FROM FORMAL TO SEMI-INFORMAL ALGORITHMS: THE PASSAGE OF A CLASSROOM INTO A NEW MATHEMATICAL REALITY

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In contrast to the traditional top- down approach, a bottom-up approach is proposed by current reform in mathematics education. According to this alternative proposal, algorithmizing is the activity in which students should be involved. What can we do when we want to enact such an algorithmizing approach in a classroom and our students have already been instructed the algorithms in a traditional way? The students have to move from a school-based to an inquiry-based mathematical reality. Is the passage from one reality to the other so easy? The focus of this paper is on the difficulties that a fifth-grade classroom met as we tried to revisit the multiplication and division algorithms, which had been taught in a traditional way. How these difficulties influenced the emergence of mathematical content?

INTRODUCTION

When we conduct a classroom teaching experiment, our general purpose is to attempt a change of the "school mathematics tradition". An "inquiry mathematics tradition" is what we are looking for (Cobb et al., 1992). Such a change implies that with our support students will come to experience mathematics in a different way. From a mathematical reality where students comprehend mathematics as a set of ready-made propositions and procedures, a new inquiry-based mathematical reality where mathematics is viewed as a human activity has to emerge.

According to Mehan & Wood (1975), realities are permeable and so the passage to a new reality is feasible. However, this passage is fraught with difficulties if we take into account their characterization of realities. As they claim, realities are sustained through the reflexive use of bodies of knowledge in interaction. Reflexivity means that a reality is not easily abandoned. Even when counter evidence is provided, it reflexively becomes evidence for the sustenance of an assumed reality. It is then interesting to investigate the difficulties that may occur as the members of a classroom are guided towards an inquiry-based mathematical reality.

Students in a fifth-grade classroom had already memorized the steps of the formal multiplication and division algorithms. Now they have to revisit these algorithms in order to extend their practice to numerals with any number of digits. In previous grades, instruction had been based on the use of concrete representations accompanying the explanation of numerical examples. Drill and practice always preceded the application of the algorithms to the solution of problems. Lack of proficiency and insight, as well as low applicability are usually mentioned as the negative effects of this approach (Hart, 1981; Resnick & Omanson, 1987). Students come to identify understanding to following the teacher's or the textbook's procedural instructions to obtain correct answers. On the other hand, as algorithms are taught out of context, students do not know when it is appropriate to apply them. A school mathematics based reality was thus well established by the students of this classroom.

To avoid the negative effects of this top-down approach, an alternative bottom-up approach would be to support students' construction of algorithms based on their own activity. Starting from contextual problems students can generate their own procedures. Through shortening and schematizing, these procedures can take the form of the conventional algorithms. Even if students don't reach formal algorithms, the quality of their understanding would counterbalance the development of semi-informal algorithms (Gravemeijer, 2003). In this way, algorithmizing becomes the main practice students are involved in (Freudenthal, 1991). The reinvention of algorithms makes students' mathematical reality inquiry-based.

For the students of the above-mentioned classroom the bottom-up teaching approach had to be adjusted, if we wanted them to experience an inquiry-based mathematical reality. As we had to revisit algorithms with these students, their difficulties in establishing such a reality would be more easily investigated. These difficulties would be greater due to their instrumental understanding of algorithms. The focus of this paper will be to examine the role of these difficulties as they influence the emergence of mathematical content in this classroom. In this process, the relations between the old and the new reality will come to the fore and the passage into a new mathematical reality will be illuminated. Apart from its theoretical importance, our attempt has also practical implications. Teachers intending to develop a bottom-up teaching approach for algorithms cannot usually enact it. Students' prior experiences inhibit their attempts and so they are skeptical about their teaching effectiveness.

THEORETICAL FRAMEWORK

Studying the difficulty of the passage of a traditional classroom into an inquiry-based mathematical reality, the emergent perspective on classroom life is of relevance (Cobb & Yackel, 1996). Any mathematical reality "is becoming" through the coordinated efforts of the individual students as they participate in diverse ways in the communal mathematical meaning-making activities of their classroom. Students' taken-as-shared meanings emerging through their own interactions can be considered as the building blocks for the construction of their reality. In turn, this reality may constrain or enable their individual constructions. The reflexivity between the individual and social aspects of their mathematical activity in constructing meanings for the multiplication and division algorithms will be assumed as we analyze individual students' contributions.

METHODOLOGY

Our data is based on a fifth grade classroom in a public school of Athens at the beginning of the school year 2003-04. It consists of 15 video recorded lessons, the 22 students' worksheets and written tests. The presenting author taught most of the lessons, aiming in students' understanding of multiplication and division algorithms. Development of students' multiplicative reasoning about quantities was considered necessary both for understanding the algorithms as well as the subsequent unit that focused on fractions.

For this purpose, we prepared an initial set of instructional activities based on Gravemeijer's (1998) heuristic of emergent models. A learning path has been anticipated through which students could be supported in developing insightful ways of reasoning with algorithms. More specifically, we expected that students' reasoning with a ratio table would emerge as a model of informal solutions to multiplication and division problems. Eventually, we anticipated that reasoning with the ratio table (see results section below) would serve as a model for the construction of semi-informal algorithms and would provide opportunities for our students' developing interpretations of the formal algorithms. However, it must be noted that the set of activities used in our classroom would be tailored to the students' needs. Their actual trajectory may not match our hypothetical learning path.

As we conduct our analysis, the construct of classroom mathematical practice developed by Cobb and his colleagues (Cobb et al., 2001) will be useful. This research group differentiates between three aspects of a mathematical practice: (a) a taken-as-shared purpose, (b) taken-as-shared ways of reasoning with tools and symbols, and (c) taken-as-shared forms of mathematical argumentation. We will be focusing on instances where individual students' ways of acting can be traced back to their school mathematics based reality. Their relationship to the above three aspects of a practice will allow us to understand the difficulties the passage to an inquiry-based mathematical reality entails. The delineation of mathematical practices as a means to describe the inquiry-based reality established in our classroom is not of our concern in this paper. However, one may notice that the instances we will be referring to may belong to different practices.

RESULTS

From formal algorithms to informal ways of operating

In our first lesson, students were asked to solve multiplication and division problems. What we noticed was that students: (1) did not have any different solutions to offer when solving a problem apart from using the standard algorithms, (2) were uncertain about which operation to perform, and (3) did not have any meaning for the steps of the algorithms used. No doubt that the school-based mathematical reality was overarching and constraining their activity. On the other hand, students appeared to have mastery of the multiplication and division formal algorithms.

For the next lesson, we distributed a worksheet with the solutions of hypothetical younger students on a multiplication problem. The problem on which students were invited to explain and justify the solutions was: “A bookcase has eight shelves. Each shelve has 23 books. How many are all of its books?” The intent of this task was to give students the opportunity to reflect on multiplicative relationships. The role of these relationships in building the algorithms might then be approached through properly designed solutions.

As an example, repeated doubling was used as a means to calculate the answer:

$$\begin{array}{r} 23 \\ \underline{+23} \\ 46 \end{array} \qquad \begin{array}{r} 46 \\ \underline{+46} \\ 92 \end{array} \qquad \begin{array}{r} 92 \\ \underline{+92} \\ 184 \end{array}$$

Below is the dialogue between the teacher and a student who was willing to explain the above solution:

1	T:	What did this child do? I mean how did she think?
2	S1:	Additions.
3	T:	Can you explain what did she do?
4	S1:	She added 23 and 23 and she found 46. Then she added once more…46 and 46 and she found 92. And then she again added 92 and 92 and she found 184.
5	T:	She found the same answer! But do you understand her way? Can someone else explain to us…why did she do these additions? [Students do not respond]

Initially, students were not in a position to see any connection between the above additions and the situation. Students merely read the additions. Searching for a reason behind a calculation was not a goal in their mathematical reality. Criteria for judging when an explanation would be appropriate were lacking. That is why their explanations were exclusively calculational.

Similar tasks along with our support (i.e. drawings, symbolizing their explanations on the board, etc.) helped students to start interpreting solutions in a multiplicative way. These interpretations were the best we could achieve from students deeply immersed in the school mathematics reality. The form of their arguments was getting a contextual character. Operating informally in multiplication and division situations was finally instigated.

Comparing the algorithms with carefully chosen exemplary solutions had also become a topic of discussion. Detection of their similarities and differences came as a result of these discussions. However, we should not forget that our students did not invent the solutions. These had been given ready-made. This significant deviation from the bottom-up approach did not guarantee the re-construction of meaningful algorithms.

From informal to semi-informal ways of operating

To support our students' development of their own semi-informal ways of multiplicative operating, we introduced the model of ratio table in the classroom (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1998). The opportunities our students had in interpreting multiplicatively solutions of hypothetical younger students might now be utilized. Reasoning with the ratio table could be based on this prior experience. Acting with this model insightfully was expected to ensure a meaning for the operations of multiplication and division, as well as for their algorithms. With these conjectures in mind, we told students about a fourth-grader who was used to organize his solutions with the help of a table. In the problem: "A crate of lemonades contains 24 bottles. If a supermarket buys 49 full crates, how many bottles has it bought?" this student's solution was presented as follows:

Crates	1	10	5	4	9	40	49
Bottles	24	240	120	96	216	960	1,176

Our students did not seem to have any difficulty understanding this fourth grader's reasoning with the ratio table. In the same lesson, starting from a multiplication problem, students recorded their different solutions on ratio tables and compared them, in terms of their efficiency. It was not but until a few lessons later, that a measuring situation involving the division 135:12, led a student to the following table:

Minibuses	1	3	5	135
Students	12	36	60	

X 3

X 5

Perhaps, the use of the ratio table was confounded with the use of place value tables like:

Hundreds	Tens	Units	The number
1	3	5	135

Other students did not strongly object his way of using the ratio table. In addition, it was not much later that similar solutions were given by a group of students in a written test. We were alerted by this instrumental use of the ratio table. It seemed that the use of this tool would come to have in this classroom a meaning related to the school based mathematical reality. Rather than students reasoning about quantities in multiplicative ways, they were looking for patterns in the numerals. From our perspective, the inquiry-based mathematical reality was at risk.

Asking students to anticipate the steps they would have to take, as they were using a ratio table, might help them to change the purpose of their activity. Questions like: "Which number are you looking for?" "Where is the unknown number going to be on your table?" and "Can you say in advance, the steps you intend to take?" were instrumental in reorienting students' actions with the ratio table. Through these questions students' activity was gradually focused on explaining the reasons for the steps they proposed.

From semi-informal ways of operating to semi-informal algorithms

Eventually, students could reason with the ratio table and solve a variety of multiplication and division problems. By the end of the instructional sequence, we surprisingly saw that there were students who were still choosing operations at random. For example, in a multiplication problem, they would try to divide the given numbers by using the division algorithm. For these students, the use of the ratio table did not evolve into a model for reasoning with the algorithms. Even if they could use this model, they could not relate it to the multiplicative relationships implied in the situation at hand. Apart from the inherent difficulty that such an undertaking involves, the vestiges of their old reality were still prevailing.

Encouraging students to estimate their answers did not prove to help students increase their awareness of structuring problems multiplicatively. To avoid the random selection of operations, we were inviting students not to be thinking of which operation to select. Starting their work from the ratio table did not encase them within the vicious circle of their school based mathematical reality. Reasoning with the ratio table came to constitute a semi-informal algorithm. Efficient and sophisticated solutions were commonly produced in our classroom. The problem: " For 5 days someone was paid 140 €. How many euros was he paid for each day?" could be solved by methods like:

Euros	140	280	28
Days	5	10	1

Euros of each day	1	2	10	20	8	28
Total amount	5	10	50	100	40	140

CONCLUSIONS

We tried to change the approach by which the multiplication and division algorithms had been taught in a fifth-grade classroom. A pure algorithmizing bottom-up approach was not feasible. Students already knew the algorithms and even more so they had constructed a school-based mathematical reality. The passage to an inquiry-based mathematical reality cannot be automatic. Students' old habits were coming to the fore and influenced the learning path of the classroom.

One may view these habits as inhibiting the enactment of an algorithmizing approach. In our classroom these difficulties functioned as opportunities to redesign the hypothetical learning trajectory we had in mind when we started our teaching experiment. However, students' development of semi-informal algorithms was our only alternative if we wanted students to walk away from their old reality. We should note that the quality of understanding in this classroom was not only a matter of the mathematical practices we tried to develop. The classroom social norms were also of our concern.

The influence of our students' school-based mathematical reality on their development of multiplicative reasoning declined. Their passage to an inquiry-based mathematical reality is still incomplete. At least, we hope that our students have already experienced the distinction between the two realities.

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