Tenth International Conference on
Geometry, Integrability and Quantization
June 6–11, 2008, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2009, pp 171–182

ON THE GEOMETRY OF BIHARMONIC SUBMANIFOLDS IN SASAKIAN SPACE FORMS*

DOREL FETCU and CEZAR ONICIUC†

Department of Mathematics, "Gh. Asachi" Technical University of Iasi
Bd. Carol I no. 11, 700506 Iasi, Romania

†*Faculty of Mathematics, "Al.I. Cuza" University of Iasi*
Bd. Carol I no. 11, 700506 Iasi, Romania

Abstract. We classify all proper-biharmonic Legendre curves in a Sasakian space form and point out some of their geometric properties. Then we provide a method for constructing anti-invariant proper-biharmonic submanifolds in Sasakian space forms. Finally, using the Boothby-Wang fibration, we determine all proper-biharmonic Hopf cylinders over homogeneous real hypersurfaces in complex projective spaces.

1. Introduction

As defined by Eells and Sampson in [14], **harmonic maps** $f : (M, g) \to (N, h)$ are the critical points of the **energy functional**

$$E(f) = \frac{1}{2}\int_M \|\mathrm{d}f\|^2 \, v_g$$

and they are solutions of the associated Euler-Lagrange equation

$$\tau(f) = \operatorname{trace}_g \nabla \mathrm{d}f = 0$$

where $\tau(f)$ is called the **tension field** of f. When f is an isometric immersion with mean curvature vector field H, then $\tau(f) = mH$ and f is harmonic if and only if it is minimal.

The **bienergy functional** (proposed also by Eells and Sampson in 1964, [14]) is defined by

$$E_2(f) = \frac{1}{2}\int_M \|\tau(f)\|^2 \, v_g.$$

*Reprinted from JGSP **14** (2009) 21–34.

The critical points of E_2 are called **biharmonic maps** and they are solutions of the Euler-Lagrange equation (derived by Jiang in 1986, [20]):

$$\tau_2(f) = -\Delta^f \tau(f) - \operatorname{trace}_g R^N(\mathrm{d}f, \tau(f))\mathrm{d}f = 0$$

where Δ^f is the Laplacian on sections of $f^{-1}TN$ and $R^N(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is the curvature operator on N; $\tau_2(f)$ is called the **bitension field** of f. Since all harmonic maps are biharmonic, we are interested in studying those which are biharmonic but non-harmonic, called *proper-biharmonic* maps.

Now, if $f : M \to N_c$ is an isometric immersion into a space form of constant sectional curvature c, then

$$\tau(f) = mH \quad \text{and} \quad \tau_2(f) = -m\Delta^f H + cm^2 H.$$

Thus f is biharmonic if and only if

$$\Delta^f H = mcH.$$

In a different way, Chen defined the biharmonic submanifolds in an Euclidean space as those with harmonic mean curvature vector field ([10]). Replacing $c = 0$ in the above equation we just reobtain Chen's definition. Moreover, let $f : M \to \mathbb{R}^n$ be an isometric immersion. Set $f = (f^1, \ldots, f^n)$ and $H = (H^1, \ldots, H^n)$. Then $\Delta^f H = (\Delta H^1, \ldots, \Delta H^n)$, where Δ is the Beltrami-Laplace operator on M, and f is biharmonic if and only if

$$\Delta^f H = \Delta(\frac{-\Delta f}{m}) = -\frac{1}{m}\Delta^2 f = 0.$$

There are several classification results for the proper-biharmonic submanifolds in Euclidean spheres and non-existence results for such submanifolds in the space forms manifolds N_c, $c \leq 0$ ([4, 5, 7–10, 13]), while in spaces of non-constant sectional curvature only a few results were obtained ([1,12,18,19,25,29]).

We recall that the proper-biharmonic curves of the unit Euclidean two-dimensional sphere $\mathbb{S}^2$ are the circles of radius $\frac{1}{\sqrt{2}}$, and the proper-biharmonic curves of $\mathbb{S}^3$ are the geodesics of the minimal Clifford torus $\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right)$ with the slope different from ± 1. The proper-biharmonic curves of $\mathbb{S}^3$ are helices. Further, the proper-biharmonic curves of $\mathbb{S}^n$, $n > 3$, are those of $\mathbb{S}^3$ (up to a totally geodesic embedding). Concerning the hypersurfaces of $\mathbb{S}^n$, it was conjectured in [4] that the only proper-biharmonic hypersurfaces are the open parts of $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right)$ or $\mathbb{S}^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m_2}\left(\frac{1}{\sqrt{2}}\right)$ with $m_1 + m_2 = n - 1$ and $m_1 \neq m_2$.

Since odd dimensional unit Euclidean spheres $\mathbb{S}^{2n+1}$ are Sasakian space forms with constant φ-sectional curvature one, the next step is to study the biharmonic submanifolds of Sasakian space forms. In this paper we mainly gather the results obtained in [15–17].

We note that the proper-biharmonic submanifolds in pseudo-Riemannian manifolds are also intensively-studied (for example, see [2, 3, 11]).

For a general account of biharmonic maps see [22] and *The Bibliography of Biharmonic Maps* [28].

Conventions. We work in the C^∞ category, that means manifolds, metrics, connections and maps are smooth. The Lie algebra of the vector fields on N is denoted by $C(TN)$.

2. Sasakian Space Forms

In this section we briefly recall some basic facts from the theory of Sasakian manifolds. For more details see [6].

A **contact metric structure** on a manifold N^{2n+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$ on N, ξ is a vector field on N, η is an one-form on N and g is a Riemannian metric, such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \varphi Y) = \mathrm{d}\eta(X, Y)$$

for any $X, Y \in C(TN)$.

A contact metric structure (φ, ξ, η, g) is **Sasakian** if it is *normal*, i.e.,

$$N_\varphi + 2\mathrm{d}\eta \otimes \xi = 0$$

where for all $X, Y \in C(TN)$

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$$

is the Nijenhuis tensor field of φ.

The *contact distribution* of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN;\ \eta(X) = 0\}$, and any integral curve of the contact distribution is called **Legendrian curve**.

A submanifold M of N which is tangent to ξ is said to be *anti-invariant* if φ maps any vector tangent to M and normal to ξ to a vector normal to M.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a two-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called *φ-sectional curvature* determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a *Sasakian space form* and it is denoted by $N(c)$.

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is called *regular* if for any point $p \in N$ there exists a cubic neighborhood of p such that any integral curve of ξ passes through the neighborhood at most once, and *strictly regular* if all integral curves are homeomorphic to each other.

Let (N,φ,ξ,η,g) be a regular contact metric manifold. Then the orbit space $\bar{N}=N/\xi$ has a natural manifold structure and, moreover, if N is compact then N is a principal circle bundle over $\bar{N}$ (the Boothby-Wang Theorem). In this case the fibration $\pi : N \to \bar{N}$ is called **Boothby-Wang fibration**. The Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a well-known example of a Boothby-Wang fibration.

Theorem 1 ([24]). *Let (N,φ,ξ,η,g) be a strictly regular Sasakian manifold. Then on $\bar{N}$ can be given the structure of a Kähler manifold. Moreover, if (N,φ,ξ,η,g) is a Sasakian space form $N(c)$, then $\bar{N}$ has constant sectional holomorphic curvature $c+3$.*

Even if N is non-compact, we still call the fibration $\pi : N \to \bar{N}$ of a strictly regular Sasakian manifold, the Boothby-Wang fibration.

3. Biharmonic Legendre Curves in Sasakian Space Forms

Let (N^n,g) be a Riemannian manifold and $\gamma : I \to N$ a curve parametrized by arc length. Then γ is called a *Frenet curve of osculating order* r, $1\leq r\leq n$, if there exists orthonormal vector fields $E_1,E_2,\ldots,E_r$ along γ such that $E_1=\gamma'=T$, $\nabla_T E_1=\kappa_1 E_2$, $\nabla_T E_2=-\kappa_1 E_1+\kappa_2 E_3$, $\ldots$, $\nabla_T E_r=-\kappa_{r-1}E_{r-1}$, where $\kappa_1,\ldots,\kappa_{r-1}$ are positive functions on I.

A geodesic is a Frenet curve of osculating order one, a *circle* is a Frenet curve of osculating order two with $\kappa_1=\text{const}$, a *helix of order* r, $r\geq 3$, is a Frenet curve of osculating order r with $\kappa_1,\ldots,\kappa_{r-1}$ constants and a helix of order three is called, simply, helix.

In [16] we studied the biharmonicity of Legendre Frenet curves and we obtained the following results.

Let $(N^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. Then γ is biharmonic if and only if

$$\begin{aligned}\tau_2(\gamma) &= \nabla^3_T T - R(T,\nabla_T T)T\\ &= (-3\kappa_1\kappa_1')E_1+\Big(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2+\frac{(c+3)\kappa_1}{4}\Big)E_2\\ &\quad+(2\kappa_1'\kappa_2+\kappa_1\kappa_2')E_3+\kappa_1\kappa_2\kappa_3E_4+\frac{3(c-1)\kappa_1}{4}g(E_2,\varphi T)\varphi T\\ &=0.\end{aligned}$$

The expression of the bitension field $\tau_2(\gamma)$ imposed a case-by-case analysis as follows.

Case I ($c = 1$)

Theorem 2 ([16]). *If $c = 1$ then γ is proper-biharmonic if and only if $n \geq 2$ and either γ is a circle with $\kappa_1 = 1$ or γ is a helix with $\kappa_1^2 + \kappa_2^2 = 1$.*

Case II ($c \neq 1$ and $E_2 \perp \varphi T$)

Theorem 3 ([16]). *Assume that $c \neq 1$ and $E_2 \perp \varphi T$. We have*

1) *if $c \leq -3$ then γ is biharmonic if and only if it is a geodesic;*
2) *if $c > -3$ then γ is proper-biharmonic if and only if either*
   a) *$n \geq 2$ and γ is a circle with $\kappa_1^2 = \frac{c+3}{4}$, or*
   b) *$n \geq 3$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4}$.*

Case III ($c \neq 1$ and $E_2 \parallel \varphi T$)

Theorem 4 ([16]). *If $c \neq 1$ and $E_2 \parallel \varphi T$, then $\{T, \varphi T, \xi\}$ is the Frenet frame field of γ and we have*

1) *if $c < 1$ then γ is biharmonic if and only if it is a geodesic*
2) *if $c > 1$ then γ is proper-biharmonic if and only if it is a helix with $\kappa_1^2 = c - 1$ and $\kappa_2 = 1$.*

Remark 1. In dimension 3 the result was obtained by Inoguchi in [19] and explicit examples are given in [15].

Case IV ($c \neq 1$ and $g(E_2, \varphi T)$ is not constant 0, 1 or -1)

Theorem 5 ([16]). *Let $c \neq 1$ and γ a Legendre Frenet curve of osculating order r such that $g(E_2, \varphi T)$ is not constant 0, 1 or -1. We have*

1) *if $c \leq -3$ then γ is biharmonic if and only if it is a geodesic;*
2) *if $c > -3$ then γ is proper-biharmonic if and only if $r \geq 4$, $\varphi T = \cos \alpha_0 E_2 + \sin \alpha_0 E_4$ and*

$$\begin{aligned} &\kappa_1, \kappa_2, \kappa_3 = \text{const} > 0 \\ &\kappa_1^2 + \kappa_2^2 = \tfrac{c+3}{4} + \tfrac{3(c-1)}{4} \cos^2 \alpha_0 \\ &\kappa_2 \kappa_3 = -\tfrac{3(c-1)}{8} \sin(2\alpha_0) \end{aligned}$$

where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that

$$c + 3 + 3(c-1)\cos^2 \alpha_0 > 0, \qquad 3(c-1)\sin(2\alpha_0) < 0.$$

In order to obtain explicit examples of proper-biharmonic Legendre curves given by Theorem 2 we used the unit Euclidean sphere $\mathbb{S}^{2n+1}$ as a model of a Sasakian space form with $c = 1$ and we proved the following

Theorem 6 ([16]). *Let $\gamma : I \to \mathbb{S}^{2n+1}(1)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the parametric equation of γ in the Euclidean space $\mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, \langle\, , \rangle)$ is either*

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}s\right)e_1 + \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}s\right)e_2 + \frac{1}{\sqrt{2}}e_3$$

where $\{e_i, \mathcal{I}e_j\}$ are constant unit vectors orthogonal to each other, or

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4$$

where $A = \sqrt{1+\kappa_1}$, $B = \sqrt{1-\kappa_1}$, $\kappa_1 \in (0,1)$, $\{e_i\}$ are constant unit vectors orthogonal to each other such that

$$\langle e_1, \mathcal{I}e_3\rangle = \langle e_1, \mathcal{I}e_4\rangle = \langle e_2, \mathcal{I}e_3\rangle = \langle e_2, \mathcal{I}e_4\rangle = 0$$

$$A\langle e_1, \mathcal{I}e_2\rangle + B\langle e_3, \mathcal{I}e_4\rangle = 0$$

and $\mathcal{I}$ is the usual complex structure on $\mathbb{R}^{2n+2}$.

Remark 2. For the Cases II and III we also obtained the explicit equations of proper-biharmonic Legendre curves in odd dimensional spheres endowed with the deformed Sasakian structure introduced in [27].

In [21] are introduced the complex torsions for a Frenet curve in a complex manifold. In the same way, for $\gamma : I \to N$ a Legendre Frenet curve of osculating order r in a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$, we define the φ-torsions $\tau_{ij} = g(E_i, \varphi E_j) = -g(\varphi E_i, E_j)$, $i, j = 1, \ldots, r$, $i < j$.
It is easy to see that we can formulate

Proposition 1. *Let $\gamma : I \to N(c)$ be a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$, $c \neq 1$. Then $c > -3$ and τ_{12} is a constant.*

Moreover

Proposition 2. *If γ is a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$, $c > -3$, $c \neq 1$, of osculating order $r < 4$, then it is a circle or a helix with constant φ-torsions.*

Proof: From Theorems 3, 4 and 5 we see that if γ is a proper-biharmonic Legendre Frenet curve of osculating order $r < 4$, then $\tau_{12} = 0$ or $\tau_{12} = \pm 1$ and, obviously, we only have to prove that when γ is a helix then τ_{13} and τ_{23} are constants.
Indeed, by using the Frenet equations of γ, we have

$$\tau_{13} = g(E_1, \varphi E_3) = -\frac{1}{\kappa_2}g(\varphi E_1, \nabla_{E_1}E_2 + \kappa_1 E_1) = -\frac{1}{\kappa_2}g(\varphi E_1, \nabla_{E_1}E_2)$$

$$= \frac{1}{\kappa_2}g(E_2, \nabla_{E_1}\varphi E_1) = \frac{1}{\kappa_2}g(E_2, \varphi\nabla_{E_1}E_1 + \xi) = 0$$

since

$$g(E_2,\xi)=\frac{1}{\kappa_1}g(\nabla_{E_1}E_1,\xi)=-\frac{1}{\kappa_1}g(E_1,\nabla_{E_1}\xi)=\frac{1}{\kappa_1}g(E_1,\varphi E_1)=0.$$

On the other hand, it is easy to see that for any Frenet curve of osculating order three we have $\tau_{23}=\frac{1}{\kappa_1}(\tau_{13}'+\kappa_2\tau_{12}+\eta(E_3))$ and

$$\eta(E_3)=g(E_3,\xi)=\frac{1}{\kappa_2}(g(\nabla_{E_1}E_2,\xi)+\kappa_1 g(E_1,\xi))=-\frac{1}{\kappa_2}g(E_2,\nabla_{E_1}\xi)$$
$$=-\frac{1}{\kappa_2}\tau_{12}.$$

In conclusion $\tau_{23}=\frac{1}{\kappa_1}(\tau_{13}'+\kappa_2\tau_{12}-\frac{1}{\kappa_2}\tau_{12})=\text{const}.$ □

Proposition 3. *If γ is a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$ of osculating order $r=4$, then $c\in(\frac{7}{3},5)$ and the curvatures of γ are*

$$\kappa_1=\frac{\sqrt{c+3}}{2},\qquad \kappa_2=\frac{1}{2}\sqrt{\frac{6(c-1)(5-c)}{c+3}},\qquad \kappa_3=\frac{1}{2}\sqrt{\frac{3(c-1)(3c-7)}{c+3}}.$$

Moreover, the φ-torsions of γ are given by

$$\tau_{12}=\mp\sqrt{\frac{2(5-c)}{c+3}},\qquad \tau_{13}=0,\qquad \tau_{14}=\pm\sqrt{\frac{3c-7}{c+3}}$$

$$\tau_{23}=\mp\frac{3c-7}{\sqrt{3(c-1)(c+3)}},\qquad \tau_{24}=0,\qquad \tau_{34}=\pm\sqrt{\frac{2(5-c)(3c-7)}{3(c-1)(c+3)}}.$$

Proof: Let γ be a proper-biharmonic Legendre Frenet curve in $N(c)$ of osculating order $r=4$. Then $c\neq 1$ and τ_{12} is different from 0, 1 or -1. From Theorem 5 we have $\varphi E_1=\cos\alpha_0 E_2+\sin\alpha_0 E_4$. It results that

$$\tau_{12}=-\cos\alpha_0,\qquad \tau_{13}=0,\qquad \tau_{14}=-\sin\alpha_0\quad\text{and}\quad \tau_{24}=0.$$

In order to prove that τ_{23} is constant we differentiate the expression of φE_1 along γ and using the Frenet equations we obtain

$$\nabla_{E_1}\varphi E_1=\cos\alpha_0\nabla_{E_1}E_2+\sin\alpha_0\nabla_{E_1}E_4$$
$$=-\kappa_1\cos\alpha_0 E_1+(\kappa_2\cos\alpha_0-\kappa_3\sin\alpha_0)E_3.$$

On the other hand, $\nabla_{E_1}\varphi E_1=\kappa_1\varphi E_2+\xi$ and therefore we have

$$\kappa_1\varphi E_2+\xi=-\kappa_1\cos\alpha_0 E_1+(\kappa_2\cos\alpha_0-\kappa_3\sin\alpha_0)E_3. \tag{1}$$

We take the scalar product in (1) with ξ and obtain

$$(\kappa_2\cos\alpha_0-\kappa_3\sin\alpha_0)\eta(E_3)=1. \tag{2}$$

In the same way as in the proof of Proposition 2 we get

$$\begin{aligned}\eta(E_3) = g(E_3,\xi) &= \frac{1}{\kappa_2}(g(\nabla_{E_1}E_2,\xi)+\kappa_1 g(E_1,\xi))\\ &= -\frac{1}{\kappa_2}g(E_2,\nabla_{E_1}\xi)\\ &= -\frac{1}{\kappa_2}\tau_{12} = \frac{\cos\alpha_0}{\kappa_2}\end{aligned}$$

and then, from (2), $\kappa_2\sin\alpha_0 = -\kappa_3\cos\alpha_0$. Therefore $\alpha_0 \in (\frac{\pi}{2},\pi)\cup(\frac{3\pi}{2},2\pi)$.
Next, from Theorem 5, we have

$$\kappa_1^2 = \frac{c+3}{4}, \qquad \kappa_2^2 = \frac{3(c-1)}{4}\cos^2\alpha_0, \qquad \kappa_3^2 = \frac{3(c-1)}{4}\sin^2\alpha_0$$

and so c must be greater than one.
Now, we take the scalar product in (1) with E_3, φE_2 and φE_4, respectively, and we get

$$\kappa_1\tau_{23} = -(\kappa_2\cos\alpha_0-\kappa_3\sin\alpha_0)+\eta(E_3) = -\frac{\kappa_2}{\cos\alpha_0}+\frac{\cos\alpha_0}{\kappa_2} \tag{3}$$

$$\kappa_1\sin^2\alpha_0 = -(\kappa_2\cos\alpha_0-\kappa_3\sin\alpha_0)\tau_{23} = -\frac{\kappa_2}{\cos\alpha_0}\tau_{23} \tag{4}$$

$$\begin{aligned}0 &= \kappa_1\cos\alpha_0\sin\alpha_0+(\kappa_2\cos\alpha_0-\kappa_3\sin\alpha_0)\tau_{34}\\ &= \kappa_1\cos\alpha_0\sin\alpha_0+\frac{\kappa_2}{\cos\alpha_0}\tau_{34}\end{aligned} \tag{5}$$

and then, equations (3) and (4) lead to $\kappa_1^2\sin^2\alpha_0 = \frac{\kappa_2^2}{\cos^2\alpha_0}-1$. We come to the conclusion that $\sin^2\alpha_0 = \frac{3c-7}{c+3}$, so $c\in\left(\frac{7}{3},5\right)$, and then we obtain the expressions of the curvatures and the φ-torsions. □

Remark 3. The proper-biharmonic Legendre curves given by Theorem 6 (for the case $c=1$) have also constant φ-torsions.

4. A Method to Obtain Biharmonic Submanifolds in a Sasakian Space Form

In [16] we gave a method to obtain proper-biharmonic anti-invariant submanifolds in a Sasakian space form from proper-biharmonic integral submanifolds.

Theorem 7 ([16]). *Let $(N^{2n+1},\varphi,\xi,\eta,g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and let $\boldsymbol{i}: M\to N$ be an r-dimensional integral submanifold of N, $1\le r\le n$. Consider*

$$F:\widetilde{M} = I\times M\to N, \qquad F(t,p) = \phi_t(p) = \phi_p(t)$$

where $I = \mathbb{S}^1$ *or* $I = \mathbb{R}$ *and* $\{\phi_t\}_{t\in I}$ *is the flow of the vector field* ξ. *Then* $F : (\widetilde{M}, \widetilde{g} = \mathrm{d}t^2 + \boldsymbol{i}^* g) \to N$ *is a Riemannian immersion and it is proper-biharmonic if and only if* M *is a proper-biharmonic submanifold of* N.

The previous Theorem provides a classification result for proper-biharmonic surfaces in a Sasakian space form, which are invariant under the flow-action of ξ.

Theorem 8 ([16]). *Let* M^2 *be a surface of* $N^{2n+1}(c)$ *invariant under the flow-action of the characteristic vector field* ξ. *Then* M *is proper-biharmonic if and only if, locally, it is given by* $x(t, s) = \phi_t(\gamma(s))$, *where* γ *is a proper-biharmonic Legendre curve.*

Also, using the standard Sasakian 3-structure on $\mathbb{S}^7$, by iteration, Theorem 7 leads to examples of three-dimensional proper-biharmonic submanifolds of $\mathbb{S}^7$.

5. Biharmonic Hopf Cylinders in a Sasakian Space Form

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold and $\bar{\mathbf{i}} : \bar{M} \to \bar{N}$ a submanifold of $\bar{N}$. Then $M = \pi^{-1}(\bar{M})$ is the Hopf cylinder over $\bar{M}$, where $\pi : N \to \bar{N} = N/\xi$ is the Boothby-Wang fibration.

In [19] the biharmonic Hopf cylinders in a three-dimensional Sasakian space form are classified.

Theorem 9 ([19]). *Let* $S_{\bar{\gamma}}$ *be a Hopf cylinder, where* $\bar{\gamma}$ *is a curve in the orbit space of* $N^3(c)$, *parametrized by arc length. We have*

1) *if* $c \leqslant 1$, *then* $S_{\bar{\gamma}}$ *is biharmonic if and only if it is minimal*;
2) *if* $c > 1$, *then* $S_{\bar{\gamma}}$ *is proper-biharmonic if and only if the curvature* $\bar{\kappa}$ *of* $\bar{\gamma}$ *is constant* $\bar{\kappa}^2 = c - 1$.

In [17] we obtained a geometric characterization of biharmonic Hopf cylinders of any codimension in an arbitrary Sasakian space form. A special case of our result is the case when $\bar{M}$ is a hypersurface.

Proposition 4 ([17]). *If* $\bar{M}$ *is a hypersurface of* $\bar{N}$, *then* $M = \pi^{-1}(\bar{M})$ *is biharmonic if and only if*

$$\Delta^{\perp} H = \left(-\|B\|^2 + \frac{c(n+1) + 3n - 1}{2}\right) H$$

$$2\,\mathrm{trace}\, A_{\nabla^{\perp}_{\cdot} H}(\cdot) + n\, \mathrm{grad}(\|H\|^2) = 0$$

where B, A *and* H *are the second fundamental form of* M *in* N, *the shape operator and the mean curvature vector field, respectively, and* $\nabla^{\perp}$ *and* $\Delta^{\perp}$ *are the normal connection and Laplacian on the normal bundle of* M *in* N.

Proposition 5 ([17]). *If $\bar{M}$ is a hypersurface and $\|\bar{H}\| = \text{const} \neq 0$, then $M = \pi^{-1}(\bar{M})$ is proper-biharmonic if and only if*

$$\|\bar{B}\|^2 = \frac{c(n+1)+3n-5}{2}.$$

Remark 4. From the last result we see that there exist no proper-biharmonic hypersurfaces of constant mean curvature $M = \pi^{-1}(\bar{M})$ in $N(c)$ if $c \leq \frac{5-3n}{n+1}$, which implies that such hypersurfaces do not exist if $c \leq -3$, whatever the dimension of N is.

In [26] Takagi classified all homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n > 1$, and found five types of such hypersurfaces (see also [23]). The first type (with subtypes $A1$ and $A2$) are described in the following.

We shall consider $u \in (0, \frac{\pi}{2})$ and r a positive constant given by $\frac{1}{r^2} = \frac{c+3}{4}$.

Theorem 10 ([26]). *The geodesic spheres $(Type\ A1)$ in complex projective space $\mathbb{C}P^n(c+3)$ have two distinct principal curvatures: $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2n-2$ and $a = \frac{2}{r}\cot 2u$ of multiplicity one.*

Theorem 11 ([26]). *The hypersurfaces of Type $A2$ in complex projective space $\mathbb{C}P^n(c+3)$ have three distinct principal curvatures: $\lambda_1 = -\frac{1}{r}\tan u$ of multiplicity $2p$, $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2q$, and $a = \frac{2}{r}\cot 2u$ of multiplicity one, where $p > 0$, $q > 0$, and $p+q = n-1$.*

We note that if $c = 1$ and $\bar{M}$ is of type $A1$ or $A2$ then $\pi^{-1}(\bar{M}) = \mathbb{S}^1(\cos u) \times \mathbb{S}^{2n-1}(\sin u) \subset \mathbb{S}^{2n+1}$ or $\pi^{-1}(\bar{M}) = \mathbb{S}^{2p+1}(\cos u) \times \mathbb{S}^{2q+1}(\sin u)$, respectively.

By using Takagi's result we classified in [17] the biharmonic Hopf cylinders $M = \pi^{-1}(\bar{M})$ in a Sasakian space form N^{2n+1} over homogeneous real hypersurfaces in $\mathbb{C}P^n$, $n > 1$.

Theorem 12 ([17]). *Let $M = \pi^{-1}(\bar{M})$ be the Hopf cylinder over $\bar{M}$.*

1) *If $\bar{M}$ is of Type $A1$, then M is proper-biharmonic if and only if either*
   a) $c = 1$ *and* $\tan^2 u = 1$, *or*
   b) $c \in \left[\frac{-3n^2+2n+1+8\sqrt{2n-1}}{n^2+2n+5}, +\infty\right) \setminus \{1\}$ *and*

$$\tan^2 u = n + \frac{2c-2}{c+3} \pm \frac{\sqrt{c^2(n^2+2n+5)+2c(3n^2-2n-1)+9n^2-30n+13}}{c+3}.$$

2) *If $\bar{M}$ is of Type $A2$, then M is proper-biharmonic if and only if either*
   a) $c = 1$, $\tan^2 u = 1$ *and* $p \neq q$, *or*

b) $c \in \left[\frac{-3(p-q)^2-4n+4+8\sqrt{(2p+1)(2q+1)}}{(p-q)^2+4n+4}, +\infty\right) \setminus \{1\}$ *and*

$$\tan^2 u = \frac{n}{2p+1} + \frac{2c-2}{(c+3)(2p+1)} \pm \frac{\sqrt{c^2((p-q)^2+4n+4)+2c(3(p-q)^2+4n-4)+9(p-q)^2-12n+4}}{(c+3)(2p+1)}.$$

Theorem 13 ([17]). *There are no proper-biharmonic hypersurfaces* $M = \pi^{-1}(\bar{M})$ *when* $\bar{M}$ *is a hypersurface of Type B, C, D or E in the complex projective space* $\mathbb{CP}^n(c+3)$.

Acknowledgements

The authors were partially supported by the Grant CEEX, ET, 5871/2006 and by the Grant CEEX, ET, 5883/2006, Romania. The first author would like to thank to the organizers, especially to Professor I. Mladenov, for the Conference Grant.

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