

## REAL FORMS OF COMPLEXIFIED HAMILTONIAN DYNAMICS

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**Abstract.** Complexified Hamiltonian dynamical systems are considered with subsequent construction of real forms of Hamiltonian dynamics by using compatible complex structures or involution operators mimicking the properties of complex conjugation. This provides a method of associating a class of real dynamical systems to a given initial (also real) one. Examples are given and the problem of integrability of the derived dynamical systems is also discussed.

### 1. Introduction

We start with a real Hamiltonian system  $(\mathcal{M}, \omega, H)$  with  $n$  degrees of freedom and Hamiltonian  $H$  depending analytically on the dynamical variables. Such systems can be complexified and then considered as Hamiltonian systems with  $2n$  (real) degrees of freedom. Our main construction relates to each compatible involutive automorphism  $\tilde{C}$  of the complexified phase space and commuting with the complex structure a real Hamiltonian form of the complexified system. Just like with each complex Lie algebra one associates several inequivalent real forms, so to each complexified dynamics we associate several inequivalent real forms which again have  $n$  real degrees of freedom just like the initial system. Provided  $\tilde{C}(H) = H$  the dynamics on the real form will be well defined and

will coincide with the dynamics on  $\mathcal{M}_{\mathbb{C}}$  restricted to  $\mathcal{M}_{\mathbb{R}}$ . If the initial system  $\mathcal{H}$  is integrable then its real Hamiltonian forms will also be integrable. We also include discussion on invariant complex polarizations, their connection with integrable systems and the possibility they offer to define a class of new integrable systems starting from an initial one.

Part of the motivation comes from the fact that the so-called **complex Toda chain** (CTC) was shown to describe  $N$ -soliton interactions in the adiabatic approximation [5, 6, 1]. The complete integrability of the CTC is a direct consequence of the integrability of the real (standard) Toda chain (TC); it was also shown that CTC allows several dynamical regimes that are qualitatively different from the one of RTC [6].

Examples of non-standard (or *twisted*) real forms have already been studied by Evans and Madsen [3] in connection with the problem of positivity of the kinetic energy terms in the Lagrangian description and with emphasis on conformal WZNW models. Different real forms of Lie algebras were used there for construction of integrable models via Hamiltonian reduction. Here we present a construction of real forms of dynamics directly in terms of symplectic manifolds and Hamiltonians and in this sense it is a generalization of the former method. Examples of *indefinite-metric Toda chain* (IMTC) has already been studied by Kodama and Ye [8]. In particular they note that while the solutions of the TC model are regular for all  $t$ , the solutions of the IMTC model develop singularities for finite values of  $t$ . The approach we follow here may also give indefinite-metric Hamiltonians but our main interest is in producing new integrable non-linear evolution equations (NLEE) starting from already known ones.

## 2. Involutions and Real Hamiltonian Forms

The approach we will follow in this paragraph is inspired by the basic idea of construction of real forms for simple Lie algebras [2]. A basic tool in the following construction is a Cartan-like involutive automorphism  $\mathcal{C}$  which will play the role of a *complex conjugation operator*. We will say that  $\mathcal{C}$  is an involutive compatible automorphism on the complexified phase space  $\mathcal{M}_{\mathbb{C}}^{(2n)}$  if:

$$\mathcal{C}(\{F, G\}) = \{\mathcal{C}(F), \mathcal{C}(G)\}, \quad \mathcal{C}^2 = \mathbb{1}, \quad (1)$$

where  $F$  and  $G$  are analytic functions on  $\mathcal{M}_{\mathbb{C}}^{(2n)}$  and involution acts on them by acting on the arguments:  $\mathcal{C}(F(z)) = F(\mathcal{C}(z))$ . In terms of vector fields  $X, Y \in T\mathcal{M}_{\mathbb{C}}$  and the lifted involution  $TC: T\mathcal{M}_{\mathbb{C}} \rightarrow T\mathcal{M}_{\mathbb{C}}$  we have:

$$\omega(TC(X), TC(Y)) = \mathcal{C}(\omega(X, Y)), \quad (TC)^2 = \mathbb{1}. \quad (2)$$

Here we shall also assume that the actions of  $\mathcal{C}$  and the proper complex conjugation commute. Obviously  $\mathcal{C}$  has eigenvalues 1 and  $-1$  and splits  $(\mathcal{M}_m^{(2n)})_{\mathbb{C}}$  into a direct sum of its two eigenspaces:

$$\mathcal{M}_{\mathbb{C}}^{(2n)} = \mathcal{M}_0 \oplus \mathcal{M}_1. \quad (3)$$

Then we define a “real” phase space of our system by:

$$\mathcal{M}_{\mathbb{R}}^{(2n)} = \mathcal{M}_0 \oplus i\mathcal{M}_1. \quad (4)$$

Let's introduce  $\vec{X} = \sum_k \xi_k \vec{x}_k \in \mathcal{M}_0$  and  $\vec{Y} = \sum_p \eta_p \vec{y}_p \in \mathcal{M}_1$  where  $\vec{x}_k$  and  $\vec{y}_p$  form bases of  $\mathcal{M}_0$  and  $\mathcal{M}_1$  respectively and  $\xi_k$  and  $\eta_p$  are complex coefficients. Then any element of the real form  $\mathcal{M}_{\mathbb{R}}^{(2n)}$  can be represented as:

$$\vec{Z} = \vec{X} + i\vec{Y} \in \mathcal{M}_{\mathbb{R}}^{(2n)}, \quad (5)$$

and reality condition means that

$$\tilde{\mathcal{C}}(\vec{Z}) \equiv \mathcal{C}(\vec{Z}^*) = \mathcal{C}(\vec{X} - i\vec{Y}) = \vec{X} + i\vec{Y} = \vec{Z}, \quad (6)$$

where by  $*$  we have denoted the complex conjugation.

Obviously  $\tilde{\mathcal{C}} \equiv \mathcal{C} \circ * \equiv * \circ \mathcal{C}$  is a compatible involution again; then  $\mathcal{M}_{\mathbb{R}}$  will be the space of  $\tilde{\mathcal{C}}$ -invariant variables embedded in  $\mathcal{M}_{\mathbb{C}}$ .

Along with these properties our  $\tilde{\mathcal{C}}$  must be compatible with the dynamics of the corresponding real Hamiltonian form. If we choose functions  $F, G$  from  $\mathcal{M}_{\mathbb{R}}^{(2n)}$  then due to (1)  $\{F, G\} \in \mathcal{M}_{\mathbb{R}}^{(2n)}$  too. Then if we choose a Hamiltonian  $H \in \mathcal{F}(\mathcal{M}_{\mathbb{R}}^{(2n)})$  and a dynamical variable  $f \in \mathcal{F}(\mathcal{M}_{\mathbb{R}}^{(2n)})$  obviously the equation of motion for  $f$

$$\frac{df}{dt} = \{H, f\}, \quad (7)$$

becomes naturally restricted to  $\mathcal{M}_{\mathbb{R}}^{(2n)}$ . Rewriting (7) into its equivalent form:

$$\omega(X_H, \cdot) = dH \cdot, \quad (8)$$

and making use of (2) we see that the vector field  $X_H$  must also satisfy

$$\tilde{\mathcal{C}}(X_H) = X_H. \quad (9)$$

Compatibility condition eq. (1) guarantees that  $\mathcal{M}_0$ ,  $\mathcal{M}_1$  and  $\mathcal{M}_{\mathbb{R}}$  will be symplectic subspaces of  $\mathcal{M}_{\mathbb{C}}^{(2n)}$ . If  $\mathcal{M}_{\mathbb{C}}^{(2n)}$  is endowed with Hamiltonian which is “real” with respect to  $\tilde{\mathcal{C}}$ , dynamics on  $(\mathcal{M}_{\mathbb{R}}, \omega|_{\mathcal{M}_{\mathbb{R}}}, H|_{\mathcal{M}_{\mathbb{R}}})$  will be well defined and will coincide with the dynamics on  $(\mathcal{M}_{\mathbb{C}}^{(2n)}, \omega, H)$  restricted to  $\mathcal{M}_{\mathbb{R}}^{(2n)}$ .

If the initial Hamiltonian is integrable (i. e. being only a function of  $n$  action variables) then this property will be preserved. Indeed  $\tilde{\mathcal{C}}$  maps Lagrangian submanifolds to Lagrangian submanifolds. Compatibility guarantees that Lagrangian submanifold will be mapped to a coisotropic submanifold and  $\tilde{\mathcal{C}}^2 = \mathbb{1}$  guarantees that it will actually be a Lagrangian one. Also, due to  $\tilde{\mathcal{C}}(H) = H$  we have

$$H(I_i) = \tilde{\mathcal{C}}(H(I_i)) = H(\tilde{\mathcal{C}}(I_i)) = H(\mathcal{I}_i)$$

where  $\mathcal{I}_i = \tilde{\mathcal{C}}(I_i)$  span a Lagrangian submanifold and  $H$  will depend on  $n$  integrals of motion. Hence we will have integrable dynamics on  $(\mathcal{M}_{\mathbb{R}}, \omega|_{\mathcal{M}_{\mathbb{R}}}, H)^{(1)}$ .

To illustrate the construction let us assume a completely integrable system with  $I$ -s and  $\phi$ -s as action-angle variables and choose involutions of the type:

$$\tilde{\mathcal{C}}(I_k) = \varepsilon_k(I_k)^*, \quad \tilde{\mathcal{C}}(\phi_k) = \varepsilon_k(\phi_k)^*, \quad \varepsilon_k = \pm 1$$

assuming that all they leave Hamiltonian invariant. As a result we will have well defined equations of motion for every real form corresponding to different choices of  $\varepsilon_k$ .

We stress that  $\tilde{\mathcal{C}}$  must satisfy both conditions (1) and (9). Let us illustrate what may happen when one of them is violated. Indeed, the case when  $H$  is not “real” is not so nice. The above mentioned arguments are no more valid and we could actually have a nonintegrable restricted Hamiltonian  $H|_{\mathcal{M}_{\mathbb{R}}}$  coming from an integrable Hamiltonian  $H$ . As an example of loss of integrability (or as a counterexample to the naïve expectations) one can take the integrable Hamiltonian  $H = \prod I_i$  with  $I$ -s and  $\phi$ -s as action and angle variables and to assume the involution  $\mathcal{C}(I_i) = \frac{1}{\sqrt{2}}(I_i + \phi_i)$ ,  $\mathcal{C}(\phi_i) = \frac{1}{\sqrt{2}}(I_i - \phi_i)$ . The resulting  $H_{\mathbb{R}}$  is obviously nonintegrable.

In such case we could follow *two* routes. *First*, we can use the restricted Hamiltonian:

$$H_{\mathbb{R}} = H|_{\mathcal{M}_{\mathbb{R}}} = \frac{1}{2} [H + \mathcal{C}(H) + i(H - \mathcal{C}(H))]$$

to define a *new* dynamics on  $(\mathcal{M}_{\mathbb{R}}, \omega|_{\mathcal{M}_{\mathbb{R}}})$  which may not inherit all of the properties of the initial one (like integrability). *Second*, we could employ the machinery of (Dirac–Bergmann style) constraints theory through which we will have consistent equations of motion on a *submanifold* of  $\mathcal{M}_{\mathbb{R}}$ .

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<sup>(1)</sup> Due to the fact that  $\mathcal{M}_{\mathbb{R}}$  is a symplectic submanifold, Poisson brackets between functions on  $\mathcal{M}_{\mathbb{R}}$  will coincide with the original Poisson brackets (i. e. we will not have Dirac brackets for these variables despite the fact that we will be using brackets corresponding to  $\omega|_{\mathcal{M}_{\mathbb{R}}}$  but not to  $\omega$ ).

### 3. Complex Polarizations and Families of Integrable Dynamics

We shall begin this paragraph with a reminder on polarizations which are best known in the context of geometric quantization [13] but have a much wider importance.

By definition a real polarization is an integrable Lagrangian distribution. A complex polarization  $P$  of a  $2n$ -dimensional symplectic manifold  $(\mathcal{M}, \omega)$  is a complex integrable Lagrangian distribution. More precisely, for every  $m \in \mathcal{M}$ ,  $P_m$  is a complex Lagrangian subspace of  $(T_m\mathcal{M})_{\mathbb{C}}$ ; there is a neighbourhood around  $m$  and a collection of smooth complex-valued functions  $\{z^1, \dots, z^n\}$  such that  $P$  is spanned there by (the complex conjugate of) the Hamiltonian vector fields of these functions  $(X_{z^i})^*$ ; and  $P_m \cap \bar{P}_m \cap T_m\mathcal{M}$  has constant dimension for all  $m$ . There is a naturally defined Hermitian form on  $P$  by

$$\langle\langle X, Y \rangle\rangle = 4i\omega(X, Y^*), \quad X, Y \in P.$$

If the signature of this form, say with  $r$  ones and  $s$  minus ones, is the same at every  $m$ ,  $P$  is said to be of  $(r, s)$  type and furthermore it is positive if  $s = 0$  and Kähler if  $r + s = n$ . There is an one-to-one correspondence (when they exist) between Kähler polarizations and complex structures  $J$  on  $\mathcal{M}$  compatible with  $\omega$ , i. e.  $J$  is an automorphism of  $T\mathcal{M}$  such that  $J^2 = -\mathbb{1}$  and  $\omega(JX, JY) = \omega(X, Y)$  for  $X, Y \in T\mathcal{M}$ . The so called holomorphic polarization  $\mathcal{P}$  is spanned at any point by all  $X - iJX$  (and the antiholomorphic one  $\bar{\mathcal{P}}$  by  $X + iJX$ ) where  $X \in T\mathcal{M}$ . The general form of  $J$  acting on a symplectic frame  $(V_a, W^b)$  such that  $\{V_a, W^b\} = \delta_a^b$  is:

$$JV_a = g_{ab}W^b \quad \text{and} \quad JW^a = -g^{ab}V_b$$

where  $g_{ab} = (g^{ab})^{-1}$  is any symmetric nonsingular  $n \times n$  matrix (which when diagonalized has  $r$  ones and  $s$  minus ones for a  $(r, s)$ -type polarization).

$J$  mimics the properties of imaginary unit and it converts  $\mathcal{M}$  into a complex manifold with local complex analytic coordinates  $z_i$  (playing the role of the defining set of functions for the holomorphic polarization) such that

$$J\left(\frac{\partial}{\partial z_i}\right) = i\left(\frac{\partial}{\partial z_i}\right), \quad J\left(\frac{\partial}{\partial z_i^*}\right) = -i\left(\frac{\partial}{\partial z_i^*}\right). \quad (10)$$

To make connection with the previous construction let's note that once we have a  $J$  playing the role of imaginary unit, we can define a "complex conjugation operator"  $TC$  being an involutive automorphism of  $T\mathcal{M}_{\mathbb{C}}$  mapping holomorphic onto antiholomorphic polarizations (and vice versa):

$$TC: \mathcal{P} \rightarrow \bar{\mathcal{P}}, \quad TC: \bar{\mathcal{P}} \rightarrow \mathcal{P}.$$

This is the common ground of the two approaches —  $J$  splits  $T\mathcal{M}_{\mathbb{C}}$  in two eigenspaces and  $TC$  maps them onto each other.

We could have different definitions of “creation and annihilation” variables by picking coordinates which diagonalize the above mentioned Hermitian form (or its equivalent  $2\omega(X, JY)$  for  $X, Y \in T_m\mathcal{M}$ ). One is free to choose  $J$  and usually the first choice is among the positive ones (i. e. with  $s = 0$  in the corresponding 2-form). As an example of a Kähler polarization of  $(1, 1)$  type which is invariant for  $H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$  we can take one defined locally by  $z_1 = q_1 + ip_1$  and  $z_2 = q_2 - ip_2$  and with “creation and annihilation” variables:

$$\begin{aligned} a_1 &= (q_1 + ip_1)/\sqrt{2}, & a_1^\dagger &= (q_1 - ip_1)/\sqrt{2} \\ a_2 &= (q_2 - ip_2)/\sqrt{2}, & a_2^\dagger &= (q_2 + ip_2)/\sqrt{2}. \end{aligned}$$

and  $J$  acting as:  $JP_1 = Q_2$ ,  $JP_2 = Q_1$  and  $JQ_1 = -P_2$ ,  $JQ_2 = -P_1$  for  $q_i, p_i \in (T_m\mathcal{M})_{\mathbb{C}}$  and  $Q_i, P_i \in (T_m\mathcal{M})$  forming symplectic frames.

In this manner we may classify our possible choices of compatible complex structures by their type (or signature) and inside each  $(r, s)$  type they are labeled by the elements of  $U(r, n - r)$ .

Invariant polarizations are naturally connected with integrable dynamical systems and offer coordinate-free approach to them. Invariant tori defined by  $I_i = \text{const}$  (where  $I_i$  and  $\phi_i$  are action-angle coordinates) are real polarizations which are preserved by the flow of the Hamiltonian. Dynamically stable splitting of the complexified phase space into creation and annihilation variables which do not mix with each other during evolution gives us an invariant complex polarization. Of course, one could easily recognize that the assumption for existence of a dynamically stable polarization means that we are dealing with an integrable system.

When written in the natural coordinates for the assumed stable Kählerian polarization the Hermitian form is automatically diagonalized and any integrable Hamiltonian which by definition depends only on  $I_i$ -s, should also be linear in  $a_i^\dagger$ -s in order to preserve the polarization [13]. As a result Hamiltonian could be written (in some neighbourhood) as a  $H = \sum \Omega_i a_i^\dagger a_i$ . Associated with the initial stable polarization we have a family of polarizations which are again stable but with different signature of the Hermitian form. Consequently, we can always choose a compatible complex structure (or Kählerian polarization with suitable signature from this family) such that

$$H = \sum \epsilon_i \Omega_i a_i^\dagger a_i \quad \text{with } \epsilon_i = \pm 1.$$

In this manner we can obtain new dynamical systems which are again integrable. The introduction of various (quasi) particles, “Bogolyubov transformation”, etc,

may be looked at as illustrations of this mechanism (see also [9] for applications in the BRST context).

#### 4. Integrability

For integrability we may require either preserving of the polarization by the Hamiltonian flow or preserving of the complex structure:  $\mathcal{L}_\Gamma J = 0$  where  $\mathcal{L}$  denotes the Lie derivative and  $\Gamma$  is the dynamical vector field:  $i_\Gamma \omega = -dH$ . In purely algebraic language integrability means existence of a maximal rank Abelian subalgebra in the commutant of the Hamiltonian.

A recursion operator method [10] gives us another route towards integrability. Recursion operator is a (1,1) tensor field

$$T = T_j^i \frac{\partial}{\partial x^i} \otimes dx^j$$

with nonvanishing Nijenhuis torsion

$$N_T(\alpha, X, Y) = \langle \alpha, (\mathcal{L}_{TX}T - T\mathcal{L}_X T)Y \rangle,$$

and doubly degenerate (nowhere vanishing) eigenvalues. For example, for a non resonant Hamiltonian integrable dynamics admitting action-angle coordinates  $(I, \phi)$  a class of such tensor fields is

$$T = \sum_h \lambda_h(\nu^h) \left( d\nu^h \otimes \frac{\partial}{\partial \nu^h} + d\phi^h \otimes \frac{\partial}{\partial \phi^h} \right).$$

where  $\nu^h = \frac{\partial H}{\partial I_h}$  are the frequencies and the  $\lambda$ 's are arbitrary and functionally independent.

If a given Hamiltonian dynamics leaves invariant this tensor field (and assuming analyticity), then also the complexified dynamics shall leave invariant the corresponding tensor. By construction, the restriction to other real forms will leave the invariance and the Nijenhuis property unchanged provided  $\tilde{\mathcal{C}}(H) = H$ . For this reasons, the complexification procedure and the projection on the real part will give a tensor field with the same properties. Due to the fact that existence of recursion operators is essentially equivalent to integrability in the non-resonant case, this line of argumentation gives us another instrument to treat the integrability of real forms.

#### 5. Examples

**Example 1.** *Let us first show how the standard complexification of dynamics (see e. g. [12]) realized by the variables:*

$$a_i = (q_i + ip_i)/\sqrt{2}, \quad a_i^* = (q_i - ip_i)/\sqrt{2}$$

fits in the present scheme. Namely we can request that

$$TC(\vec{q}) = \vec{q}, \quad TC(\vec{p}) = -\vec{p} \text{ where } \vec{q}, \vec{p} \in T\mathcal{M}_{\mathbb{C}}. \quad (11)$$

In this case  $\mathcal{M}_0 \equiv \{q\}$ ,  $\mathcal{M}_1 \equiv \{p\}$  and  $\mathcal{M}_{\mathbb{R}} \equiv \{q\} + i\{p\}$ . The action of the complex structure is simply:  $J(\vec{Q}) = \vec{P}$ ,  $J(\vec{P}) = -\vec{Q}$  for  $\vec{P}, \vec{Q} \in T\mathcal{M}$ .

Then the only nontrivial basic Poisson brackets among  $a_i$  and  $a_i^*$  which span  $\mathcal{M}_{\mathbb{R}}$  turn into the form:

$$\{a_k^*(t), a_m(t)\} = i\delta_{km}, \quad (12)$$

well known from a number of models in field theory involving complex fields. Obviously, any Hamiltonian of the type:  $H = f(\vec{p}^2 + \vec{q}^2)$  would be “real” with respect to the involution chosen.

**Example 2.** In [7] one can find a discussion on complexified Hamiltonian systems together with construction of integrals of motion for a list of two-dimensional Hamiltonian systems. Crucial part of the procedure is the (possibly puzzling) replacing of the original phase space variables:  $q \rightarrow q_1 + ip_2$ ,  $p \rightarrow p_1 + iq_2$ . This gives an example of the construction in Section 3 with polarization of (1,1) type.

**Example 3.** We shall illustrate our construction by the paradigmical example of the Toda chain related to the  $sl(n, \mathbb{C})$  algebra:

$$H_{TC} = \frac{1}{2} \sum_{k=1}^n p_k^2 + \sum_{k=1}^{n-1} e^{q_{k+1} - q_k}. \quad (13)$$

We complexify  $p_k$  and  $q_k$  and choose the involution as:

$$\tilde{\mathcal{C}}(p_k) = -p_{n+1-k}^*, \quad \tilde{\mathcal{C}}(q_k) = -q_{n+1-k}^*. \quad (14)$$

As a result we obtain the following real forms of the TC model:

(i) for  $n = 2r + 1$ :

$$H_{TC1} = \sum_{k=1}^r (p_{0,k}^2 - p_{1,k}^2) - p_{1,r+1}^2 + e^{-2q_{0,r}} \cos(q_{1,r+1} - q_{1,r}) + 2 \sum_{k=1}^{r-1} e^{(q_{0,k+1} - q_{0,k})} \cos(q_{1,k+1} - q_{1,k}), \quad (15)$$

and (ii) for  $n = 2r$ :

$$H_{TC2} = \sum_{k=1}^r (p_{0,k}^2 - p_{1,k}^2) + e^{-2q_{0,r}} + 2 \sum_{k=1}^{r-1} e^{(q_{0,k+1} - q_{0,k})} \cos(q_{1,k+1} - q_{1,k}) \quad (16)$$



Since the solutions of the CTC model are well known (see e. g. [6] and the references therein) we can easily obtain the solutions for each of the models (15) and (16) just imposing the corresponding reductions on the initial parameters. It is also easy to see that these models are generalizations of the well known Toda chain models associated to the classical Lie algebras [11]; indeed if we put  $q_{1,k} \equiv 0$  and  $p_{1,k} \equiv 0$  we find that (15) goes into the  $\mathbf{B}_r$  TC while (16) provides the  $\mathbf{C}_r$  TC.

This list of examples can be substantially extended.

Another approach to constructing real forms of Toda theories has been used by Evans and Madsen [3]. They addressed Toda fields theories in 2-dimensional space-time related to the real forms  $\mathfrak{g}_{\mathbb{R}}$  of the simple Lie algebras underlying their Lax representations. These models allow Hamiltonian formulation in which the (infinite-dimensional) phase space is identified with the properly chosen co-adjoint orbit of  $\mathfrak{g}_{\mathbb{R}}$ . The Cartan involution  $\tau$  responsible for  $\mathfrak{g}_{\mathbb{R}}$  induces an involutive automorphism on the co-adjoint orbit which is the analog of  $\tilde{\mathcal{C}}$ . Therefore the real forms of the Toda theories in [3] can be viewed as infinite-dimensional realizations of our construction.

## 6. Discussion

By using  $J^2 = -\mathbb{1}$ , it is possible to consider the eigenspaces (eigendistributions) of  $T\mathcal{M}_{\mathbb{C}}$  associated with eigenvalue  $i$  and eigenvalue  $-i$  (see Eq. (10)) which give us the holomorphic and antiholomorphic polarizations. Each compatible  $J$  will induce a similar splitting of the tangent space into transversal distributions.

In this case we will have reduced dynamics on a carrier space with half dimensions but working on a Lagrangian submanifold means that there the symplectic form will be zero and in this sense there will be *no symplectic* dynamics on it. Nevertheless, it may be quite a nice dynamics — something like  $a_k(t) = a_k(0) e^{i\Omega_k t}$  for integrable systems.

Similarly, by using  $\mathcal{C}^2 = \mathbb{1}$ , it is possible to consider the eigenspaces (eigendistributions) of  $\mathcal{M}_{\mathbb{C}}$  associated with eigenvalue  $1$  and eigenvalue  $-1$ . Here compatibility condition guarantees that  $\mathcal{M}_{\mathbb{R}}$  will be a symplectic subspace and consequently we could have a *symplectic dynamics* on it. Here integrability could be defined again as existence of a maximal rank Abelian subalgebra in the commutant of the Hamiltonian.

If we are to compare the two approaches, polarizations seem to be well tailored to integrable systems and give us a method to pass naturally from one integrable system to another. The price to pay is that this is a very restrictive case and

that in practice we have to solve our initial integrable system in order to obtain another invariant polarization.

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