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LAGUERRE'S FUNCTION OF DIRECTION IN A GENERALIZED WEYL HYPERSURFACE

GÜLER GÜRPINAR ARSAN and GÜLÇIN ÇIVI

Faculty of Science and Letters, Istanbul Technical University
80626 Maslak-Istanbul, Turkey

Abstract. In [1], the generalization of Laguerre's function of direction for a surface in ordinary space to a hypersurface of a Riemannian space is obtained. The Laguerre's function of direction for a hypersurface of a Weyl space has been derived in [2]. In this paper, the generalization of Laguerre's function of direction to a hypersurface of generalized Weyl space is made.

1. Introduction

An n-dimensional differentiable manifold W_n is said to be a Weyl space if it has a symmetric conformal metric tensor g_{ij} and a symmetric connection ∇ satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0\,, \tag{1.1}$$

where T_k are the components of a covariant vector field and ∇_k denotes the usual covariant derivative.

Let Γ^i_{jk} denote the coefficients of the connection ∇. Then, from the compatibility condition given by (1.1) we get

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - \left(\delta^i_j T_k + \delta^i_k T_j - g^{li} g_{jk} T_l\right). \tag{1.2}$$

Under a renormalization of the fundamental tensor of the form $\tilde{g}_{ij} = \lambda^2 g_{ij}$ an object A admitting a transformation of the form $\tilde{A} = \lambda^p A$ is called a **satellite with weight** $\{p\}$ of the metric tensor g_{ij}.

The **prolonged covariant derivative** of the satellite A relative to ∇, denoted by $\dot{\nabla}A$ is defined by [3]

$$\dot{\nabla}_k A = \nabla_k A - pT_k A\,. \tag{1.3}$$

An n-dimensional differentiable manifold having an asymmetric connection ∇^* and asymmetric conformal metric tensor g^*_{ij} preserved by ∇^* is called a generalized Weyl space [4]. Such a **generalized Weyl space** will be denoted by GW_n.

In local coordinates, we then have

$$\nabla^*_k g^*_{ij} - 2T^*_k g^*_{ij} = 0\,, \tag{1.4}$$

where T^*_k are the components of a covariant vector field called the complementary vector field of the generalized Weyl space.

The prolonged covariant derivative of the satellite A, with weight $\{p\}$, relative to ∇^* is defined as

$$\dot{\nabla}^*_k A = \nabla^*_k A - pT^*_k A\,, \tag{1.5}$$

where ∇^*_k denotes the usual covariant derivative.

Assume that g^*_{ij} is broken up into the sum of its symmetric and anti-symmetric parts $g^*_{(ij)}$ and $g^*_{[ij]}$, respectively, so that we have

$$g^*_{ij} = g^*_{(ij)} + g^*_{[ij]}\,. \tag{1.6}$$

Let us consider the generalized Weyl space GW_n having the same complementary vector field T as that of the Weyl space W_n having the symmetric part of g^*_{ij} as its metric tensor. The Weyl space W_n is called the associate space to the generalized Weyl space GW_n [5].

The coefficients L^i_{jk} of the connection ∇^* are obtained from the compatibility condition as [6]

$$L^i_{jk} = \Gamma^i_{jk} + \frac{1}{2}\left[\Omega^h_{kl} g^*_{(jh)} + \Omega^h_{jl} g^*_{(hk)} + \Omega^h_{jk} g^*_{(hl)}\right] g^{*(li)} \tag{1.7}$$

or, putting

$$Q^i_{jk} = \frac{1}{2}\left[\Omega^h_{kl} g^*_{(jh)} + \Omega^h_{jl} g^*_{(hk)} + \Omega^h_{jk} g^*_{(hl)}\right] g^{*(li)} \tag{1.8}$$

we have

$$L^i_{jk} = \Gamma^i_{jk} + Q^i_{jk}\,, \tag{1.9}$$

where $\Omega^i_{jk} = L^i_{jk} - L^i_{kj}$ are the components of the **torsion tensor** of the connection ∇^*.

2. Frenet Formulas in a Generalized Weyl Space

Let $\mathbf{t}$ be the tangent vector field, normalized by the condition $g^*_{(ij)}t^i t^j = 1$, to the curve C: $x^i = x^i(s)$ in the associate Weyl space W_n of the generalized Weyl space GW_n and let s be the arclength of C measured from a fixed point on C.

The prolonged derivatives of $\mathbf{t}$ along C, relative to ∇ and ∇^* denoted, respectively, by $\frac{\dot{\delta} t}{\delta s}$ and $\frac{\dot{\delta}^* t}{\delta s}$ are given by

$$\frac{\dot{\delta} t^i}{\delta s} = t^j \dot{\nabla}_j t^i \,, \qquad \frac{\dot{\delta}^* t^i}{\delta s} = t^j \dot{\nabla}^*_j t^i \,, \tag{2.1}$$

$$t^h = \frac{\mathrm{d}x^h}{\mathrm{d}s} \,.$$

Frenet formulae for W_n can be written as [3],

$$\frac{\dot{\delta} t^i_r}{\delta s} = \underset{r+1}{\kappa}\, t^i_{r+1} - \underset{r}{\kappa}\, t^i_{r-1} \,, \tag{2.2}$$

$$t^i_0 = t^i \,, \quad \underset{0}{\kappa} = \underset{n}{\kappa} = 0 \,,$$

$$r = 0, 1, \ldots, n-1 \,,$$

where $\underset{r}{\kappa}$ is the r**-th curvature** of the curve C.

Similarly, the Frenet formulae for the space GW_n can be written in the form

$$\frac{\dot{\delta}^* t^i_r}{\delta s} = \underset{r+1}{\kappa^*}\, t^i_{r+1} - \underset{r}{\kappa^*}\, t^i_{r-1}, \qquad \underset{0}{\kappa^*} = \underset{n}{\kappa^*} = 0 \tag{2.3}$$

where $\underset{r}{\kappa^*}$ is the r-th curvature of the curve C relative to GW_n.

If v^i is the contravariant components of any vector $\boldsymbol{v}$ in GW_n, by using (1.9) and (2.1), we get

$$\frac{\dot{\delta}^* v^i}{\delta s} = \frac{\dot{\delta} v^i}{\delta s} + Q^i_{jk} v^j t^k \,. \tag{2.4}$$

Replacing v^i in (2.4) by $t_0^i, t_1^i, t_2^i, \dots t_{n-1}^i$ and using (2.1) we obtain respectively

$$\begin{aligned}
\frac{\dot{\delta}^* t_0^i}{\delta s} &= \underset{1}{\kappa} t_1^i + Q_{jk}^i t^j t^k \\
\frac{\dot{\delta}^* t_1^i}{\delta s} &= (\underset{2}{\kappa} t_2^i - \underset{1}{\kappa} t^i) + Q_{jk}^i t_1^j t^k \\
\frac{\dot{\delta}^* t_2^i}{\delta s} &= (\underset{3}{\kappa} t_3^i - \underset{2}{\kappa} t_1^i) + Q_{jk}^i t_2^j t^k \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{\dot{\delta}^* t_{n-1}^i}{\delta s} &= - \underset{n-1}{\kappa} t_{n-2}^i + Q_{jk}^i t_{n-1}^j t^k \, .
\end{aligned} \tag{2.5}$$

These formulae may be replaced by the single equation

$$\frac{\dot{\delta}^* t_r^i}{\delta s} = (\underset{r+1}{\kappa} t_{r+1}^i - \underset{r}{\kappa} t_{r-1}^i) + Q_{jk}^i t_r^j t^k \, . \tag{2.6}$$

Let us find the relationship between the curvatures $\underset{n}{\kappa}$ and $\underset{n}{\kappa}^*$ of the curve C relative to W_n and GW_n.

Since the vectors $t_0, t_1, \dots, t_{n-1}$ are mutually orthogonal

$$g_{(ij)}^* t_p^i t_q^j = \delta_q^p \qquad i,j = 1,2,\dots,n\,; \quad p,q = 0,1,\dots,n-1\,. \tag{2.7}$$

Multiplying (2.2) by $g_{(ij)}^* t_{r-1}^j$ and summing over i and j we find

$$\underset{r}{\kappa} = -g_{(ij)}^* \Big(\frac{\dot{\delta} t_r^i}{\delta s} \Big) t_{r-1}^j \, . \tag{2.8}$$

Using (2.3), (2.6) and (2.8) we obtain

$$\underset{r}{\kappa}^* = \underset{r}{\kappa} - Q_{hjk}\, t_r^h\, t_{r-1}^j\, t^k \, , \tag{2.9}$$

where $g_{(ij)}^* Q_{hk}^i = Q_{hjk}$.

3. Laguerre's Function of Direction in a Generalized Weyl Hypersurface

Let GW_n be a hypersurface with coordinates $u^i (i = 1,2,\dots,n)$ in a generalized Weyl space GW_{n+1} with coordinates $x^a (a = 1,2,\dots,n,n+1)$.

Suppose that the metrics of GW_n and GW_{n+1} are elliptic and that they are given respectively, by $g_{ij}^*\, \mathrm{d}u^i\, \mathrm{d}u^j$ and $g_{ab}^*\, \mathrm{d}x^a\, \mathrm{d}x^b$ which are connected by the relations

$$g_{ij}^* = g_{ab}^* x_i^a x_j^b \, , \qquad i,j = 1,2,\dots,n; \quad a,b = 1,2,\dots,n+1$$

from which it follows that

$$g^*_{(ij)} = g^*_{(ab)} x^a_i x^b_j \,, \qquad g^*_{[ij]} = g^*_{[ab]} x^a_i x^b_j \,,$$

where x^a_i denotes the covariant derivative of x^a with respect to u^i.
Let n^a be the contravariant components of the vector field in GW_{n+1} normal to GW_n and let it be normalized by the condition $g^*_{ab} n^a n^b = 1$. Then, we have

$$g^*_{(ab)} n^a n^b = 1 \,. \tag{3.1}$$

The moving frame $\{x^i_a, n_a\}$ on GW_n reciprocal to the moving frame $\{x^a_i, n^a\}$ is defined by [7]

$$n_a n^a = 1, \quad n_a x^a_i = 0 \,, \quad n^a x^i_a = 0 \,, \quad x^a_i x^j_a = \delta^j_i \,. \tag{3.2}$$

On the other hand, differentiating covarianty x^a_i with respect to u^k, we get

$$\dot{\nabla}^*_k x^a_i = \nabla^*_k x^a_i = A_{ik} n^a + B^j_{ik} x^a_j$$

which yields, with the help of (3.1) and (3.2)

$$A_{ik} = g^*_{(ab)} (\dot{\nabla}^*_k x^a_i) n^b \,, \qquad B^j_{ik} = x^j_a (\dot{\nabla}^*_k x^a_i) \,.$$

The **normal curvature** and the **geodesic torsion** of the curve C in GW_n are respectively,

$$\rho^*_n = A_{(ij)} t^i t^j \tag{3.3}$$

$$\tau^*_g = A_{(ij)} t^i t^j_1 \,. \tag{3.4}$$

If the generalized prolonged derivative of (3.3) in the direction of C is taken and if the fact that the weight of ρ^*_n is $\{-1\}$ is used, we find that

$$\begin{aligned}
\frac{\dot{\delta}^* \rho^*_n}{\delta s} &= t^h \dot{\nabla}^*_h \rho^*_n \\
&= t^h \left(\nabla^*_h \rho^*_n + T_h \rho^*_n \right) \\
&= t^h \left[\nabla^*_h (A_{(ij)} t^i t^j) \right] + t^h T_h \rho^*_n \\
&= t^h (\nabla^*_h A_{(ij)}) t^i t^j + A_{(ij)} t^h (\nabla^*_h t^i) t^j + A_{(ij)} t^h (\nabla^*_h t^j) t^i + t^h T_h \rho^*_n
\end{aligned}$$

and hence

$$\frac{\dot{\delta}^* \rho^*_n}{\delta s} = t^h (\nabla^*_h A_{(ij)}) t^i t^j + 2 A_{(ij)} t^h (\nabla^*_h t^j) t^i + t^h T_h \rho^*_n \,. \tag{3.5}$$

By virtue of (2.1), (2.3) and (3.4) the equation (3.5) reduces to

$$\frac{\dot{\delta}^* \rho_n^*}{\delta s} = t^h (\nabla_h^* A_{(ij)}) t^i t^j + 2\tau_g^* \underset{1}{\kappa}^* + t^h T_h A_{(ij)} t^i t^j \,,$$

or, putting

$$\mathcal{L} = \frac{\dot{\delta}^* \rho_n^*}{\delta s} - 2\tau_g^* \underset{1}{\kappa}^* ,$$

we obtain

$$\mathcal{L} = t^h (\nabla_h^* A_{(ij)}) t^i t^j + t^h T_h A_{(ij)} t^i t^j \tag{3.6}$$

which is the generalized Laguerre's function of direction to a hypersurface in a generalized Weyl space. If, in particular, $T_h = 0$, i. e. if the space is Riemannian, then we obtain the expression for Laguerre's direction function of a Riemannian hypersurface.

Definition. *A curve in a hypersurface will be called a* ***Laguerre line*** *if and only if the Laguerre function of direction along the curve vanishes identically.*

The differential equation of Laguerre lines on a generalized Weyl hypersurface is, by (3.6)

$$\mathcal{L} = \Big[(\nabla_h^* A_{(ij)}) t^i t^j + T_h A_{(ij)} t^i t^j \Big] t^h = 0 \,.$$

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