{E}^{n+1}$ a *torse*. Torses are characterized in terms of the second fundamental tensor as follows [4]:

Lemma 2.1. *Let (M^n, g) be a hypersurface in $\mathbb{E}^{n+1}$ with second fundamental tensor h. Then M^n is locally a torse iff*

$$h = k\omega \otimes \omega\,,$$

where k and ω are a function and a unit one-form on M^n, respectively.

If N is a unit vector field, normal to the torse $M^n = \{\mathbb{E}^{n-1}(s)\}$, $s \in J$, then according to Lemma 2.1

$$\nabla'_X N = -k\omega(X)W\,, \qquad X \in \mathfrak{X}M^n\,,$$

where W is a unit vector field, orthogonal to the generators and corresponding to the one-form ω.

Remark 2.1. *Every hyperplane $M^n = \mathbb{E}^n$ can be regarded as a torse with $k = 0$. The hyperplanes are trivial torses.*

Now we shall consider a developable $(n-1)$-surface $Q^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ in Euclidean space $\mathbb{E}^{n+1}$. Let $\{N, \xi\}$ be an orthonormal frame, normal to Q^{n-1}. We denote by h_1 and h_2 the second fundamental tensors of Q^{n-1} corresponding to the vector fields N and ξ, respectively:

$$h_1(x,y) = g(A_1x, y)\,, \qquad h_2(x,y) = g(A_2x,y)\,, \qquad x, y \in \mathfrak{X}Q^{n-1}\,.$$

According to Gauss and Weingarten formulae

$$\begin{aligned} \nabla'_x y &= \nabla_x y + h_1(x,y)N + h_2(x,y)\xi\,, \quad x,y \in \mathfrak{X}Q^{n-1}\,,\\ \nabla'_x N &= -A_1x + D_xN\,,\\ \nabla'_x \xi &= -A_2x + D_x\xi\,. \end{aligned} \tag{2.1}$$

Let $p \in Q^{n-1}$ be an arbitrary point and $\mathbb{E}^{n-2}(s)$ be the generator of Q^{n-1} containing p. Let $\Delta_0(p)$ denote the subspace of T_pQ^{n-1}, tangent to $\mathbb{E}^{n-2}(s)$ and $\Delta_0\,;\ p \to \Delta_0(p)$ be the corresponding distribution. The unit vector field on Q^{n-1}, orthogonal to Δ_0 and its corresponding one-form are denoted by W and ω, respectively (W is determined up to a sign).

Since Q^{n-1} is a developable surface, then the normal frame $\{N, \xi\}$ is parallel (constant) along each generator $\mathbb{E}^{n-2}(s)$, i. e.

$$\begin{aligned} \nabla'_{x_0} N &= 0\,, \\ \nabla'_{x_0} \xi &= 0\,, \end{aligned} \qquad x_0 \in \Delta_0\,. \tag{2.2}$$

So, the equalities (2.1) and (2.2) imply

$$\begin{aligned} A_1 x_0 &= 0\,, \qquad & D_{x_0} N &= 0\,, \\ A_2 x_0 &= 0\,, \qquad & D_{x_0} \xi &= 0\,, \end{aligned} \qquad x_0 \in \Delta_0\,.$$

Hence,

$$\begin{aligned} h_1(x_0, y) &= 0\,, \\ h_2(x_0, y) &= 0\,, \end{aligned} \qquad x_0 \in \Delta_0\,, \quad y \in \mathfrak{X}Q^{n-1}\,. \tag{2.3}$$

If x and y are arbitrary vector fields on Q^{n-1}, then

$$x - \omega(x)W \in \Delta_0\,, \qquad y - \omega(y)W \in \Delta_0\,, \tag{2.4}$$

and using (2.3) we get

$$\begin{aligned} h_1(x, y) &= p\omega(x)\omega(y)\,, \\ h_2(x, y) &= q\omega(x)\omega(y)\,, \end{aligned}$$

where $p = h_1(W, W), q = h_2(W, W)$.

Hence, the second fundamental tensors of a developable $(n-1)$-surface Q^{n-1} are

$$\begin{aligned} A_1 x &= p\omega(x)W\,, \\ A_2 x &= q\omega(x)W\,, \end{aligned} \qquad x \in \mathfrak{X}Q^{n-1}\,.$$

So, the Weingarten formulae for a developable $(n-1)$-surface Q^{n-1} takes the form

$$\begin{aligned} \nabla'_x N &= -p\omega(x)W + D_x N\,, \\ \nabla'_x \xi &= -q\omega(x)W + D_x \xi\,, \end{aligned} \qquad x \in \mathfrak{X}Q^{n-1}\,. \tag{2.5}$$

Now we shall characterize the developable $(n-1)$-surfaces in $\mathbb{E}^{n+1}$.

Let (M^{n-1}, g, W) be a surface of codimension two in $\mathbb{E}^{n+1}$ endowed with a unit vector field W and let ω denote the unit one-form corresponding to W.

Lemma 2.2. *Let* (M^{n-1}, g, W) *be a surface in* $\mathbb{E}^{n+1}$ *with normal frame* $\{N, \xi\}$. *Then* M^{n-1} *is locally a developable* $(n-1)$*-surface iff*

$$\begin{aligned} \nabla'_x N &= -p\omega(x)W - \mu\omega(x)\xi\,, \\ \nabla'_x \xi &= -q\omega(x)W + \mu\omega(x)N\,, \end{aligned} \qquad x \in \mathfrak{X}M^{n-1}\,, \tag{2.6}$$

where μ, p *and* q *are functions on* M^{n-1}, *such that* $p^2 + q^2 > 0$.

Proof: I. Let M^{n-1} be a developable $(n-1)$-surface $Q^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$. Hence, the equalities (2.5) hold good. Since N and ξ are unit vector fields, then $g(\nabla'_x N, N) = 0$, $g(\nabla'_x \xi, \xi) = 0$. Denoting

$$\mu = g(\nabla'_W \xi, N) = -g(\nabla'_W N, \xi)\,,$$

we get

$$\begin{aligned} \nabla'_W N &= -pW - \mu\xi\,, \\ \nabla'_W \xi &= -qW + \mu N\,. \end{aligned} \tag{2.7}$$

Taking into account the presentation (2.4) of arbitrary vector fields x and y from the equalities (2.2) and (2.7) we obtain (2.6).

II. Let the equalities (2.6) hold good for a surface M^{n-1} in $\mathbb{E}^{n+1}$. If $R' = 0$ is the curvature tensor of the canonical connection ∇' in $\mathbb{E}^{n+1}$, then calculating $R'(x,y)N$ and $R'(x,y)\xi$ from (2.6) we find

$$\begin{aligned} &\{\mu\,\mathrm{d}\omega(x,y) - (\omega \wedge \mathrm{d}\mu)(x,y)\}\xi + \{p\,\mathrm{d}\omega(x,y) - (\omega \wedge \mathrm{d}p)(x,y)\}W \\ &\qquad - p\omega(x)\nabla'_y W + p\omega(y)\nabla'_x W = 0\,, \\ &\{\mu\,\mathrm{d}\omega(x,y) - (\omega \wedge \mathrm{d}\mu)(x,y)\}N - \{q\,\mathrm{d}\omega(x,y) - (\omega \wedge \mathrm{d}q)(x,y)\}W \\ &\qquad + q\omega(x)\nabla'_y W - q\omega(y)\nabla'_x W = 0\,. \end{aligned} \tag{2.8}$$

From (2.8) it follows that for each $x_0 \in \Delta_0$ we have

$$\mathrm{d}\omega(x_0, y_0) = 0\,, \qquad \nabla'_{x_0} W = 0\,. \tag{2.9}$$

The first equality of (2.9) implies that the distribution Δ_0 is involutive. Hence, for every point $p \in M^{n-1}$ there exists a unique maximal integral submanifold S_p^{n-2} of Δ_0 containing p. Using (2.6) and the second equality of (2.9) we get

$$\nabla'_{x_0} N = 0\,, \qquad \nabla'_{x_0} \xi = 0\,, \qquad \nabla'_{x_0} W = 0\,, \qquad x_0 \in \Delta_0\,,$$

which shows that S_p^{n-2} lies on a $(n-2)$-dimensional plane $\mathbb{E}_p^{n-2}$ with canonical normal frame $\{N, \xi, W\}$. Hence, M^{n-1} lies on a one-parameter system $\{\mathbb{E}^{n-2}(s)\}$, $s \in J$ of planes of codimension three. Besides that, the normal frame $\{N, \xi\}$ of M^{n-1} is constant along each generator $\mathbb{E}^{n-2}(s)$ ($\nabla'_{x_0} N = 0$, $\nabla'_{x_0} \xi = 0$), which implies that M^{n-1} is locally a developable $(n-1)$-surface. □

Remark 2.2. *Let Q^{n-1} be a developable $(n-1)$-surface. From the equalities (2.6) and (2.8) it follows that*

$$\mu\, \mathrm{d}\omega(x,y) - (\omega \wedge \mathrm{d}\mu)(x,y) = 0\,.$$

Hence, $\mathrm{d}(\mu\omega) = 0$, i. e. the 1-form $\mu\omega$ is closed. Then there exists locally a function φ on Q^{n-1} such that $\mu\omega = \mathrm{d}\varphi$. Now the equalities (2.6) take the form

$$\begin{aligned} \nabla'_x N &= -p\omega(x)W - \mathrm{d}\varphi(x)\xi\,, \\ \nabla'_x \xi &= -q\omega(x)W + \mathrm{d}\varphi(x)N\,. \end{aligned} \tag{2.10}$$

Setting

$$\begin{aligned} l &= \cos\varphi\xi - \sin\varphi N\,, \\ n &= \sin\varphi\xi + \cos\varphi N\,, \end{aligned}$$

from (2.10) we find

$$\begin{aligned} \nabla'_x l &= -k_1\omega(x)W\,, \\ \nabla'_x n &= -k_2\omega(x)W\,, \end{aligned} \tag{2.11}$$

where $k_1 = q\cos\varphi - p\sin\varphi$, $k_2 = p\cos\varphi + q\sin\varphi$.

Hence, for each developable $(n-1)$-surface Q^{n-1} there exists locally a normal frame $\{l,n\}$, satisfying the equalities (2.11), where k_1 and k_2 are functions on Q^{n-1}. We shall call $\{l,n\}$ the **canonical normal frame** of Q^{n-1}.

3. Hypersurfaces of Conullity Two, which Are One-parameter Systems of Developable $(n-1)$-surfaces

Let (M^n, g, W, ξ) be a hypersurface of conullity two in $\mathbb{E}^{n+1}$ with second fundamental tensor (1.1). The integrability conditions 1)–8), given in Section 1 hold good. We denote by Δ_0 the distribution on M^n, orthogonal to W and ξ, i. e.

$$\Delta_0(p) = \{x_0 \in T_pM^n\,;\, x_0 \perp W, x_0 \perp \xi\}\,, \quad p \in M^n$$

and by Δ the distribution on M^n, orthogonal to ξ, i. e.

$$\Delta(p) = \{x \in T_pM^n\,;\, x \perp \xi\}\,, \quad p \in M^n\,.$$

Now we shall prove the main theorem in our paper

Theorem 3.1. *Let (M^n, g, W, ξ) be a hypersurface of conullity two in $\mathbb{E}^{n+1}$. Then M^n is locally a one-parameter system of developable $(n-1)$-surfaces iff*

i) *the distribution Δ is involutive;*

ii) $\gamma = 0$.

Proof: I. Let $M^n = \{Q^{n-1}(s)\}$, $s \in J$ be a one-parameter system of developable $(n-1)$-surfaces $Q^{n-1}(s)$, defined in an interval J. For an arbitrary $s \in J$ the surface $Q^{n-1}(s)$ has a canonical normal frame $\{l(s), n(s)\}$, satisfying (2.11).

Let N be a unit vector field, normal to M^n and ξ be a vector field on M^n, orthogonal to the surfaces $Q^{n-1}(s)$. Then $\{N, \xi\}$ is a frame, normal to each surface $Q^{n-1}(s)$. Hence,

$$\begin{aligned} \xi &= \cos\varphi l + \sin\varphi n\,, \\ N &= -\sin\varphi l + \cos\varphi n\,, \end{aligned}$$

where $\varphi = \angle(l, \xi)$. The equalities (2.11) and (3.1) imply

$$\begin{aligned} \nabla'_x N &= -(k_2\cos\varphi - k_1\sin\varphi)\omega(x)W - \mathrm{d}\varphi(x)\xi\,, \\ \nabla'_x \xi &= -(k_1\cos\varphi + k_2\sin\varphi)\omega(x)W + \mathrm{d}\varphi(x)N\,, \end{aligned} \quad x \in \mathfrak{X}Q^{n-1}\,. \tag{3.1}$$

If $h(X,Y) = g(AX,Y)$, $X, Y \in \mathfrak{X}M^n$ is the second fundamental tensor of M^n, then the first equation of (3.2) gives

$$Ax = (k_2\cos\varphi - k_1\sin\varphi)\omega(x)W + \mathrm{d}\varphi(x)\xi\,, \quad x \in \mathfrak{X}Q^{n-1}\,.$$

So, for $x_0 \in \Delta_0$ we have $Ax_0 = \mathrm{d}\varphi(x_0)\xi$. Since M^n is a hypersurface of conullity two, then $\mathrm{d}\varphi(x_0) = 0, x_0 \in \Delta_0$. Hence,

$$\mathrm{d}\varphi(x) = \mathrm{d}\varphi(W)\omega(x)\,. \tag{3.2}$$

So, the second fundamental tensor h of a one-parameter system of developable $(n-1)$-surfaces is:

$$h = (k_2\cos\varphi - k_1\sin\varphi)\omega\otimes\omega + \mathrm{d}\varphi(W)(\omega\otimes\eta + \eta\otimes\omega) + \nu\eta\otimes\eta\,,$$

where $\nu = h(\xi,\xi)$.

Using (3.3) and the second equality of (3.2), we get $\nabla'_{x_0}\xi = 0, x_0 \in \Delta_0$. Hence, the one-form γ on Δ_0 is zero.

Since the developable $(n-1)$-surfaces $Q^{n-1}(s)$ are the integral submanifolds of the distribution Δ, orthogonal to ξ, then Δ is involutive.

II. Let M^n be a hypersurface of conullity two, for which the conditions i) and ii) hold good. Since Δ is an involutive distribution, then for every point $p \in M^n$ there exists a unique maximal integral submanifold S^{n-1}_p of Δ containing p. Hence, M^n lies on a one-parameter system $\{S^{n-1}(s)\}$, $s \in J$ of surfaces

$S^{n-1}(s)$ of codimension two. Taking into account i), ii) and the integrability condition 4) for M^n we obtain

$$\begin{array}{ll} \nabla'_W N = -\lambda W - \mu\xi\,, & \nabla'_{x_0} N = 0\,, \\ \nabla'_W \xi = \operatorname{div}\xi W + \mu N\,, & \nabla'_{x_0}\xi = 0\,, \end{array} \qquad x_0 \in \Delta_0\,. \tag{3.3}$$

If $x \in \Delta$, then $x - \omega(x)W \in \Delta_0$ and using (3.4) we get

$$\begin{array}{l} \nabla'_x N = -\lambda\omega(x)W - \mu\omega(x)\xi\,, \\ \nabla'_x \xi = \operatorname{div}\xi\omega(x)W + \mu\omega(x)N\,, \end{array} \qquad x \in \Delta\,.$$

If $\lambda = 0$ and $\operatorname{div}\xi = 0$, then according to Theorem 1.1 M^n is locally a ruled hypersurface $\{\mathbb{E}^{n-1}(s)\}$, $s \in J$ (the integral submanifolds S^{n-1} of the distribution Δ are planes $\mathbb{E}^{n-1}$).

If $\lambda \neq 0$ or $\operatorname{div}\xi \neq 0$, then denoting $p = \lambda$, $q = -\operatorname{div}\xi$ and applying Lemma 2.2, we obtain that the integral submanifolds $S^{n-1}(s)$ of Δ are locally developable $(n-1)$-surfaces $Q^{n-1}(s)$. Hence, M^n is locally a one-parameter system $\{Q^{n-1}(s)\}$, $s \in J$ of developable $(n-1)$-surfaces. □

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