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WALLMAN-FRINK PROXIMITIES

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Abstract. This is a survey of compactification extension results and problems for a special class of proximities.

The compatible Efremovič proximities on a Tichonov space are ordered by $\delta \leq \rho$ if for all $A, B \subseteq X$, $A\delta B \rightarrow A\rho B$. It is well known that the Smirnov completions of the Efremovič proximities give a one-to-one, order reversing correspondence between the Hausdorff compactifications and the proximities. This merely means that the proximities share the compactification lattice order problem ("find a necessary and sufficient condition on X that the Hausdorff compactifications form a lattice").

While it is not practical to work on the order problems by embedding the Hausdorff compactifications in the larger family of T_1 compactifications, there simply being no end to the latter, the Efremovič proximities generalize nicely to the complete lattice of Ladato proximities. When Gagrat and Naimpally showed that the compatible separated proximities on a space complete to T_1 compactifications, it seemed a solution to the order problem was at last in sight.

Unfortunately, troubles remain. The Gagrat-Naimpally compactifications are among the T_1 compactifications, yes, but which ones are they? Moreover, the corresponding order between the completions breaks down. Even a p-map between proximity spaces will only lift, in general, to a continuous extension from the Gagrat-Naimpally completion of the domain to the "bunch space" of the range — something rich and strange.

What follows is an attempt to solve these problems for the special case of Lodato proximities in the style of Wallman-Frink.

Definition 1. Let X be a T_1 space with at least two points. Let $\mathcal{B}$ be any base for the closed sets such that

(1) $\mathcal{B}$ is a network (i.e. $x \in G$, open, implies there is some $B \in \mathcal{B}$ with $x \in B \subseteq G$), and
(2) $\mathcal{B}$ is a ring of sets (i.e. $\mathcal{B}$ is closed under finite unions and finite intersections).

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Then $\mathcal{B}$ is a *normal* base for the closed sets.

N. b. since x has at least two points, $\emptyset \in \mathcal{B}$.

Definition 2. The *Wallman-Frink* proximity δ_b associated with normal base $\mathcal{B}$ is given by:

$$A \not\delta_b B \equiv \exists F_1, F_2 \in \mathcal{B} \text{ with } A \subseteq F_1, B \subseteq F_2 \text{ and } F_1 \cap F_2 = \emptyset.$$

Hence $A \delta_b B \iff \forall F_1, F_2 \in \mathcal{B}, A \subseteq F_1$ and $B \subseteq F_2 \Rightarrow F_1 \cap F_2 \neq \emptyset$.

Theorem 3. *δ_b is a compatible, separated Lodato proximity on X.*

Proof. Since $\emptyset \in \mathcal{B}$, clearly $\emptyset \not\delta_b A$ for any $A \subseteq X$. It is also clear that $A \not\delta_b B \Rightarrow B \not\delta_b A$, and $A \not\delta_b B \Rightarrow A \cap B = \emptyset$.

Since $\mathcal{B}$ is a ring of sets, if $A \not\delta_b B$ and $A \not\delta_b C$, then $A \not\delta_b (B \cup C)$.

Since $\mathcal{B}$ is a network, $x \neq y \Rightarrow x \not\delta_b Y$, and $x \notin cl(A) \iff x \not\delta_b A$. Thus $A \not\delta_b C$ and $B \subseteq cl(C) \Rightarrow \not\delta_b B$. □

Corollary 4. *δ_b is Efremovič $\iff$ whenever F_1, $F_2 \in \mathcal{B}$ such that $F_1 \cap F_2 = \emptyset$, then there exists some C, $D \in \mathcal{B}$ such that $F_1 \subseteq X \backslash C$, $F_2 \subseteq X \backslash D$, and $(X \backslash C) \cap (X \backslash D) = \emptyset$.*

A *b-filter* is a filterbase $\mathcal{F}$ of sets from $\mathcal{B}$ such that whenever $B \in \mathcal{F}$ and $B \subseteq F \in \mathcal{B}$, then $F \in \mathcal{F}$. A *b-ultrafilter* is a maximal b-filter. By Zorn's Lemma, every b-filter is contained in at least one b-ultrafilter. N.b., for any ultrafilter μ on X, $\mu \cap \mathcal{B}$ is a b-filter.

Definition 5. An ultrafilter μ on X is an *ultrafilter of type b* if $\mu \cap \mathcal{B}$ is a b-ultrafilter.

Lemma 6. *For any normal base $\mathcal{B}$ on T_1-space X and each $x \in X$, the point ultrafilter $\mu_x = \{A \colon x \in A\}$ is an ultrafilter of type b.*

Proof. We need only check that the b-filter $\mu_x \cap \mathcal{B}$ is a maximal among the b-filters. Suppose φ is a b-filter with $\mu_x \cap \mathcal{B} \subseteq \varphi$. Then each $F \in \varphi \subseteq \mathcal{B}$, $F \cap B \neq \emptyset$ for all $B \in \mu_x \cap \mathcal{B}$. Since $\mathcal{B}$ is a network, we see that $x \in cl(F) = F$ for each $F \in \varphi$. Hence $\mu_x \cap \mathcal{B} = \varphi$. □

Lemma 7. *If μ is any b-ultrafilter, then there is an ultrafilter α of type b such that $\mu = \alpha \cap \mathcal{B}$.*

Proof. μ is a filterbase of sets, so it is contained an ultrafilter α. Thus $\mu \subseteq \alpha \cap \mathcal{B}$. Since μ is maximal, $\mu = \alpha \cap \mathcal{B}$. □

Lemma 8. *Let δ be any compatible Lodato proximity with $\delta_b \leq \delta$. For every ultrafilter μ on X, there is an ultrafilter α of type b such that $\mu \delta \alpha$.*

Proof. Since $\mu \cap \mathcal{B}$ is a b-filter, we must have $\mu \cap \mathcal{B} \subseteq \varphi$ for some b-ultrafilter φ. By the last lemma, there is an ultrafilter α of type b with $\varphi = \alpha \cap \mathcal{B}$. Suppose $\mu \not\delta \alpha$. Then $\mu \not\delta_b \alpha$, so we must have an $M \in \mu$, and $A \in \alpha$ and some F_1, $F_2 \in \mathcal{B}$ with $M \subseteq F_1$, $A \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$. But since $\mu \cap \mathcal{B} \subseteq \alpha \cap \mathcal{B}$, we have F_1 and $F_2 \subseteq \alpha$, contradiction to filter. □

Lemma 9. *On (X, δ_b), given any ultrafilter μ and any ultrafilter α of type b, $\mu\delta_b\alpha$ if and only if $\mu \cap \mathcal{B} \subseteq \alpha \cap \mathcal{B}$. Hence any two ultrafilters near an ultrafilter of type b are near each other. In particular, δ_b is transitive over ultrafilters of type b.*

Proof. Suppose $\mu\delta_b\alpha$. Then for each $B \in \mu\cap\mathcal{B}$, $B\cap F \neq \emptyset$ or every $F \in \alpha\cap\mathcal{B}$. Since $\alpha \cap \mathcal{B}$ is a b-ultrafilter, we have that $B \in \alpha \cap \mathcal{B}$. The converse is clear from the definition of δ_b. □

Definition 10. Let γ be any filterbase on proximity space (X, δ) and set $\Pi_\delta(\gamma) = \bigcup\{\upsilon \colon \upsilon \text{ is an ultrafilter and } \upsilon\delta\gamma\}$.

Definition 11.

a) A grill γ on a proximity space (X, δ) is a *precluster* if whenever $A \subseteq X$ and $\{A\}\delta\gamma$, then $A \in \gamma$.
b) A pre-cluster σ on (X, δ) is a *cluster* if it is a clan.

Example 12. *Let α be any filterbase on proximity space (X, δ). Then $\Pi(\varphi)$ is a pre-cluster.*

Theorem 13. *On (X, δ_b), if α is an ultrafilter of type b, then $\Pi(\alpha)$ is a cluster.*

Proof. By Example 12, $\Pi(\alpha)$ is a pre-cluster and by lemma 9, $\Pi(\alpha)$ is a clan. □

Theorem 14. *The subspace T_bX of the Gagrat-Naimpally completion $\alpha_{\delta_b}X$ given by the set of all maximal clans is a T_1 compactification on X.*

Proof. By Theorem 11,

$$T_bX = \{\Pi(\alpha) \colon \alpha \text{ is an ultrafilter of type } b\}$$

is a subset of $\alpha_\delta X$, the set of all maximal clans on X. Hence T_bX is a T_1 space, which by Lemma 6 contains a dense copy of X. It remains to show that T_bX is compact.

For each $F \in \mathcal{B}$, let

$$F' = \{\Pi(\alpha) \colon F \in \Pi(\alpha)\}.$$

These will be the basic closed sets of the topology of T_bX. Let $\mathcal{L} = \{F'_j \colon j \in \Gamma\}$ be a family of basic closed sets with the finite intersection property. Let

$$\mathcal{F} = \{F_j \colon j \in \Gamma\} \subseteq \mathcal{B}.$$

Let

$$\mathcal{F}' = \{\bigcap_{j\in\forall} F_j \colon \Lambda \text{ a non-empty, finite subset of } \Gamma\}.$$

Then $\mathcal{F}'$ is a filterbase of sets from $\mathcal{B}$ since $\mathcal{B}$ is a ring, and $\mathcal{F} \subseteq \mathcal{F}'$. By Zorn's Lemma, $\mathcal{F}'$ is contained in some b-ultrafilter, which by Lemma 7 we may write as $\alpha \cap \mathcal{B}$ of some ultrafilter α of type b. But then $\mathcal{F} \subseteq \Pi(\alpha)$, so for each $j \in \Gamma$, $\Pi(\alpha) \in F'_j$, and thus $\bigcap \mathcal{L} \neq \emptyset$. □

If $\mathcal{B}$ is a *normal* base of closed sets for T_1 space X, we may constuct a "Wallman-Frink" compactification bX of X in the usual way (See Willard's "General Topology", Exercise 19K p 142):

a) Let bX be the set of all b-ultrafilters on X.
b) For each $B \in \mathcal{B}$, let $B' = \{\varphi \in bX : B \in \varphi\}$. This is a base for closed sets for a topology on bX. Call it the "absorption" topology for $\mathcal{B}$.
c) Under the absorption topology, bX is compact and T_1.
d) $\tau : X \to bX$ by $t(x) = \mu_x \cap \mathcal{B}$ is a homeomorphism from X onto a dense subset of bX.

Theorem 15. *There is a homeomorphism between the T_1 compactification T_bX of (X, δ_b) and the Wallman-Frink compactification bX, which fixes X point-wise.*

Proof. Let f be the bijection assigning maximal b-clan $\sigma = \Pi(\alpha)$ to the b-ultrafilter $\sigma \cap \mathcal{B}$. Set $B \in \mathcal{B}$ gives us both basic closed set

$$B* = \{\sigma \in T_bX : B \in \sigma\} \text{ and } B' = \{\varphi \in bX : B \in \varphi\}.$$

But clearly $B \in \varphi$ if and only if $B\delta\alpha$ for every ultrafilter α of type b for which $\varphi = \alpha \cap \mathcal{B}$, and $F \in \alpha$ if and only if $F \in \sigma_\alpha$ for $\sigma_\alpha = \bigcup\{\mu\colon \text{ ultrafilter } \mu\delta_b\alpha\}$. Hence f is one-to-one between basic closed sets of bX and T_bX, and is therefore a homeomorphism. That f is $1 - 1$ on X follows from Lemma 6. □

Problem. This identifies the Wallman compactification of a T_1 space with a subspace of the Gagrat-Naimpally compactification of (X, δ_w). Conceivably this is possible, yet it does not rule out the equality of the two extensions. This would amount to showing the reverse of Theorem 11, that every maximal clan is of form $\Pi(\alpha)$ for some ultrafilter α of type b.

Specifically, let σ be any maximal clan on (X, δ_b) and consider

$$\sigma \cap \mathcal{B} = \bigcap\{\mu \cap \mathcal{B}\colon \text{ ultrafilter } \mu \subseteq \sigma\}.$$

Let $F \in \sigma \cap \mathcal{B}$ and $F \subseteq B \in \mathcal{B}$. Then there is some ultrafilter $\mu \subseteq \sigma$ such that $F \in \mu$. Therefore $B \in \mu$, so $B \in \sigma \cap \mathcal{B}$. Now let F_1 and $F_2 \in \sigma \cap \mathcal{B}$. Then $F_1\delta_bF_2$ since σ is a clan, so $F_1 \cap F_2 \neq \emptyset$. Suppose $F_1 \cap F_2 \in \sigma$. Then $\sigma \cap \mathcal{B}$ is a b-filter, so there is an ultra-filter α of type b such that $\sigma \cap \mathcal{B} \subseteq \alpha \cap \mathcal{B}$. Hence for every $\mu \subseteq \sigma$, $\mu\delta_b\alpha$, so $\sigma \subseteq \Pi(\alpha)$. By the maximality of σ, $\sigma = \Pi(\alpha)$. Thus

Theorem 16. *Let σ be a maximal clan on (X, δ_b). Then $\sigma = \Pi(\alpha)$ for some ultrafilter α of type b $\iff$ $[F_1, F_2 \in \sigma \Rightarrow F_1 \cap F_2 \in \sigma]$.*

Now suppose X to be a T_1-space for which $\mathcal{B}$ and $\mathcal{D}$ are *normal* bases with $\mathcal{D} \subseteq \mathcal{B}$. [Hence $\delta_b \leq \delta_d$.]

Lemma 17. *Let σ be any element of T_d. Then there is an ultrafilter β of both types b and d such that $\sigma = \Pi_d(\beta)$.*

Proof. Since σ is an element of T_d, there is an ultrafilter γ of type d such that $\sigma = \Pi_d(\gamma)$. But γ is an ultrafilter in (X, δ_b), so there exists an ultrafilter β of type b such that $\gamma\delta_b\beta$. This means that $\gamma \cap \mathcal{B} \subseteq \beta$. But $\mathcal{D} \subseteq \mathcal{B}$ implies $\gamma \cap \mathcal{D} \subseteq \mathcal{B} \cap \mathcal{D}$. By maximality of $\gamma \cap \mathcal{D}$, we have $\gamma \cap \mathcal{D} = \beta \cap \mathcal{D}$. Thus β is also an ultrafilter of type d.

Now ultrafilter $\mu \subseteq \Pi_d(\gamma)$ if and only if $\mu \cap \mathcal{D} \subseteq \gamma \cap \mathcal{D} = \beta \cap \mathcal{D}$. Therefore $\mu \subseteq \Pi_d(\gamma)$ if and only if $\mu \subseteq \Pi_d(\beta)$, and $\sigma = \Pi_d(\beta)$. □

Notation 18. An ultrafilter β of both types b and d will be called an ultrafilter of joint type.

Lemma 19. *Let α be any ultrafilter of type b. Then there is an ultrafilter β of joint type such that $\alpha\delta_d\beta$.*

Proof. There is an ultrafilter γ of type d such that $\sigma\delta_d\gamma$. By the last lemma, there is an ultrafilter β of joint type with $\gamma\delta_d\beta$. Hence $\alpha\delta_d\beta$. □

Definition 20. Let β be an ultrafilter of joint type and put

$$\Gamma_\beta = \{\Pi_b(\alpha)\colon \alpha \text{ is an ultrafilter of type } b \text{ and } \alpha\delta_d\beta\}.$$

Theorem 21. *Γ_β is a δ_d-clan, and is contained in a unique maximal δ_d-clan, namely $\Pi_d(\beta)$.*

Proof. First, Γ_β is a d-clan. Let S and T be sets in Γ_β. By definition of Γ_β, we can find ultrafilters α_1 and α_2 so that $\alpha_1\delta_d\beta$, $\alpha_2\delta_d\beta$ and $S \in \alpha_1$, $T \in \alpha_2$. Suppose $S \not\delta_d T$. Then there exist D_1, $D_2 \in \mathcal{D}$ such that $S \subseteq D_1$, $T \subseteq D_2$ and $D_1 \cap D_2 = \emptyset$. Since α_1 and α_2 are ultrafilters, we must have $D_1 \in \alpha_1$ and $D_2 \in \alpha_2$. Now $\alpha_1 \cap \mathcal{D} \subseteq \beta \cap \mathcal{D}$ and $\alpha_2 \cap \mathcal{D} \subseteq \beta \cap \mathcal{D}$. Thus D_1 and D_2 are in $\beta \cap \mathcal{D}$, contradiction to $D_1 \cap D_2 = \emptyset$. Clearly $\Gamma_\beta \subseteq \Pi_d(\beta)$, which is a maximal d-clan. Suppose Π is a maximal d-clan with $\Gamma_\beta \subseteq \Pi \neq \Pi_d(\beta)$. Then there must an $A \in \Pi$ and some $G \in \beta$ for which $A \not\delta_d G$. But $A\delta_d B$ for all $B \in \Gamma_\beta$, and, because $\beta \subseteq \Pi_b(\beta) \subseteq \Gamma_\beta$, $G \in \Gamma$, contradiction. Thus $\Pi_d(\gamma)$ is unique. □

Theorem 22. *The following are clearly equivalent:*

a) *For each β of joint type, $\Gamma_\beta = \Pi_d(\beta)$.*
b) *If α is of type b and β_1, β_2 are of joint type such that $\beta_1\delta_d\alpha$ and $\beta_2\delta_d\alpha$, then $\beta_1\delta_d\beta_2$.*
c) *$\{\Gamma_\beta\colon \beta$ is of joint type$\}$ partitions T_bX.*

Theorem 23. *If any (hence, all) of the conditions of* 22 *are met, then there exists a continuous function $f\colon T_bX \to T_dX$ which is the identity on X.*

Proof. Define $f\colon T_bX \to T_dX$ by the following: For $\Pi_b(\alpha) \in T_bX$, there is some β of joint type such that $\alpha\delta_d\beta$. Let $f[\Pi_b(\alpha)] = \Pi_d(\beta)$. By 19, f is a well defined function.

To show that f is continuous, let $\mathcal{A} \subseteq T_bX$ and suppose $\sigma_0 \in T_bX$ with $f(\sigma_0) \notin cl_d f[\mathcal{A}]$. We shall show $\sigma_0 \notin cl_b(\mathcal{A})$. Now

$$cl_d f[\mathcal{A}] = \cap\{D'\colon f[\mathcal{A}] \subseteq D'\}.$$

In particular, there is some $D_0 \in \mathcal{D}$ with $f[\mathcal{A}] \subseteq D_0'$, yet $f(\sigma_0) \notin D_0'$. Now

$$D_0' = \{\Pi_d(\gamma) \colon D_0 \in \Pi_d(\gamma)\}.$$

So $\sigma \in \mathcal{A}$ implies $f(\sigma) \in D_0'$, whence $D_0 \in f(\sigma)$ and $f^{\leftarrow}(D_0) \in \sigma$. Therefore $\sigma \in [f^{\leftarrow}(D_0)]'$. Hence $\mathcal{A} \subseteq [f^{\leftarrow}(D_0)]'$, closed in $T_b X$, so $cl_b(\mathcal{A}) \subseteq [f^{\leftarrow}(D_0)]'$. But $f(\sigma_0) \notin D_0'$. Therefore $D_0 \notin f(\sigma_0)$ and $\sigma_0 \notin [f^{\leftarrow}(D_0)]'$. □

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