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GREEN'S FUNCTION, WAVEFUNCTION AND WIGNER FUNCTION OF THE MIC-KEPLER PROBLEM

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Abstract. The phase-space formulation of the nonrelativistic quantum mechanics is constructed on the basis of a deformation of the classical mechanics by the $*$-product. We have taken up the MIC-Kepler problem in which Iwai and Uwano have interpreted its wave-function as the cross section of complex line bundle associated with a principal fibre bundle in the conventional operator formalism. We show that its Green's function, which is derived from the $*$-exponential corresponds to unitary operator through the Weyl application, is equal to the infinite series that consists of its wave-functions. Finally, we obtain its Wigner function.

1. Introduction

We come to the reluctant conclusion that in our previous paper [5] we obtained only a piece of the local expression of the Green's function for the MIC-Kepler problem. There (Theorem 12) we have presented two expressions denoted by $G_+(\boldsymbol{r}_f,\, \boldsymbol{r}_i;\, E)$ and $G_-(\tilde{\boldsymbol{r}}_f,\, \tilde{\boldsymbol{r}}_i;\, E)$ where $\boldsymbol{r} = \tilde{\boldsymbol{r}}$ means the position vector $\boldsymbol{x}$ in $\dot{\mathbb{R}}^3 = \mathbb{R}^3 \backslash \{0\}$ i.e., $\boldsymbol{r} = (x,\, y,\, z)$. However, $G_-(\tilde{\boldsymbol{r}}_f,\, \tilde{\boldsymbol{r}}_i;\, E)$ is actually identical with $G_+(\boldsymbol{r}_f,\, \boldsymbol{r}_i;\, E)$ because the transition function is constant (independent of $\boldsymbol{x}$) and therefore, despite the difference in appearance, τ_- is essentially the same local trivialization as τ_+. This is the reason why $G_-(\tilde{\boldsymbol{r}}_f,\, \tilde{\boldsymbol{r}}_i;\, E)$ became equivalent to $G_+(\boldsymbol{r}_f,\, \boldsymbol{r}_i;\, E)$ in the case of iii). After that we have succeeded in obtaining the other piece of the local expression denoted by $G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E)$ via of finding another local trivialization τ_- which is transformed into τ_+ by the transition function of principal S^1 bundle varying with the position (more precisely, the longitudinal angle) of point $\boldsymbol{x}$ (see [4]). We have found, in addition, the wave-function of the MIC-Kepler problem. In this paper, by turning the right-hand system of orthogonal curvilinear local coordinates on U_- into the left-hand one, we obtain the Green's function and wave-function in a new form. In this way we end up with two left-handed coordinate systems bringing the two local trivializations which are transformed into each other by the transition function of the principal S^1 bundle. Thus it becomes possible to obtain its Wigner function on $T^*(U_+ \cap U_-) \subset T^*\dot{\mathbb{R}}^3$. The energy-eigenfunction on the phase space is called Wigner function, and we found that of the MIC-Kepler problem on the reduced phase-space of $T^*\dot{\mathbb{R}}^4$ by solving the $*$-characteristic equations for its energy and angular momentum where $*$ denotes the Moyal product generated from the canonical coordinates bringing the standard symplectic form on $T^*\dot{\mathbb{R}}^4$ (see [5, Theorem 10]). How they could be expressed in each of the local coordinate systems on $T^*\dot{\mathbb{R}}^3$ is an interesting question, which we have succeeded to answer.

The contents of this paper is as follows. In Section 2 we indicate our conclusive results of the Green's function and wave-function for the MIC-Kepler problem, where it comes to be apparent that the Green's function is not a function existing globaly on the configuration space $\dot{\mathbb{R}}^3$ but a cross section in the complex line bundles over $\dot{\mathbb{R}}^3$ which have been introduced by Iwai and Uwano [3]. In Section 3, we express the Hamiltonian system of the MIC-Kepler problem in terms of the spherical coordinates and their conjugate momentums to obtain its Wigner function on the phase space $T^*\dot{\mathbb{R}}^3$ without z-axis.

2. Green's Function and Wavefunction

The MIC-Kepler problem is the reduced Hamiltonian system of the four-dimensional conformal Kepler problem by an S^1 action, if the associated momentum mapping equals to some fixed value μ which stands for the strength of Dirac's monopole field [2]. Then the Green's function of the MIC-Kepler problem is obtained by reducing that one of the conformal Kepler problem which have been already shown in [4] and [5]. Here we have found another kind of local coordinate system especially reconsidering one side of the local trivializations, denoted by τ_-, derived from the open subset U_- in $\dot{\mathbb{R}}^3 = U_+ \cup U_-$ as it is necessary to make it transformable into the other side τ_+ through the transition function of the principal fibre bundle $g_{-+} : U_+ \cap U_- \ni \boldsymbol{x} \longmapsto \mathrm{e}^{\mathrm{i}\phi(\boldsymbol{x})} = \mathrm{e}^{-\mathrm{i}\tilde{\phi}(\boldsymbol{x})} \in S^1$ (see [4]) and it is also necessary to alter the orientation of the orthogonal curvilinear local coordinate system on U_- as that on one U_+ i.e., anti-clockwise. The details of the latest local coordinate system are as follows (see Fig. 1).

Let U_+ be an open subset without negative z-axis such that

$$U_+ = \left\{ \boldsymbol{x}(r,\, \theta,\, \phi) \in \dot{\mathbb{R}}^3 ;\, r > 0\,,\, 0 \le \theta < \pi\,,\, 0 \le \phi < 2\pi \right\}$$

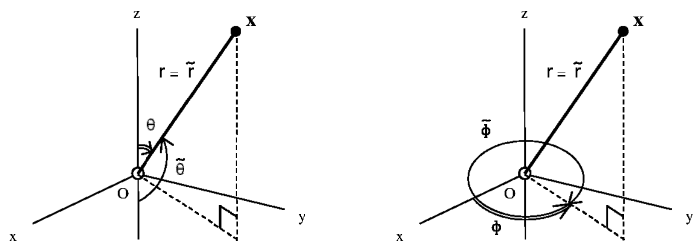

Figure 1. The configuration space $\dot{\mathbb{R}}^3 = \mathbb{R}^3 \backslash \{0\}$.

bringing the following local trivialization where π is the bundle projection and ν has the range of values $0 \le \nu < 4\pi$

$$\tau_+ : \pi^{-1}(U_+) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}),\, \varphi_+(\boldsymbol{u})) = (\boldsymbol{x}(r,\, \theta,\, \phi),\, \exp(\mathrm{i}\nu/2)) \in U_+ \times S^1$$

$$\begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array} \quad \begin{cases} u_1 = \sqrt{r}\cos\dfrac{\theta}{2}\cos\dfrac{\nu+\phi}{2}, & u_2 = \sqrt{r}\cos\dfrac{\theta}{2}\sin\dfrac{\nu+\phi}{2} \\ u_3 = \sqrt{r}\sin\dfrac{\theta}{2}\cos\dfrac{\nu-\phi}{2}, & u_4 = \sqrt{r}\sin\dfrac{\theta}{2}\sin\dfrac{\nu-\phi}{2}. \end{cases}$$

Similarly U_- is an other open subset without positive z-axis such that

$$U_- = \left\{ \boldsymbol{x}(\tilde{r},\, \tilde{\theta},\, \tilde{\phi}) \in \dot{\mathbb{R}}^3 ;\, \tilde{r} > 0\,,\, 0 \le \tilde{\theta} < \pi\,,\, 0 < \tilde{\phi} \le 2\pi \right\}$$

bringing the following local trivialization where $0 \le \tilde{\nu} < 4\pi$

$$\tau_- : \pi^{-1}(U_-) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}),\, \varphi_-(\boldsymbol{u})) = (\boldsymbol{x}(\tilde{r},\, \tilde{\theta},\, \tilde{\phi}),\, \exp(\mathrm{i}\tilde{\nu}/2)) \in U_- \times S^1$$

$$\begin{array}{l} x = \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi} \\ y = -\tilde{r}\sin\tilde{\theta}\sin\tilde{\phi} \\ z = -\tilde{r}\cos\tilde{\theta} \end{array} \quad \begin{cases} u_1 = -\sqrt{\tilde{r}}\sin\dfrac{\tilde{\theta}}{2}\cos\dfrac{\tilde{\nu}+\tilde{\phi}}{2}, & u_2 = -\sqrt{\tilde{r}}\sin\dfrac{\tilde{\theta}}{2}\sin\dfrac{\tilde{\nu}+\tilde{\phi}}{2} \\ u_3 = -\sqrt{\tilde{r}}\cos\dfrac{\tilde{\theta}}{2}\cos\dfrac{\tilde{\nu}+3\tilde{\phi}}{2}, & u_4 = -\sqrt{\tilde{r}}\cos\dfrac{\tilde{\theta}}{2}\sin\dfrac{\tilde{\nu}+3\tilde{\phi}}{2}. \end{cases}$$

We suppose that the real parameter $E < 0$ denoting the energy of the MIC-Kepler problem do not coinside with some of its eigenvalues as given in [6] and [5, Theorem 10]

$$E \ne E_N = -\frac{2mk^2}{\hbar^2\,(N+2)^2}, \qquad N = 0, 1, 2, \ldots$$

The positive constant m is the mass of the electron. The charge of the electron equals $-e$ where $e > 0$ is the charge of the proton called an 'elementary charge', and k is the positive constant defined as

$$k \equiv \frac{e^2}{4\pi\varepsilon}$$

where $\varepsilon > 0$ means electric permittivity. The positive parameter $\hbar$ is defined as $\hbar \equiv h/2\pi$ where h is Planck's constant. Then we reduce the Green's function of the conformal Kepler problem denoted by $G(\boldsymbol{u}_f,\, \boldsymbol{u}_i;\, 4k)$ to the Green's functions of the MIC-Kepler problem on U_+ and U_- denoted by $G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E)$ and $G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E)$ respectively. The subscripts i and f say that the points with each of them denote initial and final points of motion in a proper configuration space respectively. The final results are in the following proposition where l is an arbitrary integer quantizing μ by the relation of $\mu = l\hbar/2$, and $J_l(v)$ denote the **Bessel functions**.

Proposition 1. i) *When $\boldsymbol{x}_i,\, \boldsymbol{x}_f \in U_+$, the Green's function of the* MIC-Kepler *problem is*

$$\begin{aligned}
G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E &= -m\omega^2/8) \\
&= r_f \lim_{\chi \to 4\pi - 0} \int_0^{\chi} G(\boldsymbol{u}_f,\, \boldsymbol{u}_i;\, 4k)\, \exp\left(\mathrm{i}\, l\, \frac{\nu_i - \nu_f}{2}\right) \mathrm{d}\nu_i \\
&= -\frac{\mathrm{i}^{l+1}\, m^2\omega^2}{16\pi\hbar^3} \lim_{y' \to +0} \int_0^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(4k - \mathrm{i}y')(t + \mathrm{i}y')} \mathrm{cosec}^2(\omega t + \mathrm{i}\omega y') \\
&\quad \times \exp\left[-\mathrm{i}\, \frac{m\omega}{2\hbar}(r_i + r_f)\, \cot\,(\omega t + \mathrm{i}\omega y') - \mathrm{i}\, l \cdot \frac{\Theta}{2}\right] \\
&\quad \times J_l\left(\frac{m\omega}{2\hbar}\sqrt{2\boldsymbol{x}_i \cdot \boldsymbol{x}_f + 2 r_i r_f}\; \mathrm{cosec}\,(\omega t + \mathrm{i}\omega y')\right)\, \mathrm{d}t
\end{aligned}$$

where $$\frac{\Theta}{2} \equiv \tan^{-1}\left[\frac{\sin\dfrac{\phi_i - \phi_f}{2}}{\cos\dfrac{\phi_i - \phi_f}{2}} \cdot \frac{\cos\dfrac{\theta_i + \theta_f}{2}}{\cos\dfrac{\theta_i - \theta_f}{2}}\right].$$

ii) *When $\boldsymbol{x}_i\,,\, \boldsymbol{x}_f \in U_-$, the Green's function is*

$$\begin{aligned}
G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E &= -m\omega^2/8) \\
&= \tilde{r}_f \lim_{\chi \to 4\pi - 0} \int_0^{\chi} G(\boldsymbol{u}_f,\, \boldsymbol{u}_i;\, 4k)\, \exp\left(\mathrm{i}\, l\, \frac{\tilde{\nu}_i - \tilde{\nu}_f}{2}\right) \mathrm{d}\tilde{\nu}_i \\
&= -\frac{\mathrm{i}^{l+1}\, m^2\omega^2}{16\pi\hbar^3} \lim_{y' \to +0} \int_0^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(4k - \mathrm{i}y')(t + \mathrm{i}y')} \mathrm{cosec}^2(\omega t + \mathrm{i}\omega y') \\
&\quad \times \exp\left[-\mathrm{i}\, \frac{m\omega}{2\hbar}(\tilde{r}_i + \tilde{r}_f)\, \cot\,(\omega t + \mathrm{i}\omega y') - \mathrm{i}\, l \cdot \frac{\tilde{\Theta}}{2}\right] \\
&\quad \times J_l\left(\frac{m\omega}{2\hbar}\sqrt{2\boldsymbol{x}_i \cdot \boldsymbol{x}_f + 2 \tilde{r}_i \tilde{r}_f}\; \mathrm{cosec}\,(\omega t + \mathrm{i}\omega y')\right)\, \mathrm{d}t
\end{aligned}$$

where

$$\frac{\tilde{\Theta}}{2} \equiv \tan^{-1}\left[\frac{\sin\dfrac{\tilde{\phi}_i - \tilde{\phi}_f}{2}}{\cos\dfrac{\tilde{\phi}_i - \tilde{\phi}_f}{2}} \cdot \frac{2\cos(\tilde{\phi}_i - \tilde{\phi}_f)\cos\dfrac{\tilde{\theta}_i}{2}\cos\dfrac{\tilde{\theta}_f}{2} + \cos\dfrac{\tilde{\theta}_i - \tilde{\theta}_f}{2}}{2\cos(\tilde{\phi}_i - \tilde{\phi}_f)\cos\dfrac{\tilde{\theta}_i}{2}\cos\dfrac{\tilde{\theta}_f}{2} - \cos\dfrac{\tilde{\theta}_i + \tilde{\theta}_f}{2}}\right].$$

iii) *When* $\boldsymbol{x}_i,\ \boldsymbol{x}_f \in U_+ \cap U_-$, $\theta = \pi - \tilde{\theta}$, $\phi = 2\pi - \tilde{\phi}$ *and* $\tilde{\Theta}$ *is given by*

$$\frac{\tilde{\Theta}}{2} = \tan^{-1}\left[-\frac{\sin\dfrac{\phi_i - \phi_f}{2}}{\cos\dfrac{\phi_i - \phi_f}{2}} \cdot \frac{2\cos(\phi_i - \phi_f)\sin\dfrac{\theta_i}{2}\sin\dfrac{\theta_f}{2} + \cos\dfrac{\theta_i - \theta_f}{2}}{2\cos(\phi_i - \phi_f)\sin\dfrac{\theta_i}{2}\sin\dfrac{\theta_f}{2} + \cos\dfrac{\theta_i + \theta_f}{2}}\right].$$

As it can be seen, the Green's functions on $U_+ \cap U_-$ are not equivalent with each other since the difference between G_+ and G_- is the one between Θ and $\tilde{\Theta}$ because of the equality $r = \tilde{r}$. Besides $\tilde{\Theta}$ is not equal to Θ obviously. Right then, what is the concrete relation between G_+ and G_- on the common part? An answer to this question is the following proposition established by similar methods as that in the earlier paper [4, §5].

Proposition 2. i) *When* $\boldsymbol{x} \in U_+$, *the wave function of the* MIC-Kepler *problem is*

$$\Psi_N^+(\boldsymbol{x}) = \frac{m\omega}{2\sqrt{\pi}\hbar}\left(\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathcal{P}\left(r\cos^2\frac{\theta}{2},\ r\sin^2\frac{\theta}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}}\ \exp\left(-\frac{m\omega}{2\hbar}r\right)$$
$$\left(\sqrt{r}\cos\frac{\theta}{2}\right)^{k_1+k_3}\left(\sqrt{r}\sin\frac{\theta}{2}\right)^{k_2+k_4}\exp\left[-\mathrm{i}\,(k_1 - k_2 - k_3 + k_4)\frac{\phi}{2}\right]$$

and if $\boldsymbol{x}_i,\ \boldsymbol{x}_f \in U_+$, *the Green's function of the* MIC-Kepler *problem is also written by*

$$G_+(\boldsymbol{x}_f,\ \boldsymbol{x}_i;\ E) = \sum_{N=0}^{\infty}\frac{1}{4k - (N+2)\hbar\omega}\ \Psi_N^+(\boldsymbol{x}_f)\,\overline{\Psi_N^+(\boldsymbol{x}_i)}$$

where $\omega \equiv \sqrt{-8E/m}$, *the infinite sum of* N *includes the finite sum of all terms whose non-negative integers* $(k_1,\ k_2,\ k_3,\ k_4)$ *that satisfy the following conditions for a fixed* $N \in \mathbb{N} \cup \{0\}$

$$k_1 + k_2 + k_3 + k_4 = N, \qquad k_1 + k_2 - k_3 - k_4 = -l, \qquad \mathbb{Z} \ni l = 2\mu/\hbar$$

and $\mathcal{P}(X,\ Y)$ *is the polynomial*

$$\mathcal{P}(X,\ Y) = \sum_{j=0}^{k_1}\sum_{s=0}^{k_2} j!s!\left(-\frac{\hbar}{m\omega}\right)^{j+s}\ {}_{k_1}C_j \cdot {}_{k_3}C_j \cdot {}_{k_2}C_s \cdot {}_{k_4}C_s\, X^{-j}\,Y^{-s}.$$

ii) *When* $\boldsymbol{x} \in U_-$, *the wave function of the* MIC-Kepler *problem is*

$$\Psi_N^-(\boldsymbol{x}) = \frac{m\omega}{2\sqrt{\pi}\hbar}\left(-\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathcal{P}\left(\tilde{r}\sin^2\frac{\tilde{\theta}}{2},\, \tilde{r}\cos^2\frac{\tilde{\theta}}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} \exp\left(-\frac{m\omega}{2\hbar}\tilde{r}\right)$$
$$\left(\sqrt{\tilde{r}}\sin\frac{\tilde{\theta}}{2}\right)^{k_1+k_3}\left(\sqrt{\tilde{r}}\cos\frac{\tilde{\theta}}{2}\right)^{k_2+k_4} \exp\left[-\mathrm{i}\,(k_1+3k_2-k_3-3k_4)\frac{\tilde{\phi}}{2}\right]$$

and for $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_-$, *the Green's function of the* MIC-Kepler *problem is*

$$G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E) = \sum_{N=0}^{\infty} \frac{1}{4k-(N+2)\hbar\omega}\, \Psi_N^-(\boldsymbol{x}_f)\, \bar{\Psi}_N^-(\boldsymbol{x}_i).$$

iii) *When* $\boldsymbol{x} \in U_+ \cap U_-$, *the relation between* Ψ_N^+ *and* Ψ_N^- *is*

$$\Psi_N^-(\boldsymbol{x}) = \Psi_N^+(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\,l\,\phi}$$

and if $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_+ \cap U_-$, *the relation between* G_+ *and* G_- *is*

$$G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E) = G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i;\, E)\, \mathrm{e}^{-\mathrm{i}\,l\,(\phi_f-\phi_i)}\,.$$

3. Wigner Function

When the z-axis is excluded from the configuration space $\dot{\mathbb{R}}^3$ the Hamiltonian system of the **conformal Kepler problem** $(T^*(\pi^{-1}(U_+ \cap U_-)),\, \mathrm{d}\vartheta,\, H)$ where ϑ denotes the canonical one-form of $T^*\dot{\mathbb{R}}^4 \supset T^*(\pi^{-1}(U_+ \cap U_-))$ is described in the above-mentioned local trivializations τ_+ and τ_- (see also §2) as follows[1]

$$H(r,\, \theta,\, \phi,\, \nu,\, \rho_r\,,\, \rho_\theta\,,\, \rho_\phi\,, \rho_\nu) = \frac{1}{2m}\left(\rho_r^2 + \frac{\rho_\theta^2}{r^2} + \frac{\rho_\phi^2 + \rho_\nu^2 - 2\rho_\phi\rho_\nu\cos\theta}{r^2\sin^2\theta}\right) - \frac{k}{r}$$

where ρ_r, ρ_θ, ρ_ϕ and ρ_ν are the conjugate momentums of r, θ, ϕ and ν respectively such that $\vartheta \equiv \rho_1\mathrm{d}u_1 + \rho_2\mathrm{d}u_2 + \rho_3\mathrm{d}u_3 + \rho_4\mathrm{d}u_4 = \rho_r\mathrm{d}r + \rho_\theta\mathrm{d}\theta + \rho_\phi\mathrm{d}\phi + \rho_\nu\mathrm{d}\nu$, or

$$H(\tilde{r},\, \tilde{\theta},\, \tilde{\phi},\, \tilde{\nu},\, \rho_{\tilde{r}}\,,\, \rho_{\tilde{\theta}}\,,\, \rho_{\tilde{\phi}}\,,\, \rho_{\tilde{\nu}})$$
$$= \frac{1}{2m}\left(\rho_{\tilde{r}}^2 + \frac{\rho_{\tilde{\theta}}^2}{\tilde{r}^2} + \frac{\rho_{\tilde{\phi}}^2 + \rho_{\tilde{\nu}}^2(4\cos\tilde{\theta}+5) - 2\rho_{\tilde{\phi}}\rho_{\tilde{\nu}}(\cos\tilde{\theta}+2)}{\tilde{r}^2\sin^2\tilde{\theta}}\right) - \frac{k}{\tilde{r}}$$

[1]The forms written by the trivialization τ_+ equal the ones given by Iwai [1, §.3].

where $\rho_{\tilde r}$, $\rho_{\tilde\theta}$, $\rho_{\tilde\phi}$ and $\rho_{\tilde\nu}$ are the conjugate momentums of $\tilde r$, $\tilde\theta$, $\tilde\phi$ and $\tilde\nu$ respectively such that $\vartheta \equiv \rho_1 \mathrm{d}u_1 + \rho_2 \mathrm{d}u_2 + \rho_3 \mathrm{d}u_3 + \rho_4 \mathrm{d}u_4 = \rho_{\tilde r}\mathrm{d}\tilde r + \rho_{\tilde\theta}\mathrm{d}\tilde\theta + \rho_{\tilde\phi}\mathrm{d}\tilde\phi + \rho_{\tilde\nu}\mathrm{d}\tilde\nu$. Moreover verifying the following equalities

$$\begin{aligned} &\tilde r = r, && \tilde\theta + \theta = \pi, && \tilde\phi + \phi = 2\pi, && \tilde\nu - \nu = 2\phi \\ &\rho_{\tilde r} = \rho_r, && \rho_{\tilde\theta} = -\rho_\theta, && \rho_{\tilde\phi} = 2\rho_\nu - \rho_\phi, && \rho_{\tilde\nu} = \rho_\nu \end{aligned}$$

we have the following equivalence on the common part $T^*(\pi^{-1}(U_+ \cap U_-)) \subset T^*\dot{\mathbb{R}}^4$

$$H(r,\, \theta,\, \phi,\, \nu,\, \rho_r\,,\, \rho_\theta\,,\, \rho_\phi\,, \rho_\nu) = H(\tilde r,\, \tilde\theta,\, \tilde\phi,\, \tilde\nu,\, \rho_{\tilde r}\,,\, \rho_{\tilde\theta}\,,\, \rho_{\tilde\phi}\,,\, \rho_{\tilde\nu}).$$

Additionally $\rho_{\tilde\nu} = \rho_\nu$ coincides with the associated momentum mapping $\psi(\boldsymbol{u}\,,\, \boldsymbol{\rho})$

$$\rho_{\tilde\nu} = \rho_\nu = \frac{1}{2}(-u_2\rho_1 + u_1\rho_2 - u_4\rho_3 + u_3\rho_4) = \psi(\boldsymbol{u}\,,\, \boldsymbol{\rho})$$

and therefore restricting the Hamiltonian system $(T^*(\pi^{-1}(U_+ \cap U_-)),\, \mathrm{d}\vartheta,\, H)$ on the subset $\psi^{-1}(\mu) \subset T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4$

$$\psi^{-1}(\mu) = \Big\{(\boldsymbol{u}\,,\, \boldsymbol{\rho}) \in T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4;\, \psi(\boldsymbol{u}\,,\, \boldsymbol{\rho}) = \mu\Big\}$$

is easily done by setting each of the conjugate momentum ρ_ν or $\rho_{\tilde\nu}$ to the fixed value μ. Further, according to Iwai & Uwano, $T^*\dot{\mathbb{R}}^3$ is diffeomorphic with the quotient space $\psi^{-1}(\mu)/S^1$ (see [2, Lemma 2.4]) i.e., $\pi_\mu^*\sigma_\mu = \iota_\mu^*\mathrm{d}\vartheta$ and $\pi_\mu^* H_\mu = \iota_\mu^* H$ where $\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$ is the inclusion and $\pi_\mu : \psi^{-1}(\mu) \ni (\boldsymbol{u}\,,\, \boldsymbol{\rho}) \to (\boldsymbol{x}\,,\, \boldsymbol{p}) \in T^*\dot{\mathbb{R}}^3$ is the map which provides a principal S^1 bundle such that $\boldsymbol{x} = \pi(\boldsymbol{u})$ and

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ -\mu/r \end{pmatrix} = \frac{1}{4r}\begin{pmatrix} 2u_3 & 2u_4 & 2u_1 & 2u_2 \\ -2u_4 & 2u_3 & 2u_2 & -2u_1 \\ 2u_1 & 2u_2 & -2u_3 & -2u_4 \\ 2u_2 & -2u_1 & 2u_4 & -2u_3 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix}.$$

Then, we have the following equations

$$\begin{aligned} &\rho_r = p_r, && \rho_\theta = p_\theta, && \rho_\phi = p_\phi + \mu\cos\theta \\ &\rho_{\tilde r} = p_{\tilde r}, && \rho_{\tilde\theta} = p_{\tilde\theta}, && \rho_{\tilde\phi} = p_{\tilde\phi} + \mu\cos\tilde\theta + 2\mu \end{aligned}$$

where p_r, p_θ, p_ϕ, $p_{\tilde r}$, $p_{\tilde\theta}$ and $p_{\tilde\phi}$ are the conjugate momentums of r, θ, ϕ, $\tilde r$, $\tilde\theta$ and $\tilde\phi$ defined by

$$p_x\mathrm{d}x + p_y\mathrm{d}y + p_z\mathrm{d}z \;=\; p_r\mathrm{d}r + p_\theta\mathrm{d}\theta + p_\phi\mathrm{d}\phi \;=\; p_{\tilde r}\mathrm{d}\tilde r + p_{\tilde\theta}\mathrm{d}\tilde\theta + p_{\tilde\phi}\mathrm{d}\tilde\phi\,.$$

In this way the reduced Hamiltonian system $(T^*(U_+ \cap U_-), \sigma_\mu, H_\mu)$ that is referred from now on as the **MIC-Kepler problem** is

$$\begin{cases} \sigma_\mu = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}p_\theta \wedge \mathrm{d}\theta + \mathrm{d}(p_\phi + \mu \cos\theta) \wedge \mathrm{d}\phi \\ H_\mu(r,\, \theta,\, \phi,\, p_r\,,\, p_\theta\,,\, p_\phi) = \dfrac{1}{2m}\left(p_r^2 + \dfrac{p_\theta^2}{r^2} + \dfrac{p_\phi^2}{r^2 \sin^2\theta}\right) - \dfrac{k}{r} + \dfrac{\mu^2}{2mr^2} \end{cases}$$

or

$$\begin{cases} \sigma_\mu = \mathrm{d}p_{\tilde r} \wedge \mathrm{d}\tilde r + \mathrm{d}p_{\tilde\theta} \wedge \mathrm{d}\tilde\theta + \mathrm{d}(p_{\tilde\phi} + \mu \cos\tilde\theta + 2\mu) \wedge \mathrm{d}\tilde\phi \\ H_\mu(\tilde r,\, \tilde\theta,\, \tilde\phi,\, p_{\tilde r}\,,\, p_{\tilde\theta}\,,\, p_{\tilde\phi}) = \dfrac{1}{2m}\left(p_{\tilde r}^2 + \dfrac{p_{\tilde\theta}^2}{\tilde r^2} + \dfrac{p_{\tilde\phi}^2}{\tilde r^2 \sin^2\tilde\theta}\right) - \dfrac{k}{\tilde r} + \dfrac{\mu^2}{2m\tilde r^2}. \end{cases}$$

Moreover provided that the following equalities

$$p_{\tilde r} = p_r, \qquad p_{\tilde\theta} = -p_\theta, \qquad p_{\tilde\phi} = -p_\phi$$

are satisfied we have the following equivalence on the common part $T^*(U_+ \cap U_-) \subset T^*\dot{\mathbb{R}}^3$.

$$H_\mu(r,\, \theta,\, \phi,\, p_r\,,\, p_\theta\,,\, p_\phi) = H_\mu(\tilde r,\, \tilde\theta,\, \tilde\phi,\, p_{\tilde r}\,,\, p_{\tilde\theta}\,,\, p_{\tilde\phi})$$

Similarly, the Wigner function of the MIC-Kepler problem (see [5, Theorem 10]) can be rewritten as a function on $T^*(U_+ \cap U_-)$ which is almost the phase space $T^*\dot{\mathbb{R}}^3$. Finally if N is an arbitrary non-negative integer called 'principal quantum number', l is an arbitrary integer quantizing μ by the relation of $\mu = l\hbar/2$ and $L_n(X)$ denotes the **Laguerre polynomial** of degree n, i.e.,

$$L_n(X) = \sum_{\alpha=0}^{n} (-1)^\alpha \frac{n!}{(\alpha!)^2 (n-\alpha)!}\, X^\alpha$$

$$\sum_{n=0}^{\infty} L_n(X)\, \xi^n = \frac{1}{1-\xi} \exp\left(-\frac{\xi}{1-\xi}\, X\right)$$

we can formulate the following

Proposition 3. *Suppose that the point* $\boldsymbol{x} \in \mathbb{R}^3$ *is not on the* z*-axis. Then the Wigner function of the* MIC-Kepler *problem is given as follows*

$$\begin{aligned} f_N(r,\, \theta,\, \phi,\, p_r,\, p_\theta,\, p_\phi) &= \frac{(-1)^N}{(\pi\hbar)^4}\, \mathrm{e}^{-2(N+2)} \\ &\times L_{n_a}\left(\frac{N+2}{2mk}\left[\mathcal{A}^2 + \frac{\mathcal{C}^2}{r(1+\cos\theta)}\right]\right) L_{n_b}\left(\frac{N+2}{2mk}\left[\mathcal{A}^2 + \frac{\mathcal{D}^2}{r(1+\cos\theta)}\right]\right) \\ &\times L_{n_c}\left(\frac{N+2}{2mk}\left[\mathcal{B}^2 + \frac{\mathcal{E}^2}{r(1-\cos\theta)}\right]\right) L_{n_d}\left(\frac{N+2}{2mk}\left[\mathcal{B}^2 + \frac{\mathcal{F}^2}{r(1-\cos\theta)}\right]\right) \end{aligned}$$

or

$$
\begin{aligned}
f_N(\tilde{r},\,\tilde{\theta},\,\tilde{\phi},\,p_{\tilde{r}},\,p_{\tilde{\theta}},\,p_{\tilde{\phi}}) &= \frac{(-1)^N}{(\pi\hbar)^4}\,\mathrm{e}^{-2(N+2)} \\
&\times L_{n_a}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{A}}^2+\frac{\tilde{\mathcal{C}}^2}{\tilde{r}(1-\cos\tilde{\theta})}\right]\right) L_{n_b}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{A}}^2+\frac{\tilde{\mathcal{D}}^2}{\tilde{r}(1-\cos\tilde{\theta})}\right]\right) \\
&\times L_{n_c}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{B}}^2+\frac{\tilde{\mathcal{E}}^2}{\tilde{r}(1+\cos\tilde{\theta})}\right]\right) L_{n_d}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{B}}^2+\frac{\tilde{\mathcal{F}}^2}{\tilde{r}(1+\cos\tilde{\theta})}\right]\right)
\end{aligned}
$$

where n_a, n_b, n_c *and* n_d *are non-negative integers such that*

$$
\begin{cases} 2(n_a+n_d) = N+l \\ 2(n_b+n_c) = N-l \end{cases} \quad \textit{i.e.,} \quad \begin{cases} |\,l\,| \leq N \\ N \textit{ and } l \textit{ are simultaneously even or odd} \end{cases}
$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$, $\tilde{\mathcal{E}}$ *and* $\tilde{\mathcal{F}}$ *are given by the formulas*

$$
\begin{aligned}
\mathcal{A}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) &= p_r\sqrt{r(1+\cos\theta)} - p_\theta\sqrt{\frac{1-\cos\theta}{r}} \\
\mathcal{B}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) &= p_r\sqrt{r(1-\cos\theta)} + p_\theta\sqrt{\frac{1+\cos\theta}{r}} \\
\mathcal{C}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) &= p_\phi + r(1+\cos\theta)\left(\frac{2mk}{\hbar(N+2)}+\frac{\mu}{r}\right) \\
\mathcal{D}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) &= p_\phi - r(1+\cos\theta)\left(\frac{2mk}{\hbar(N+2)}-\frac{\mu}{r}\right) \\
\mathcal{E}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) &= p_\phi + r(1-\cos\theta)\left(\frac{2mk}{\hbar(N+2)}-\frac{\mu}{r}\right) \\
\mathcal{F}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) &= p_\phi - r(1-\cos\theta)\left(\frac{2mk}{\hbar(N+2)}+\frac{\mu}{r}\right) \\
\tilde{\mathcal{A}}(\tilde{r},\,\tilde{\theta},\,\tilde{\phi},\,p_{\tilde{r}},\,p_{\tilde{\theta}},\,p_{\tilde{\phi}}) &= p_{\tilde{r}}\sqrt{\tilde{r}(1-\cos\tilde{\theta})} + p_{\tilde{\theta}}\sqrt{\frac{1+\cos\tilde{\theta}}{\tilde{r}}} \\
\tilde{\mathcal{B}}(\tilde{r},\,\tilde{\theta},\,\tilde{\phi},\,p_{\tilde{r}},\,p_{\tilde{\theta}},\,p_{\tilde{\phi}}) &= p_{\tilde{r}}\sqrt{\tilde{r}(1+\cos\tilde{\theta})} - p_{\tilde{\theta}}\sqrt{\frac{1-\cos\tilde{\theta}}{\tilde{r}}}
\end{aligned}
$$

$$\tilde{\mathcal{C}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} - \tilde{r}(1 - \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} + \frac{\mu}{\tilde{r}}\right)$$

$$\tilde{\mathcal{D}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} + \tilde{r}(1 - \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} - \frac{\mu}{\tilde{r}}\right)$$

$$\tilde{\mathcal{E}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} - \tilde{r}(1 + \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} - \frac{\mu}{\tilde{r}}\right)$$

$$\tilde{\mathcal{F}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} + \tilde{r}(1 + \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} + \frac{\mu}{\tilde{r}}\right).$$

Furthermore we have the equalities

$$\tilde{\mathcal{A}} = \mathcal{A}, \quad \tilde{\mathcal{B}} = \mathcal{B}, \quad \tilde{\mathcal{C}} = -\mathcal{C}, \quad \tilde{\mathcal{D}} = -\mathcal{D}, \quad \tilde{\mathcal{E}} = -\mathcal{E}, \quad \tilde{\mathcal{F}} = -\mathcal{F}$$

leading to the following equivalence

$$f_N(r, \theta, \phi, p_r, p_\theta, p_\phi) = f_N(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}).$$

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