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EQUIVARIANT COHOMOLOGY RINGS OF TORIC HYPERKÄHLER MANIFOLDS

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ABSTRACT. A smooth hyper Kähler quotient of a quaternionic vector space $\mathbf{H}^N$ by a subtorus of T^N is called a toric hyperKäler manifold. We determine the ring structure of the integral equivariant cohomology of a toric hyperKäler manifold.

1. INTRODUCTION

The topology of symplectic quotients has been studied intensively for recent two decades, using Morse theory and equivariant cohomology theory [Ki][JK]. However, the topology of hyperKähler quotients has not been studied well. In this note we study the topology of smooth hyperKähler quotients of a quaternionic vector space $\mathbf{H}^N$ by subtori of T^N, which we call toric hyperKähler manifolds. Originally Bielawski and Dancer introduced and studied toric hyperKähler manifolds in [BD], being influenced by [D], [A] and [Gu]. Especially they computed their Betti numbers. In [Ko] we determined the ring structure of the integral cohomology of toric hyperKähler manifolds. In this note we show that the method in [Ko] is enough to compute the integral equivariant cohomology rings. Since the topology of toric hyperKähler manifolds depends only on the subtori of T^N, we describe the ring structures of thier equivariant cohomology in terms of the subtori (Teorem 2.4). We also describe them in terms of the arrangement of hyperplanes which were associated to toric hyperKähler manifolds (Theorem 2.6).

In section 2 we review some basic properties of toric hyperKähler manifolds and state our main results. In section 3 we prove them.

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2. Main results

First let us recall the hyperKähler structure on a quaternionic vector space $\mathbf{H}^N$. Let $\{1, I_1, I_2, I_3\}$ be the standard basis of $\mathbf{H}$. We define three complex structures on $\mathbf{H}^N$ by the multiplication of I_1, I_2, I_3 from the left respectively. We fix the identification $i\colon \mathbf{H}^N \to \mathbf{C}^N \times \mathbf{C}^N$ by $i(\xi) = (z, w)$, where $\xi = (\xi_1, \dots, \xi_N) \in \mathbf{H}^N$, $z = (z_1, \dots, z_N), w = (w_1, \dots, w_N) \in \mathbf{C}^N$ and $\xi_j = z_j + w_j I_2$ for $j = 1, \dots, N$.

The real torus $T^N = \{\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{C}^N | |\alpha_i| = 1\}$ acts on $\mathbf{H}^N$ by

$$(z, w)\alpha = (z\alpha, w\alpha^{-1}).$$

This action preserves the hyperKähler structure. Let $\mathrm{Exp}_{T^N}\colon t^N \to T^N$ be the exponential map and $\{X_1, \dots, X_N\} \subset t^N$ be the basis satisfying $\mathrm{Exp}_{T^N}(\sum_{i=1}^N a_i X_i) = (e^{2\pi\sqrt{-1}a_1}, \dots, e^{2\pi\sqrt{-1}a_N}) \in T^N$. We define $\{u_1, \dots, u_N\} \subset (t^N)^*$ to be the dual basis of $\{X_1, \dots, X_N\} \subset t^N$. Then the hyperKähler moment map

$$\mu_{T^N} = (\mu_{T^N,1}, \mu_{T^N,2}, \mu_{T^N,3})\colon \mathbf{H}^N \to (t^N)^* \otimes \mathbf{R}^3,$$

is given by

$$\mu_{T^N,1}(z, w) = \pi \sum_{i=1}^N (|z_i|^2 - |w_i|^2) u_i$$

$$(\mu_{T^N,2} + \sqrt{-1}\mu_{T^N,3})(z, w) = -2\pi\sqrt{-1} \sum_{i=1}^N z_i w_i u_i.$$

Let K be a subtorus of T^N with the Lie algebra $k \subset t^N$. Then we have the torus $T^n = T^N/K$ and its Lie algebra $t^n = t^N/k$. We also have the following exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \stackrel{\iota}{\longrightarrow} & t^N & \stackrel{\pi}{\longrightarrow} & t^n & \longrightarrow & 0 \\ 0 & \longleftarrow & k^* & \stackrel{\iota^*}{\longleftarrow} & (t^N)^* & \stackrel{\pi^*}{\longleftarrow} & (t^n)^* & \longleftarrow & 0 \end{array}$$

We remark that some $\pi(X_i)$ may be zero and some $\iota^* u_j$ may be zero. Since the torus K acts on $\mathbf{H}^N$ preserving its hyperKähler structure, we obtain the hyperKähler moment map

$$\mu_K = (\iota^* \otimes id) \circ \mu_{T^N}\colon \mathbf{H}^N \to k^* \otimes \mathbf{R}^3.$$

Now we define a toric hyperKähler manifold.

Definition If $\nu \in k^* \otimes \mathbf{R}^3$ is a regular value of the hyperKähler moment map μ_K and if the action of K on $\mu_K^{-1}(\nu)$ is free, we call the hyperKähler quotient

$$X(\nu) = \mu_K^{-1}(\nu)/K$$

a toric hyperKähler manifold. □

In fact $X(\nu)$ is a $4n$ dimensional hyperKähler manifold. The torus $T^n = T^N/K$ acts on $X(\nu)$, preserving its hyperKähler structure.

In [Ko] we discussed when the hyperKähler quotient becomes a toric hyperKähler manifold as follows.

Lemma 2.1. *Fix an element $\nu = (\nu_1, \nu_2, \nu_3) \in k^* \otimes \mathbf{R}^3$. Then the following* (1) *and* (2) *are equivalent.*
(1) *$\nu \in k^* \otimes \mathbf{R}^3$ is a regular value of the hyperKähler moment map μ_K.*
(2) *For any $J \subset \{1, \dots, N\}$, whose number $\#J$ is less than $\dim k = N - n$, ν_1, ν_2 and ν_3 are not simultaneously contained in the subspace of k^* which is spanned by $\{\iota^* u_j | j \in J\}$.*

Lemma 2.2. *Let $\nu \in k^* \otimes \mathbf{R}^3$ be a regular value of the hyperKähler moment map μ_K. Then the following* (1) *and* (2) *are equivalent.*
(1) *The action of K on $\mu_K^{-1}(\nu)$ is free.*
(2) *For any $J \subset \{1, \dots, N\}$ such that $\{\iota^* u_j | j \in J\}$ forms a basis of k^**

$$t^N_{\mathbf{Z}} = k_{\mathbf{Z}} + \sum_{j \in J^c} \mathbf{Z} X_j \quad \textit{as a } \mathbf{Z}\textit{-module.}$$

These lemmas lead the following property of a toric hyperKähler manifold, which was originally proved by [BD].

Proposition 2.3. *Let $X(\nu)$ be a toric hyperKähler manifold. Then its diffeomorphism type is independent of the choice of ν.*

Next we construct line bundles L_i on a toric hyperKähler manifold $X(\nu)$ for $i = 1, \dots, N$. Let $\chi_i \colon T^N \to S^1$ be a character defined by $\chi_i(\alpha) = \alpha_i$. Define the action of T^N on $\mu_K^{-1}(\nu) \times \mathbf{C}$ by

$$((z, w), v)\alpha = ((z\alpha, w\alpha^{-1}), v\chi_i(\alpha)).$$

This action induces a T^n-equivariant line bundle $L_i = (\mu_K^{-1}(\nu) \times \mathbf{C})/K$ on $X(\nu)$.

Let $ET^n \to BT^n$ be a universal T^n-bundle. Then we define the homotopy quotient of a T^n-space M by $M_{T^n} = (ET^n \times M)/T^n$. The equivariant cohomology $H^*_{T^n}(M; \mathbf{Z})$ is by definition $H^*(M_{T^n}; \mathbf{Z})$. The T^n-equivariant line bundle L_i on $X(\nu)$ induces a line bundle $\mathcal{L}_i$ on $X(\nu)_{T^n}$. The equivariant first Chern class of L_i is defined by $c_1(\mathcal{L}_i) \in H^*_{T^n}(X(\nu); \mathbf{Z})$.

Now we can state main result of this paper, which determines the ring structure of the integral equivariant cohomology of $X(\nu)$.

Theorem 2.4. *Let $X(\nu)$ be a toric hyperKähler manifold and*

$$\Psi \colon \mathbf{Z}[u_1, \dots, u_N] \to H^*_{T^n}(X(\nu); \mathbf{Z})$$

be a ring homomorphism, which is defined by $\Psi(u_i) = c_1(\mathcal{L}_i)$. Then the following holds.
(1) *The map Ψ is a surjective ring homomorphism.*
(2) *Let I be the ideal generated by all $\prod_{a_i \neq 0} u_i$ for $\sum_{i=1}^N a_i X_i \in k \setminus \{0\}$. Then $I = \ker \Psi$.*

Example Let $\pi\colon t^5 \to t^3$ be a map such that

1. $\{\pi(X_1), \pi(X_2), \pi(X_3)\}$ forms a basis of t^3
2. $\pi(X_4) = -\pi(X_1) - \pi(X_2)$
3. $\pi(X_5) = -\pi(X_1) - \pi(X_3)$.

Then we have a toric hyperKähler manifold $X(\nu)$ for $\nu \in k^* \otimes \mathbf{R}^3$ satisfying the condition mentioned above. Since k is spanned by

$$\{X_1 + X_2 + X_4, X_1 + X_3 + X_5\},$$

there are 4 types of elements in k as follows:

$$X_1 + X_2 + X_4, X_1 + X_3 + X_5, X_2 - X_3 + X_4 - X_5,$$
$$\sum_{i=1}^{5} a_i X_i \text{where } a_i \neq 0 \text{ for } i = 1, \ldots, 5.$$

Therefore Theorem 2.4 implies

$$H^*_{T^3}(X(\nu); \mathbf{Z}) \cong \mathbf{Z}[u_1, \ldots, u_5]/I,$$

where the ideal I is generated by $\{u_1u_2u_4, u_1u_3u_5, u_2u_3u_4u_5\}$. $\square$

It is worth mentioning the results in [Ko] here. Let $p\colon X(\nu)_{T^n} \to BT^n$ be the natural projection. If we fix $x \in BT^n$, then we can identify the fiber $F_x = p^{-1}(x)$ with $X(\nu)$ and the ristriction of $\mathcal{L}_i$ to F_x is isomorphic to L_i. If we denote the embedding of F_x into $X(\nu)_{T^n}$ by $i\colon F_x \to X(\nu)_{T^n}$, then we have the map

$$\Phi = i^* \circ \Psi\colon \mathbf{Z}[u_1, \ldots, u_N] \to H^*(X(\nu); \mathbf{Z}).$$

Then we proved the following theorem in [Ko], which determines the structure of the integral cohomology ring of $X(\nu)$.

Theorem 2.5. *The map Φ is a surjective ring homomorphism. Moreover* $\ker\Phi$ *is generated by* $\ker\Psi$ *and* $\pi^*((t^n)^*) \cap \sum_{i=1}^{N} \mathbf{Z}u_i$.

Next we give another description of the ideal I in Theorem 2.4. We assume that $X(\nu)$ is a toric hyperKähler manifold with $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$. We fix an element $h \in (t^N)^*$ such that $\iota^* h = \nu_1$. Since every toric hyperKähler manifold $X(\nu')$ can be deformed to $X(\nu)$ with $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$, this assumption does not lose any generality.

We associate an arragement of hyperplanes $F_1, \ldots, F_N$ in $(t^n)^*$ to a toric hyperKähler manifold $X(\nu)$ with $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$ by

$$F_i = \{p \in (t^n)^* | \langle \pi^* p + h, X_i \rangle = 0\} \quad \text{for } i = 1, \ldots, N.$$

We note that $F_i = \emptyset$ in ths case $\pi(X_i) = 0$, because $\nu = (\nu_1, 0, 0)$ is a regular value of μ_K. We also note that the above arrangement of hyperplanes is determond by $\nu = (\nu_1, 0, 0)$ up to parallel translation.

Then we have the following theorem.

Theorem 2.6. *Let $X(\nu)$ be a toric hyperKähler manifold with $\nu = (\nu_1, 0, 0)$. Let $I \subset \mathbf{Z}[u_1, \ldots, u_N]$ be the ideal in Theorem 2.4 and $I' \subset \mathbf{Z}[u_1, \ldots, u_N]$ be the ideal generated by all $\prod_{j \in S} u_j$ for $\emptyset \neq S \subset \{1, \ldots, N\}$ such that $\bigcap_{j \in S} F_j = \emptyset$. Then we have $I = I'$.*

Since this theorem is essentially proved in [Ko], we omit it.

3. Proof of Theorem 2.4

In this section we prove Theorem 2.4. We prove it by induction on N.

First we prove the theorem for $N = 1$. In this case we have $k = \{0\}$ or $k = t^1$.

Soppose $k = \{0\}$. In this case $X(\nu)$ is $\mathbf{H}$ with S^1-action. So we have $H^*_{S^1}(X(\nu); \mathbf{Z}) \cong H^*_{S^1}(point; \mathbf{Z})$, which is generated by $\Psi(u_1)$. Therefore Ψ is surjective and $\ker \Psi = \{0\}$. On the other hand, since $k = \{0\}$, we have $I = \{0\}$. So in this case Theorem 2.4 is true.

Suppose $k = t^1$. It is easy to see that the hyperKähler quotient $X(\nu)$ is a point without torus action. So we have $\ker \Psi = (u_1)$. On the other hand, since $X_1 \in k$, we have $I = (u_1)$. So in this case Theorem 2.4 is true. Thus we proved Theorem 2.4 for $N = 1$.

From now on we assume that Theorem 2.4 is true up to $N - 1$. So we prove the theorem for N.

We begin with a few remarks.

Suppose that $\iota^* u_N = 0$, that is, $k \subset t^{N-1} = \sum_{i=1}^{N-1} \mathbf{R} X_i$. In this case the action of K preserves the product structure $\mathbf{H}^{N-1} \times \mathbf{H}$, where $\mathbf{H} = \{(z_N, w_N)\}$. Moreover K acts on $\mathbf{H}$ trivially. Therefore the hyperKähler quotient $X(\nu)$ of $\mathbf{H}^N$ by K is a product of the hyperKähler quotient $X_{(1)}(\nu^{(1)})$ of $\mathbf{H}^{N-1}$ by K and $\mathbf{H}$ itself. Here we note that $X(\nu)$ is a T^n-space, $X_{(1)}(\nu^{(1)})$ is a $T^{n-1} = T^{N-1}/K$-space and $\mathbf{H}$ is a S^1-space, where S^1 is the group with the Lie algebra $\mathbf{R}\pi(X_N)$. Let I_1 be the ideal and $\Psi_1 \colon \mathbf{Z}[u_1, \ldots, u_{N-1}] \to H^*_{T^{n-1}}(X_{(1)}(\nu^{(1)}); \mathbf{Z})$ be the map in Theorem 2.4 for $X_{(1)}(\nu^{(1)})$. Since $X(\nu)_{T^n}$ is homotopy equivalent to $X_{(1)}(\nu^{(1)})_{T^{n-1}} \times \mathbf{H}_{S^1}$, $H^*_{S^1}(\mathbf{H}; \mathbf{Z}) \cong \mathbf{Z}[u_N]$ and Ψ_1 is surjective, we see that Ψ is surjective and $\ker \Psi$ is generated by $\ker \Psi_1$. On the other hand, since $k \subset t^{N-1}$, we see that I is generated by I_1. By the assumption of the induction we have $\ker \Psi = I$. That is, Theorem 2.4 is true in this case.

Suppose that $\pi(X_N) = 0$, that is, $X_N \in k$. In this case the Lie algebra k is decomposed into the direct sum $k = k_2 \oplus \mathbf{R} X_N$, where $k_2 = k \cap \sum_{i=1}^{N-1} \mathbf{R} X_i$. Let K_2 be the corresponding Lie group to k_2. Then the hyperKähler quotient $X(\nu)$ of $\mathbf{H}^N$ by K is just the hyperKähler quotient $X_{(2)}(\nu^{(2)})$ of $\mathbf{H}^{N-1}$ by K_2. Here we note that both $X(\nu)$ and $X_{(2)}(\nu^{(2)})$ are T^n-spaces. Let I_2 be the ideal and $\Psi_2 \colon \mathbf{Z}[u_1, \ldots, u_{N-1}] \to H^*(X_{(2)}(\nu^{(2)}); \mathbf{Z})$ be the map in Theorem 2.4 for $X_{(2)}(\nu^{(2)})$. Since $X(\nu)_{T^n} = X_{(2)}(\nu^{(2)})_{T^n}$ and Ψ_2 is surjective, we see that Ψ is surjective and $\ker \Psi$ is generated by $\ker \Psi_2$ and u_N. On the other hand, since $k = k_2 \oplus \mathbf{R} X_N$, we see that the ideal I is generated by I_2 and u_N. By the assumption of the induction we have $\ker \Psi = I$. That is, Theorem 2.4 is true in this case.

¿From now on we assume that we have a fixed toric hyperKähler manifold $X(\nu)$ with $\pi(X_i) \neq 0, \iota^* u_i \neq 0$ for any $i = 1, \dots, N$. We may also assume that $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$. We fix $h \in (t^N)^*$ such that $\iota^* h = \nu_1$. Then we have an arrangement of hyperplanes $F_1, \dots, F_N$ in $(t^n)^*$.

To proceed the induction argument, we will recover the equivariant cohomology ring of $X(\nu)$ from the equivariant cohomology rings of $X_{(1)}(\nu^{(1)})$ and $X_{(2)}(\nu^{(2)})$, whose associated arrangements of hyperplanes are $F_1 \cap F_N, \dots, F_{N-1} \cap F_N$ in F_N and $F_1, \dots, F_{N-1}$ in $(t^n)^*$ respectively. In [BD] Bielawski and Dancer computed the Betti numbers of $X(\nu)$ from the information of $X_{(1)}(\nu^{(1)})$ and $X_{(2)}(\nu^{(2)})$ by Mayer-Vietoris argument. Since we study the ring structure of the equivariant cohomology, we need futher argument as we explain below.

We begin with constructing $X_{(1)}(\nu^{(1)})$. Let $\rho\colon t^N \to t^{N-1}$ be the projection such that $\rho(X_i) = X_i$ for $i = 1, \dots, N-1$ and $\rho(X_N) = 0$. Since $\pi(X_N) \neq 0$, we have an isomorphism $\rho|_k\colon k \to k_1$, where $k_1 = \rho(k)$. Then we have the following diagrams.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n & \longrightarrow & 0 \\ & & \rho|_k \downarrow & & \rho \downarrow & & \overline{\rho} \downarrow & & \\ 0 & \longrightarrow & k_1 & \xrightarrow{\iota_1} & t^{N-1} & \xrightarrow{\pi_1} & t^{n-1} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* & \longleftarrow & 0 \\ & & (\rho|_k)^* \uparrow & & \rho^* \uparrow & & \overline{\rho}^* \uparrow & & \\ 0 & \longleftarrow & k_1^* & \xleftarrow{\iota_1^*} & (t^{N-1})^* & \xleftarrow{\pi_1^*} & (t^{n-1})^* & \longleftarrow & 0 \end{array}$$

Since $(\rho|_k)^*\colon k_1^* \to k^*$ is an isomorphism, there exists $\nu_1^{(1)} \in k_1^*$ uniquely such that $(\rho|_k)^*(\nu_1^{(1)}) = \nu_1$. Let K_1 be the torus corresponding to k_1. Then the action of K_1 on $\mathbf{H}^{N-1}$ gives the hyperKähler moment map

$$\mu_{K_1}\colon \mathbf{H}^{N-1} \to k_1^* \otimes \mathbf{R}^3.$$

We set $\nu^{(1)} = (\nu_1^{(1)}, 0, 0)$. In [Ko] we showed the following.

Claim 1 (1) *$X_{(1)}(\nu^{(1)}) = \mu_{K_1}^{-1}(\nu^{(1)})/K_1$ is a toric hyperKähler manifold with T^{n-1}-action.*

(2) *$X_{(1)}(\nu^{(1)})$ is a hyperKähler submanifold of $X(\nu)$, which is preserved by T^n-action.*

(3) *We can identify $(t^{n-1})^*$ with $F_N \subset (t^n)^*$ naturally. Moreover under this identification the associated arrangement of hyperplanes for $X_{(1)}(\nu^{(1)})$ is $F_1 \cap F_N, \dots, F_{N-1} \cap F_N$ in F_N.*

Next we construct $X_{(2)}(\nu^{(2)})$. Let $t^{N-1} = \sum_{i=1}^{N-1} \mathbf{R}X_i$ and $j\colon t^{N-1} \to t^N$ be the inclusion. If we set $k_2 = k \cap t^{N-1}$, then we have the following diagrams.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k_2 & \xrightarrow{\iota_2} & t^{N-1} & & & & \\ & & j|_{k_2} \downarrow & & j \downarrow & \searrow^{\pi_2} & & & \\ 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longleftarrow & k_2^* & \xleftarrow{\iota_2^*} & (t^{N-1})^* & & & & \\ & & (j|_{k_2})^* \uparrow & & j^* \uparrow & \nwarrow^{\pi_2^*} & & & \\ 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* & \longleftarrow & 0 \end{array}$$

We remark that π_2 is surjective, because $\iota^* u_N \neq 0$, that is, $k_2 \subsetneq k$. Let K_2 be the torus corresponding to k_2. Then the action of K_2 on $\mathbf{H}^{N-1}$ gives the hyperKähler moment map

$$\mu_{K_2}\colon \mathbf{H}^{N-1} \to k_2^* \otimes \mathbf{R}^3.$$

We set $\nu_1^{(2)} = (j|_{k_2})^* \nu_1 \in k_2^*$ and $\nu^{(2)} = (\nu_1^{(2)}, 0, 0)$. In [Ko] we showed the following.

Claim 2 (1) *$X_{(2)}(\nu^{(2)}) = \mu_{K_2}^{-1}(\nu^{(2)})/K_2$ is a toric hyperKähler manifold with T^n-action.*

(2) *The associated arrangement of hyperplanes for $X_{(2)}(\nu^{(2)})$ is $F_1, \ldots, F_{N-1}$ in $(t^n)^*$.*

Let I_1, I_2 be the ideal in Theorem 2.4 and

$$\Psi_1\colon \mathbf{Z}[u_1, \ldots, u_{N-1}] \to H^*_{T^{n-1}}(X_{(1)}(\nu^{(1)}); \mathbf{Z})$$
$$\Psi_2\colon \mathbf{Z}[u_1, \ldots, u_{N-1}] \to H^*_{T^n}(X_{(2)}(\nu^{(2)}); \mathbf{Z})$$

be the map in Theorem 2.4 for $X_{(1)}(\nu^{(1)})$ and $X_{(2)}(\nu^{(2)})$ respectively. Then by the assumption of the induction Ψ_i is surjective and $\ker \Psi_i = I_i$. Moreover $X(\nu)_{T^n}$ is homotopy equivalent to $X_{(1)}(\nu^{(1)})_{T^{n-1}} \times BS^1$, where S^1 is the group with the Lie algebra $\mathbf{R}\pi(X_N)$. Therefore we have

$$\widetilde{\Psi}_1\colon \mathbf{Z}[u_1, \ldots, u_N] \to H^*_{T^n}(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \cong H^*_{T^{n-1}}(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \otimes H^*_{S^1}(point; \mathbf{Z})$$

and $\widetilde{\Psi}_1$ is surjective and $\ker \widetilde{\Psi}_1$ is generated by $\ker \Psi_1 = I_1$.

The following claim shows how we can recover the equivariant cohomology ring of $X(\nu)$ from the equivariant cohomology rings of $X_{(1)}(\nu^{(1)})$ and $X_{(2)}(\nu^{(2)})$.

Claim 3 (1) *$H^{2k+1}_{T^n}(X(\nu); \mathbf{Z}) = 0$ for each $k \in \mathbf{Z}$.*

(2) *There exists the following short exact sequence for each $k \in \mathbf{Z}$.*

$$0 \longrightarrow H^{2k-2}_{T^n}(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \xrightarrow{e} H^{2k}_{T^n}(X(\nu); \mathbf{Z}) \xrightarrow{r} H^{2k}_{T^n}(X_{(2)}(\nu^{(2)}); \mathbf{Z}) \longrightarrow 0$$

(3) *$e(\widetilde{\Psi}_1(f(u_1, \ldots, u_N))) = \Psi(f(u_1, \ldots, u_N)u_N)$ for $f \in \mathbf{Z}[u_1, \ldots, u_N]$.*

(4) *$r(\Psi(g(u_1, \ldots, u_N))) = \Psi_2(g(u_1, \ldots, u_{N-1}, 0))$ for $g \in \mathbf{Z}[u_1, \ldots, u_N]$.*

Proof We have fixed $h \in (t^N)^*$ and $\nu_1 = \iota^* h$. In Claim 1 and Claim 2 we only assumed that $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$ is a regular value of μ_K. Now we can choose

h and $\nu_1 = \iota^* h$ such that all vertices $\bigcap_{i\in S} F_i$ for $S \subset \{1,\dots,N\}$ with $\#S = n$ are contained in $\{p \in (t^n)^* | \langle \pi^* p + h, X_N\rangle \geq 0\}$. We set

$$\begin{aligned} U_0 &= \{p \in (t^n)^* | \langle \pi^* p + h, X_N\rangle = 0\} = F_N \\ U_1 &= \{p \in (t^n)^* | \langle \pi^* p + h, X_N\rangle \geq 0\} \\ U_2 &= \{p \in (t^n)^* | \langle \pi^* p + h, X_N\rangle > 0\} \\ \widetilde{V}_i &= \mu_{T^n}^{-1}(U_i, (t^n)^*, (t^n)^*) \quad \text{for } i = 0,1,2 \\ V_i &= \mu_{T^n}^{-1}(U_i, 0, 0) \quad \text{for } i = 0,1,2, \end{aligned}$$

where $\mu_{T^n}\colon X(\nu) \to (t^n)^* \otimes \mathbf{R}^3$ is a hyperKähler moment map for the action of T^n on $X(\nu)$.

Now we consider the cohomology exact sequence for $(X(\nu), \widetilde{V}_2)$.

$$\longrightarrow H^l_{T^n}(X(\nu), \widetilde{V}_2; \mathbf{Z}) \longrightarrow H^l_{T^n}(X(\nu); \mathbf{Z}) \longrightarrow H^l_{T^n}(\widetilde{V}_2; \mathbf{Z}) \longrightarrow H^{l+1}_{T^n}(X(\nu), \widetilde{V}_2; \mathbf{Z}) \longrightarrow$$

In [BD] it was shown that $\widetilde{V}_2$ is T^n-equivariant homotopy equivalent to $X_{(2)}(\nu^{(2)})$. Therefore we have

$$H^l_{T^n}(\widetilde{V}_2; \mathbf{Z}) \cong H^l_{T^n}(X_{(2)}(\nu^{(2)}); \mathbf{Z}).$$

By the same argument we also showed in [Ko] that (V_1, V_2) is a T^n-equivariant deformation retract of $(X(\nu), \widetilde{V}_2)$. Moreover the neighbourhood W of V_0 in V_1 can be identified with the neighborhood of V_0 in $E = L_N|_{V_0}$ by the T^n-equivariant map $i\colon W \to E$, which is defined by

$$i([z_1, w_1, \dots, z_{N-1}, w_{N-1}, z_N, 0]) = [(z_1, w_1, \dots, z_{N-1}, w_{N-1}, 0, 0), z_N e^{2\pi\sqrt{-1}\phi(z,w)}],$$

where $[\dots]$ denotes equivalence class. If $[z,w] \in V_1$, then $z_i w_i = 0$ for $i = 1,\dots,N-1$ and $w_N = 0$. Moreover V_0 is defined by the equation $z_N = 0$ in V_1. So we have

$$H^l_{T^n}(X(\nu), \widetilde{V}_2; \mathbf{Z}) \cong H^l_{T^n}(V_1, V_2; \mathbf{Z}) \cong H^l_{T^n}(E, E \setminus V_0; \mathbf{Z}).$$

V_0 is not smooth, but it is a T^n-equivariant deformation retract of $X_{(1)}(\nu^{(1)})$. Moreover $E = L_N|_{V_0}$ is the ristriction of $\widetilde{E} = L_N|_{X_{(1)}(\nu^{(1)})}$. Therefore we have

$$H^l_{T^n}(E, E \setminus V_0; \mathbf{Z}) \cong H^l_{T^n}(\widetilde{E}, \widetilde{E} \setminus X_{(1)}(\nu^{(1)}); \mathbf{Z}) \cong H^{l-2}_{T^n}(X_{(1)}(\nu^{(1)}); \mathbf{Z}),$$

where the second isomorphism is Thom isomorphism. Thus we have

$$H^l_{T^n}(X(\nu), \widetilde{V}_2; \mathbf{Z}) \cong H^{l-2}_{T^n}(X_{(1)}(\nu^{(1)}); \mathbf{Z}).$$

Moreover by the assumption of the induction we have

$$H^{2k+1}_{T^n}(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \cong H^{2k+1}_{T^n}(X_{(2)}(\nu^{(2)}); \mathbf{Z}) \cong 0.$$

Therefore we proved the claim. □

Now we prove the first part of Theorem 2.4.

Claim 4 Ψ *is surjective.*

Proof Fix any $a \in H^{2k}_{T^n}(X(\nu); \mathbf{Z})$. Since Ψ_2 is surjective, there exists $f \in \mathbf{Z}[u_1, \dots, u_{N-1}]$ such that

$$r(a) = \Psi_2(f(u_1, \dots, u_{N-1})) = r(\Psi(f(u_1, \dots, u_{N-1}))).$$

Since $a - \Psi(f(u_1, \dots, u_{N-1})) \in \ker r$, there exists $b \in H^{2k-2}_{T^n}(X_{(1)}(\nu^{(1)}); \mathbf{Z})$ such that

$$e(b) = a - \Psi(f(u_1, \dots, u_{N-1})).$$

Since $\widetilde{\Psi}_1$ is surjective, there exists $g \in \mathbf{Z}[u_1, \dots, u_N]$ such that

$$b = \widetilde{\Psi}_1(g(u_1, \dots, u_N)).$$

Thus we have

$$\begin{aligned} a &= \Psi(f(u_1, \dots, u_{N-1})) + e(b) \\ &= \Psi(f(u_1, \dots, u_{N-1})) + e(\widetilde{\Psi}_1(g(u_1, \dots, u_N))) \\ &= \Psi(f(u_1, \dots, u_{N-1}) + g(u_1, \dots, u_N)u_N). \quad \square \end{aligned}$$

Finally we show the second part of Theorem 2.4.

Claim 5 $I = \ker \Psi$.

Proof First we show $I \subset \ker \Psi$. To do this, we show that all generators of I belong to $\ker \Psi$. Take $\emptyset \neq S \subset \{1, \dots, N\}$ such that $\bigcap_{j \in S} F_j = \emptyset$. According to Theorem 2.6, it is sufficient to show $\prod_{j \in S} u_j \in \ker \Psi$.

Suppose $N \in S$. Since $\bigcap_{j \in S \setminus \{N\}} (F_j \cap F_N) = \bigcap_{j \in S} F_j = \emptyset$, according to Claim 1 and Theorem 2.6, we have

$$\prod_{j \in S \setminus \{N\}} u_j \in I_1 = \ker \Psi_1 \subset \ker \widetilde{\Psi}_1.$$

Therefore, according to Claim 3, we have

$$\Psi(\prod_{j \in S} u_j) = e(\widetilde{\Psi}_1(\prod_{j \in S \setminus \{N\}} u_j)) = 0.$$

Thus we proved $\prod_{j \in S} u_j \in \ker \Psi$ in the case $N \in S$. Here we used the assumption $\pi(X_N) \neq 0$. However we have assumed $\pi(X_i) \neq 0$ for any i. So we use the same argument in the case $i \in S$. Thus we proved $I \subset \ker \Psi$.

Next we show $\ker \Psi \subset I$. For $f(u_1, \dots, u_N) \in \ker \Psi$ we have to show $f \in I$. First we rewrite

$$f(u_1, \dots, u_N) = g_1(u_1, \dots, u_{N-1}) + g_2(u_1, \dots, u_N)u_N.$$

Since

$$0 = r(\Psi(f(u_1, \dots, u_N))) = \Psi_2(g_1(u_1, \dots, u_{N-1})),$$

we have that $g_1(u_1, \dots, u_{N-1}) \in \ker \Psi_2 = I_2$. Since $I_2 \subset I \subset \ker \Psi$, we have $g_2(u_1, \dots, u_N)u_N \in \ker \Psi$. We have to show $g_2(u_1, \dots, u_N)u_N \in I$.

Since

$$0 = \Psi(g_2(u_1, \ldots, u_N)u_N) = e(\widetilde{\Psi}_1(g_2(u_1, \ldots, u_N)))$$

and e is injective, we have $g_2(u_1, \ldots, u_N) \in \ker \widetilde{\Psi}_1 = I_1\mathbf{Z}[u_N]$. According to Claim 1 and Theorem 2.6, we have $I_1 u_N \subset I$. So we have $g_2(u_1, \ldots, u_N)u_N \in I$. Thus we proved $\ker \Psi \subset I$. So we finish the proof of Claim 5. □

Thus we finish the proof of Theorem 2.4.

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