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A NOTE ON POISSON LIE ALGEBROIDS*

LIVIU POPESCU

Department of Applied Mathematics in Economy, University of Craiova
13, A. I. Cuza Str., 200585 Craiova, Romania

Abstract. In this paper we study some properties of a Lie algebroid and its prolongation over the vector bundle projection of the dual bundle. We generalize some results on Poisson manifolds to the level of a Lie algebroid. The notions of canonical Poisson bivector and horizontal lift are studied and their compatibility conditions are pointed out.

1. Introduction

The Lie algebroid [10] is a generalization of both concepts of Lie algebra and integrable distribution, being a vector bundle (E, π, M) with a Lie bracket on his space of sections with properties very similar to those of a tangent bundle. The Poisson manifolds are the smooth manifolds equipped with a Poisson bracket on their ring of functions. I have to remark that the cotangent bundle of a Poisson manifold has the natural structure of a Lie algebroid [13]. In the last years diverse aspects of these subjects have been studied in a lot of papers (see for instance [13], [14], [12], [1] and [7]). In the present paper we study some geometrical structures on the prolongation of a Lie algebroid to its dual bundle and investigate some aspects of the Lie algebroid geometry endowed with a Poisson structure. In this way we generalize some results on Poisson manifolds.

The paper is organized as follows. In the Section 2 we recall the Cartan calculus and the Schouten-Nijenhuis bracket at the level of a Lie algebroid and present the Poisson structures on the Lie algebroid. The Section 3 deals with the prolongation of a Lie algebroid [5], [8] to its dual bundle and continue the investigation starting in [6]. We study the properties of the canonical Poisson bivector and introduce the notion of horizontal lift. Finally, the compatibility conditions of these bivectors

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are given. We remark that in the particular case of the standard Lie algebroid ($E = TM$, $\sigma = \mathrm{Id}$) some results of Mitric [12] are obtained.

2. Preliminaries on Lie Algebroids

Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. A Lie algebroid over the manifold M is the triple $(E, [\cdot,\cdot], \sigma)$ where $\pi : E \to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]$ and $\sigma : E \to TM$ is a vector bundle map (called *the anchor*) which induces a Lie algebra homomorphism (also denoted σ) from $\Gamma(E)$ to $\chi(M)$, satisfying the Leibniz rule

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2$$

for every $f \in C^\infty(M)$ and s_1, $s_2 \in \Gamma(E)$. Therefore, we have

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \qquad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.$$

If $\omega \in \bigwedge^k(E^*)$ then the **exterior derivative** $\mathrm{d}^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned}\mathrm{d}^E\omega(s_1, \dots, s_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, \dots, \hat{s}_i, \dots, s_{k+1})\\ &+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([s_i, s_j], s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{k+1})\end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it follows that $(\mathrm{d}^E)^2 = 0$. Also, for $\xi \in \Gamma(E)$ one can define the **Lie derivative** with respect to ξ by

$$\mathcal{L}_\xi = i_\xi \circ \mathrm{d}^E + \mathrm{d}^E \circ i_\xi$$

where i_ξ is the contraction with ξ.

If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U) \to U$ generates the local coordinates (x^i, y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M defined by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_{\alpha\beta} s_\gamma, \quad i = 1, \dots, n, \quad \alpha, \beta, \gamma = 1, \dots, m$$

are called the **structure functions** of the Lie algebroid and satisfy the so called **structure equations** on the Lie algebroid

$$\sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \qquad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma}\right) = 0. \quad (1)$$

Locally, if $f \in C^\infty(M)$ then $\mathrm{d}^E f = \frac{\partial f}{\partial x^i}\sigma^i_\alpha s^\alpha$, where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$ and, if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$\mathrm{d}^E\theta = (\sigma^i_\alpha \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2}\theta_\gamma L^\gamma_{\alpha\beta}) s^\alpha \wedge s^\beta.$$

Particularly, we get

$$\mathrm{d}^E x^i = \sigma^i_\alpha s^\alpha, \qquad \mathrm{d}^E s^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma.$$

The Schouten-Nijenhuis bracket is given by [13]

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q]$$
$$= (-1)^{p+1}\sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{i+j}[X_i, Y_j]\wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_q$$

where $X_i, Y_j \in \Gamma(E)$ and a hat means the absence of a factor.

2.1. Lie Algebroids with Poisson Structure

Let us consider the bivector on E (i.e., contravariant, skew-symmetric, 2-section) $W \in \Gamma(\wedge^2 E)$ given by

$$W = \frac{1}{2} w^{\alpha\beta}(x) s_\alpha \wedge s_\beta. \tag{2}$$

Definition 1. *The bivector W is a Poisson bivector on E if and only if we have the relation $[W, W] = 0$, where $[\cdot, \cdot]$ is Schouten-Nijenhuis bracket.*

Proposition 2. *The relation $[W, W] = 0$ implies locally that*

$$\sum_{(\alpha,\varepsilon,\delta)} \left(w^{\alpha\beta}\sigma^i_\beta \frac{\partial w^{\varepsilon\delta}}{\partial x^i} + w^{\alpha\beta} w^{\gamma\delta} L^\varepsilon_{\beta\gamma} \right) = 0. \tag{3}$$

If W is a Poisson bivector then the pair (E, W) is called a *Lie algebroid with Poisson structure*. The Poisson bracket on E is given by

$$\{f_1, f_2\} = W(\mathrm{d}^E f_1, \mathrm{d}^E f_2), \qquad f_1, f_2 \in C^\infty(E).$$

We have the bundle map $\pi^\# : E^* \to E$ defined by

$$\pi^\# \rho = i_\rho W, \qquad \rho \in \Gamma(E^*).$$

Let us consider the bracket

$$[\rho, \theta]_\pi = \mathcal{L}_{\pi^\#\rho}\theta - \mathcal{L}_{\pi^\#\theta}\rho - \mathrm{d}^E(W(\rho, \theta))$$

where $\mathcal{L}$ is the Lie derivative and $\rho, \theta \in \Gamma(E^*)$. With respect to this bracket and the usual Lie bracket on vector fields, the map $\widetilde{\sigma} : E^* \to TM$ given by

$$\widetilde{\sigma} = \sigma \circ \pi^\#$$

is a Lie algebra homomorphism

$$\widetilde{\sigma}[\rho,\theta]_\pi = [\widetilde{\sigma}\rho,\widetilde{\sigma}\theta].$$

The bracket $[\cdot,\cdot]_\pi$ satisfies also the Leibnitz rule

$$[\rho, f\theta]_\pi = f[\rho,\theta]_\pi + \widetilde{\sigma}(\rho)(f)\theta$$

and it results that $(E^*,[\cdot,\cdot]_\pi,\widetilde{\sigma})$ is a Lie algebroid [14].
Next, we can define the contravariant exterior differential $\mathrm{d}^\pi\colon \bigwedge^k(E^*)\rightarrow \bigwedge^{k+1}(E^*)$ by

$$\begin{aligned}\mathrm{d}^\pi\omega(s_1,\ldots,s_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\widetilde{\sigma}(s_i)\omega(s_1,\ldots,\hat{s}_i,\ldots,s_{k+1})\\ &+ \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\omega([s_i,s_j]_\pi,s_1,\ldots,\hat{s}_i,\ldots,\hat{s}_j,\ldots,s_{k+1}).\end{aligned}$$

In fact, is obtained the cohomology of the Lie algebroid E^* with the anchor $\widetilde{\sigma}$ and the bracket $[\cdot,\cdot]_\pi$ which generalize the Poisson cohomology of Lichnerowicz for Poisson manifolds [9].

3. The Prolongation of a Lie Algebroid to Its Dual Bundle

Let $\tau : E^* \rightarrow M$ be the dual of $\pi : E \rightarrow M$ and $(E,[\cdot,\cdot],\sigma)$ a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation of $(E,[\cdot,\cdot],\sigma)$ over $\tau : E^* \rightarrow M$ (see [5], [8], [11] and [6]). This structure is given by the following objects:

- The associated vector bundle is $(\mathcal{T}E^*,\tau_1,E^*)$ where $\mathcal{T}E^* = \cup_{u^*\in E^*}\mathcal{T}_{u^*}E^*$ with

$$\mathcal{T}_{u^*}E^* = \{(u_x,v_{u^*})\in E_x\times T_{u^*}E^* | \sigma(u_x) = T_{u^*}\tau(v_{u^*}),\tau\left(u^*\right) = x\in M\}$$

  and the projection $\tau_1 : \mathcal{T}E^* \rightarrow E^*$, $\tau_1(u_x,v_{u^*}) = u^*$.
- The Lie algebra structure $[\cdot,\cdot]$ on $\Gamma(\mathcal{T}E^*)$ is defined in the following way: if $\rho_1,\rho_2 \in \Gamma(\mathcal{T}E^*)$ are such that $\rho_i(u^*) = (X_i(\tau(u^*)),U_i(u^*))$ where $X_i\in\Gamma(E), U_i\in\chi(E^*)$ and $\sigma(X_i(\tau(u^*)) = T_{u^*}\tau(U_i\left(u^*\right)), i=1,2$, then

$$[\rho_1,\rho_2](u^*) = ([X_1,X_2](\tau(u^*)),[U_1,U_2](u^*)).$$

- The anchor is the projection $\sigma^1 : \mathcal{T}E^* \rightarrow TE^*$, $\sigma^1(u,v) = v$.

Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \rightarrow E$, $\mathcal{T}\tau(u,v) = u$ then $(V\mathcal{T}E^*,\tau_{1|V\mathcal{T}E^*},E^*)$ with $V\mathcal{T}E^* := \ker\mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*,\tau_1,E^*)$, called the **vertical subbundle**.

If (x^i, μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \to M$ then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$ where

$$\mathcal{X}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma_\alpha^i \left.\frac{\partial}{\partial x^i}\right|_{u^*}\right), \qquad \mathcal{P}^\alpha(u^*) = \left(0, \left.\frac{\partial}{\partial \mu_\alpha}\right|_{u^*}\right). \tag{4}$$

The Lie brackets on the elements of this basis are:

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L_{\alpha\beta}^\gamma \mathcal{X}_\gamma, \qquad [\mathcal{X}_\alpha, \mathcal{P}^\alpha] = 0, \qquad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0 \tag{5}$$

and

$$\sigma^1(\mathcal{X}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \qquad \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha}$$

$$\mathrm{d}^E x^i = \sigma_\alpha^i \mathcal{X}^\alpha, \qquad \mathrm{d}^E \mu_\alpha = \mathcal{P}_\alpha, \quad \mathrm{d}^E \mathcal{X}^\gamma = -\frac{1}{2} L_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \qquad \mathrm{d}^E \mathcal{P}_\alpha = 0$$

where $\{\mathcal{X}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$. Also, if $\rho = \rho^\alpha \mathcal{X}_\alpha + \zeta_\alpha \mathcal{P}^\alpha$ is a section of $\mathcal{T}E^*$, then

$$\sigma^1(\rho) = \sigma_\alpha^i \rho^\alpha \frac{\partial}{\partial x^i} + \zeta_\alpha \frac{\partial}{\partial \mu_\alpha}.$$

If $u^* \in E^*$ and $(u_x, v_{u^*}) \in E_x \times T_{u^*}E^*$ then

$$\theta_E(u^*)(u_x, v_{u^*}) = u^*(u_x)$$

is called the **Liouville section**. The *canonical symplectic section* ω_E is defined by

$$\omega_E = -\mathrm{d}^E \theta_E$$

and it results that this is a nondegenerate two form and $\mathrm{d}^E \omega_E = 0$.
In the local coordinates it follows that the Liouville section is given by

$$\theta_E = \mu_\alpha \mathcal{X}^\alpha$$

and we obtain

$$\omega_E = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma. \tag{6}$$

We remark that $V\mathcal{T}E^*$ is Lagrangian for ω_E, i.e., $\omega_E(\rho_1, \rho_2) = 0$, for every vertical sections $\rho_1, \rho_2 \in \Gamma(V\mathcal{T}E^*)$.

Definition 3. *The Ehresmann nonlinear connection on $\mathcal{T}E^*$ is an almost product structure $\mathcal{N}$ on $\tau_1 : \mathcal{T}E^* \to E^*$ (i.e., a bundle morphism $\mathcal{N} : \mathcal{T}E^* \to \mathcal{T}E^*$, such that $\mathcal{N}^2 = \mathrm{Id}$) smooth on $\mathcal{T}E^* \setminus \{0\}$ such that $V\mathcal{T}E^* = \ker(\mathrm{Id} + \mathcal{N})$.*

If $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^* = \ker(\mathrm{Id} - \mathcal{N})$ is the horizontal distribution associated to $\mathcal{N}$ and

$$\mathcal{T}E^* = V\mathcal{T}E^* \oplus H\mathcal{T}E^*.$$

Each $\rho \in \Gamma(\mathcal{T}E^*)$ can be written as $\rho = \rho^h + \rho^v$ where ρ^h, ρ^v are sections in the horizontal and respective, vertical subbundles. A connection $\mathcal{N}$ on $\mathcal{T}E^*$ induces two projectors $h, v : \mathcal{T}E^* \to \mathcal{T}E^*$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in \Gamma(\mathcal{T}E^*)$. We have

$$h = \frac{1}{2}(\mathrm{Id} + \mathcal{N}), \qquad v = \frac{1}{2}(\mathrm{Id} - \mathcal{N})$$
$$\ker h = \operatorname{im} v = V\mathcal{T}E^*, \qquad \operatorname{im} h = \ker v = H\mathcal{T}E^*$$
$$h^2 = h, \qquad v^2 = v, \qquad hv = vh = 0, \qquad h + v = \mathrm{Id}\,.$$

Locally, a nonlinear connection is expressed as $\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + 2\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta$ and $\mathcal{N}(\mathcal{P}^\alpha) = -\mathcal{P}^\alpha$, where $\mathcal{N}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(x, \mu)$ are the local coefficients of $\mathcal{N}$. The local sections $\mathcal{P}^\alpha$, $\alpha = 1, \ldots, m$ define a local frame of $V\mathcal{T}E^*$, and the sections

$$\delta^*_\alpha = (\mathcal{X}_\alpha)^h = \mathcal{X}_\alpha + \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta \tag{7}$$

generate a local frame of $H\mathcal{T}E^*$. The frame $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *adapted* to the direct sum decomposition. The respective dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{P}_\alpha\}$ where

$$\delta\mathcal{P}_\alpha = \mathcal{P}_\alpha - \mathcal{N}_{\alpha\beta}\mathcal{X}^\beta. \tag{8}$$

Definition 4. *A connection $\mathcal{N}$ is called symmetric if $H\mathcal{T}E^*$ is Lagrangian for ω_E.*

By a straightforward computation, using (6) and (7) we get

$$\omega_E(\delta^*_\alpha, \delta^*_\beta) = \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} - \mu_\gamma L^\gamma_{\alpha\beta} \tag{9}$$

and it result that $\mathcal{N}$ is symmetric if and only if

$$\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}.$$

Proposition 5. *With respect to a symmetric nonlinear connection, the canonical symplectic structure ω_E can be written in the following form*

$$\omega_E = \mathcal{X}^\alpha \wedge \delta\mathcal{P}_\alpha + \mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma.$$

Proof: Using (6) and (8) we get

$$\omega_E = \mathcal{X}^\alpha \wedge \delta\mathcal{P}_\alpha + \frac{1}{2}(\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha})\mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma$$

which ends the proof. □

Proposition 6. *The Lie brackets of the adapted basis $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ are*

$$[\delta^*_\alpha, \delta^*_\beta] = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma, \qquad [\delta^*_\alpha, \mathcal{P}^\beta] = -\frac{\partial\mathcal{N}_{\alpha\gamma}}{\partial\mu_\beta}\mathcal{P}^\gamma, \qquad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0$$

where

$$\mathcal{R}_{\alpha\beta\gamma} = \delta^*_\alpha(\mathcal{N}_{\beta\gamma}) - \delta^*_\beta(\mathcal{N}_{\alpha\gamma}) - L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}. \tag{10}$$

Proof: Using (7) we obtain

$$[\delta^*_\alpha, \delta^*_\beta] = \left(\sigma^i_\alpha \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial x^i} - \sigma^i_\beta \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial x^i} + \mathcal{N}_{\alpha\delta}\frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\delta} - \mathcal{N}_{\beta\delta}\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\delta}\right)\mathcal{P}^\gamma + L^\varepsilon_{\alpha\beta}\mathcal{X}_\varepsilon$$

and putting $\mathcal{X}_\varepsilon = \delta^*_\varepsilon - \mathcal{N}_{\varepsilon\gamma}\mathcal{P}^\gamma$ we get $[\delta^*_\alpha, \delta^*_\beta] = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma$. □

The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E^*$ is given by $\Omega = -\mathrm{N}_h$ where h is horizontal projector and N_h is the Nijenhuis tensor of h, given by

$$\mathrm{N}_h(\theta, \rho) = [h\theta, h\rho] - h[h\theta, \rho] - h[\theta, h\rho] + h^2[\theta, \rho].$$

Remark 7. *In the local coordinates we get*

$$\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta\gamma}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma$$

where $\mathcal{R}_{\alpha\beta\gamma}$ *is given by* (10) *and is called the* curvature tensor *of* $\mathcal{N}$.

Proof: Since $h^2 = h$ we obtain

$$\Omega(h\rho_1, h\rho_2) = -v[h\rho_1, h\rho_2], \qquad \Omega(h\rho_1, v\rho_2) = \Omega(v\rho_1, v\rho_2) = 0$$

and in local coordinates we get

$$\Omega(\delta^*_\alpha, \delta^*_\beta) = -v[\delta^*_\alpha, \delta^*_\beta] = -\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma$$

which concludes the proof. □

Remark 8. *The curvature satisfies the Bianchi identity*

$$\mathcal{R}_{\alpha\beta\gamma} + \mathcal{R}_{\beta\gamma\alpha} + \mathcal{R}_{\gamma\alpha\beta} = 0.$$

Proof: By direct computation, using relation (10) and structure equations given by (1). □

The curvature is an obstruction to the integrability of $H\mathcal{T}E^*$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E^*$. We have

Remark 9. $H\mathcal{T}E^*$ *is integrable if and only if the curvature vanishes.*

The integrability conditions for the almost product structure $\mathcal{N}$ is given by the vanishing of the associated Nijenhuis tensor $\mathrm{N}_\mathcal{N}$. By a straightforward computation we obtain

$$\mathrm{N}_\mathcal{N}(\mathcal{P}^\alpha, \mathcal{P}^\beta) = 0, \qquad \mathrm{N}_\mathcal{N}(\delta^*_\alpha, \mathcal{P}^\beta) = 0, \qquad \mathrm{N}_\mathcal{N}(\delta^*_\alpha, \delta^*_\beta) = 4\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma.$$

Thus

$$\mathrm{N}_\mathcal{N} = -2\mathcal{R}_{\alpha\beta\gamma}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma$$

and it results that the distribution $H\mathcal{T}E^*$ is integrable if and only if the almost product structure $\mathcal{N}$ is integrable.

3.1. Canonical Poisson Structure

On the Lie algebroid $(\mathcal{T}E^*, [\,,], \sigma^1)$ we have the canonical symplectic section ω_E given by (6) which induces a vector bundle isomorphism

$$\natural_{\omega_E} : E^* \to E, \qquad i_\zeta \omega_E \in E^* \to \zeta \in E.$$

Definition 10. *The canonical Poisson bivector is given by*

$$\Lambda = \natural_{\omega_E} \omega_E.$$

It follows that

$$\Lambda(\mathrm{d}F, \mathrm{d}G) = -\omega_E(\natural(\mathrm{d}F), \natural(\mathrm{d}G)), \qquad F, G \in C^\infty(E^*)$$

and in local coordinates we get

$$\Lambda = \mathcal{P}^\alpha \wedge \mathcal{X}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma} \mathcal{P}^\beta \wedge \mathcal{P}^\gamma.$$

Remark 11. *The Schouten-Nijenhuis bracket* $[\Lambda, \Lambda]$ *leads, locally, to the expression*

$$\frac{1}{3} \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\varepsilon_{\beta\gamma}}{\partial x^i} + L^\varepsilon_{\alpha\delta} L^\delta_{\beta\gamma} \right) \mu_\varepsilon \mathcal{P}^\beta \wedge \mathcal{P}^\alpha \wedge \mathcal{P}^\gamma$$

and $[\Lambda, \Lambda] = 0$ *follows from the structure equations on the Lie algebroid* (1).

Definition 12. *Let us consider a Poisson bivector on* E *given by* (2)*, then the horizontal lift of* W *to* $\mathcal{T}E^*$ *is the bivector defined by*

$$W^H = \frac{1}{2} w^{\alpha\beta}(x) \delta^*_\alpha \wedge \delta^*_\beta.$$

Proposition 13. *The horizontal lift* W^H *is a Poisson bivector if and only if* W *is a Poisson bivector on* E *and*

$$w^{\alpha\beta} w^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} = 0.$$

Proof: The Poisson condition $[W, W] = 0$ leads to the relation

$$\sum_{(\alpha,\varepsilon,\delta)} \left(w^{\alpha\beta} w^{\gamma\delta} L^\varepsilon_{\beta\gamma} + w^{\alpha\beta} \sigma^i_\beta \frac{\partial w^{\varepsilon\delta}}{\partial x^i} \right) = 0$$

and $[W^H, W^H] = 0$ yields

$$\sum_{(\varepsilon,\delta,\alpha)} \left(w^{\alpha\beta} w^{\gamma\delta} L^\varepsilon_{\beta\gamma} + w^{\alpha\beta} \sigma^i_\beta \frac{\partial w^{\varepsilon\delta}}{\partial x^i} \right) \delta^*_\varepsilon \wedge \delta^*_\alpha \wedge \delta^*_\delta + w^{\alpha\beta} w^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} \mathcal{P}^\varepsilon \wedge \delta^*_\alpha \wedge \delta^*_\gamma = 0$$

which ends the proof. □

Recall that two Poisson structures are said to be *compatible* if the bivectors w_1 and w_2 satisfy the condition

$$[w_1, w_2] = 0.$$

Proposition 14. *If W^H is a Poisson bivector and $\mathcal{N}$ is a symmetric nonlinear connection, then W^H is compatible with the canonical Poisson structure Λ if and only if the following relations fulfilled*

$$\sigma^i_\gamma \frac{\partial \omega^{\alpha\beta}}{\partial x^i} + \omega^{\varepsilon\alpha}\left(\frac{\partial N_{\varepsilon\gamma}}{\partial \mu_\beta} - L^\beta_{\varepsilon\gamma}\right) - \omega^{\varepsilon\beta}\left(\frac{\partial N_{\varepsilon\gamma}}{\partial \mu_\alpha} - L^\alpha_{\varepsilon\gamma}\right) = 0 \tag{11}$$

$$\omega^{\varepsilon\alpha}\mathcal{R}_{\alpha\gamma\delta} = 0. \tag{12}$$

Proof: If $\mathcal{N}$ is symmetric then $\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}$ and with respect with the basis $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ it results

$$\Lambda = \mathcal{P}^\alpha \wedge \delta^*_\alpha.$$

By a straightforward computation we obtain that the relation $[W^H, \Lambda] = 0$ is equivalent with relations (11) and (12). □

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