On continuous major and minor functions for the Henstock-Kurzweil integral

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Abstract

We give a direct proof that the multidimensional Henstock-Kurzweil integral is equivalent to the Perron integral defined by continuous major and minor functions.

1. INTRODUCTION

It is well-known that the classical Henstock-Kurzweil integral is equivalent to the Perron integral (see for example [4]). However, this equivalence is based on the definition of the Perron integral in which the major and minor functions are not supposed to be continuous. It is unclear whether this Perron integral is equivalent to the one defined by continuous major and minor functions. For the one-dimensional Henstock-Kurzweil integral, this equivalence can be proved by several methods, which are real-line dependent [5, 6, 7]. For the multidimensional case, B. Bongiorno et al. [1] used an indirect method to prove this equivalence. In this note, we shall give a constructive proof of their result.

2. PRELIMINARIES

Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the real line and the positive real line respectively, m a fixed positive integer and $\mathbb{R}^m$ the m-dimensional Euclidean space. Unless otherwise stated, an interval will always be a compact nondegenerate interval of the form $[s,t] = \prod_{i=1}^{m}[s_i,t_i]$ where $s = (s_1, s_2, \ldots, s_m)$. Also, $E = \prod_{i=1}^{m}[a_i,b_i]$ will denote a fixed interval in $\mathbb{R}^m$, and $B(x,\delta)$ denotes an open ball in $\mathbb{R}^m$ with center x and radius δ. A finite collection of intervals whose interiors are disjoint is called a nonoverlapping collection. A partial division $D = \{I\}$ of E is a finite collection of non-overlapping intervals. If, in addition, the union of I from D gives E, we say that D is a division of E. Let $\delta : E \longrightarrow \mathbb{R}^+$

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be given. A partial division $D = \{(I, \xi)\}$ is said to be *δ-fine* if for each $(I, \xi) \in D$ with $\xi \in I$, we have $I \subset B(\xi, \delta(\xi))$.

Unless otherwise stated, all functions will be assumed to be real-valued, and often the same letter is used to denote a function on E as well as its restriction to a set $Z \subset E$. A function $f : E \longrightarrow \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* to a real number A on E if for every $\epsilon > 0$, there exists $\delta : E \longrightarrow \mathbb{R}^+$ such that for any δ-fine division $D = \{(I, \xi)\}$ of E, we have

$$\left|(D) \sum f(\xi)\,|I| - A\right| < \epsilon.$$

We write $A = (HK) \displaystyle\int_E f$. If f is Henstock-Kurzweil integrable on E, then f is also Henstock-Kurzweil integrable on each subinterval I of E. If $F(I) = (HK) \int_I f$ for each subinterval I of E, we say that F is the primitive of f on E. For the definition of Perron integral, see [1, p.322]. If g is Lebesgue integrable on E, we write the Lebesgue integral of g over E as $(L) \displaystyle\int_E g$. It is known that if g is Lebesgue integrable on E, then g is Henstock-Kurzweil integrable there with the same integral value [2]. The words "measure", "measurable" and "almost everywhere" always refer to the m-dimensional Lebesgue measure. If X is measurable, we shall write $|X|$ as the m-dimensional Lebesgue measure of X.

3. MAIN RESULT

We shall begin with some known lemmas and theorems. Lemma 1 is a special case of [3, Lemma 4].

Lemma 1 *Suppose f is Henstock-Kurzweil integrable on E, and f is Lebesgue integrable on some closed subset Y of E. Then given $\epsilon > 0$, there exists $\delta : Y \longrightarrow \mathbb{R}^+$ such that for any δ-fine partial division $D = \{(I, \xi)\}$ with $\xi \in Y$, we have*

$$(D) \sum \left|(L) \int_{I \cap Y} f - (HK) \int_I f\right| < \epsilon.$$

Theorem 2 is a reformulation of [3, Theorem 6].

Theorem 2 *If f is Henstock-Kurzweil integrable on the interval E, then there exists an increasing sequence of closed sets $\{X_k\}$ such that*

(i) $\bigcup\limits_{k=1}^{\infty} X_k = E$*;*

(ii) f is Lebesgue integrable on X_k for each k;

(iii) the sequence $\{(L)\int_{X_k \cap I} f\}$ *converges uniformly over any subinterval I of E as $k \to \infty$ with*

$$\lim_{n\to\infty} (L) \int_{X_n \cap I} f = (HK) \int_I f.$$

Definition 3[1, p.320] An interval function F is said to be $\mathcal{F}$-continuous at $x \in E$ if for any given $\epsilon > 0$, there exists $\delta > 0$ such that $|F(I)| < \epsilon$ whenever $x \in I$ with $I \subset B(x, \delta(x))$.

It is easy to verify the next lemma.

Lemma 4 *Suppose $\{H_k\}$ is a sequence of interval functions such that each H_k is $\mathcal{F}$-continuous at each $x \in E$. If the series $\sum_{k=1}^{\infty} H_k(I)$ converges uniformly to $H(I)$ over any subinterval I of E, then H is $\mathcal{F}$-continuous at each $x \in E$.*

In what follows, the *lower derivative* of an interval function F at a point x is denoted by $\underline{D}F(x)$. Similarly, the upper derivative of F at a point x is denoted by $\overline{D}F(x)$.

We are now ready to give an alternative proof of [1, Theorem 5].

Theorem 5 *Suppose f is Henstock-Kurzweil integrable on E with primitive F. Then for $\epsilon > 0$, there exists a $\mathcal{F}$-continuous major function M and a $\mathcal{F}$-continuous minor function m such that*

(i) $0 \leq F(I) - m(I) < \epsilon$ and $0 \leq M(I) - F(I) < \epsilon$ for each subinterval I of E ;

(ii) $f(x) \leq \underline{D}M(x)$ and $\overline{D}m(x) \leq f(x)$ for each $x \in E$.

Proof. Since f is Henstock-Kurzweil integrable on E, we may choose a sequence of closed sets $\{X_k\}$ satisfying all the conditions in Theorem 2. Let $\epsilon > 0$ be given. We shall first construct a major function M with the required properties.

Put $Y_k = X_k - X_{k-1}$ for $k = 1, 2, \ldots$ with $X_0 = \emptyset$. Then f is Lebesgue integrable on each of the measurable set Y_k. Denoting the primitive of $f\chi_{Y_k}$ by F_k, it follows from [5, p.191] that there exists a $\mathcal{F}$-continuous major function M_k such that

$$0 \leq M_k(I) - F_k(I) < \frac{\epsilon}{2^{k+1}} \tag{1}$$

for each subinterval I of E, and

$$(f\chi_{Y_k})(x) \leq \underline{D}M_k(x) \tag{2}$$

for each $x \in E$.

By Lemma 1, there exists $\delta_k : Y_k \longrightarrow \mathbb{R}^+$ such that for any δ_k-fine partial division $D = \{(I, \xi)\}$ of Y_k, we have

$$(D)\sum |F_k(I) - F(I)| < \frac{\epsilon}{2^{k+1}} \tag{3}$$

Define the interval function V_{δ_k} by

$$V_{\delta_k}(I) = \sup \sum |F_k(J) - F(J)| \tag{4}$$

whenever $I \cap Y_k$ is non-empty, and the supremum is taken over all δ_k-fine partial division $\{(J, \xi)\}$ of $I \cap Y_k$. If $I \cap Y_k$ is empty, then we put $V_{\delta_k}(I) = 0$. It follows from [1, Proposition 1] that each V_{δ_k} is $\mathcal{F}$-continuous at each point x of E. Define an interval function M on E by $M = \sum_{k=1}^{\infty}(M_k + V_{\delta_k})$. In view of Theorem 2, the series $\sum_{k=1}^{\infty} F_k(I)$ converges uniformly for every subinterval I of E, so it follows from (1), (3) and (4) that the series $\sum_{k=1}^{\infty}(M_k(I) + V_{\delta_k}(I))$ converges uniformly to $M(I)$ for every subinterval I of E. An application of Lemma 4 shows that M is $\mathcal{F}$-continuous at each point x of E. By our definition of M, we have

$$
\begin{aligned}
&M(I) - F(I) \\
&= \sum_{k=1}^{\infty} \{M_k(I) + V_{\delta_k}(I) - F_k(I)\} \\
&\geq \sum_{k=1}^{\infty} \{M_k(I) - F_k(I)\} \\
&\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&M(I) - F(I) \\
&= \sum_{k=1}^{\infty} \{M_k(I) + V_{\delta_k}(I) - F_k(I)\} \\
&= \sum_{k=1}^{\infty} \{M_k(I) - F_k(I)\} + \sum_{k=1}^{\infty} V_{\delta_k}(I) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

We see that $0 \leq M(I) - F(I) < \epsilon$ for each subinterval I of E. It remains to show that

$$f(x) \leq \underline{D}M(x)$$

for each $x \in E$. Suppose $x \in E$. Then $x \in Y_k$ for some $k \in \mathbb{Z}^+$. If $x \in I$ with $I \subset B(x, \delta_k(x))$, then we have

$$
\begin{aligned}
&M(I) - F(I) \\
&= \sum_{k=1}^{\infty} (M_k(I) + V_{\delta_k}(I)) - F(I) \\
&= \sum_{k=1}^{\infty} (M_k(I) + V_{\delta_k}(I) - F_k(I)) \\
&\geq M_k(I) + V_{\delta_k}(I) - F_k(I) \\
&\geq M_k(I) - F(I)
\end{aligned}
$$

Consequently, we have

$$\underline{D}M(I) \geq \underline{D}M_k(I) = f\chi_{Y_k}(x) = f(x)$$

for all $x \in E$, proving that M is a required major function with the required properties. Similarly, if m_k is a $\mathcal{F}$-continuous minor function such that

$$0 \leq F_k(I) - m_k(I) < \frac{\epsilon}{2^{k+1}}$$

for each subinterval I of E, and

$$\overline{D}m_k(x) \leq (f\chi_{Y_k})(x)$$

for each $x \in E$, then we can also verify that the minor function $m = \sum_{k=1}^{\infty}(m_k - V_{\delta_k})$ is a minor function with the required properties. The proof is complete.

We remark that the converse of Theorem 5 holds, and M is not additive. Is it possible of finding an additive interval function M that satisfies the conditions of Theorem 5 ?

References

[1] B. Bongiorno, L. Di Piazza and V. Skvortsov, On continuous major and minor functions for the n-dimensional Perron integral *Real Anal. Exchange* **22** (1995/96), no. 1, 318−327.

[2] Lee Peng Yee and Rudolf Vyborny, The integral, An Easy Approach after Kurzweil and Henstock, Australian Mathematical Society Lecture Series 14, Cambridge University Press 2000.

[3] Lee Tuo Yeong, Chew Tuan Seng and Lee Peng Yee, On Henstock integrability in Euclidean spaces, *Real Anal. Exchange* **22** (1996/97), no. 1, 382−389.

[4] K.M. Ostaszewski, Henstock integration in the plane, Memoirs Amer. Math. Soc., Providence, 353, 1986.

[5] S. Saks, Theory of the integral, 2nd edition, New York, 1964.

[6] Lu Shipan, On the construction of major and minor functions, *Journal of Mathem. Study*, 27(1) (1994), 124−126.

[7] V.A. Skvortsov, Continuity of δ -variation and construction of major and minor functions for the Perron integral, *Real Anal. Exchange* **21** (1995), no 1, 270−278.