SOLUTION - WHAT DOES IT MEAN? HELPING LINEAR ALGEBRA STUDENTS DEVELOP THE CONCEPT WHILE IMPROVING RESEARCH TOOLS

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Twelve linear algebra students were interviewed about the concept of a Solution of a System of Equations. The interviews were analyzed using APOS tools, in particular the ideas of Action, Process, Object and Schema, and Genetic Decomposition. The analysis of the interviews revealed several misconceptions of Solution. The analysis also revealed shortcomings of the questionnaire that was used in the interviews: It did not permit making a distinction between lack of knowledge and partial knowledge. Research tools were improved (questionnaire, GD, and suggestions for teaching materials) and prepared for the next cycle of research.

THEORETIC PERSPECTIVE

The research reported here is part of a broader research effort conducted by the RUMEC (Research in Undergraduate Mathematics Education) group, which is dedicated to research within the scope of the theory named APOS. APOS is an acronym for the ideas of Action, Process, Object and Schema. This theory is an elaboration of Piaget's cognitive theory (Piaget, 1975) for learning mathematics. Detailed description of this theory can be found, for example, in Asiala et al., (1996). Here we will only describe such elements of this theory that are used in this report.

Action

According to APOS, the development of every concept begins in the learner's mind with an action. At this level the learner can only perform the action one step at a time. For example, given a system of linear equations with n unknowns, as well as several tuples and matrices of different sizes, students are asked which of the givens is a possible solution. If the students start substituting each tuple separately, we suspect that they cannot imagine in advance whether a given tuple can be substituted and hence be a prospective solution. The theory accounts for such inability by the explanation that at the action level, the learners are able to complete the action step after step, but cannot think of it as a whole and predict its outcome; Sometimes they can also not describe it verbally.

On the other hand, the behavior described above might indicate that *substitution in order to check equality* is the action used by these students as the starting point for

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conceptualizing *solution*. We will show findings indicating the possibility that some of our students used a different action for the development of the concept *solution.*

Process

When a learner successfully predicts the outcome, invents shortcuts and can describe the action verbally without actually performing it, we say that the concept has developed in his or her mind to the level of a process. For example, in the situation described above, if students can point to appropriate tuples as possible candidates for solutions, as well as explain the relation between the tuples' length and the number of unknowns - we might infer that their conception of *solution* is at least at the process level. They are able to envision the action of substitution without actually performing it.

Object

When the learner can already perform a new mathematical operation on the process itself, or consider the process as an element of a set of processes of its type, with rules or operations within that set – we say that the concept has developed in the learner's mind into an Object. Examples:

a) If students, when confronted with a system of linear equations are asked "what does a solution look like", are able to describe the form of *a solution* of that system - we may conclude that their understanding of *solution* is at the object level.

b) Another example of object level of *solution* concerns understanding the rule that the sum of two solutions of a homogeneous system is also a solution. Such understanding also requires an object understanding of solution, for otherwise the student would not be able to relate to the binary operation "sum of two solutions".

Genetic Decomposition

Learners can begin the development of a certain concept out of different actions. These will result in different understandings of the concept (as opposed to different levels of understanding). A genetic decomposition of a specific concept consists of a detailed description of such possible action, and the typical mathematical behaviors and reactions of a student who has developed that same concept, beginning with that action, throughout the different levels (Action, Process, Object and Schema). Hence, a satisfactory GD can first of all be used as a diagnostic tool, providing the teacher and investigator with insight into the learner's situation in the development of the concept. In addition, it helps the teacher and material developer to provide the student with activities which will enhance his progress in developing his understanding of the concept through the different levels Action-Process-Object-Schema.

It should be emphasized that a GD of a concept is in itself a developing structure. Also, it cannot be assumed to be unique. (DeVries et al., 2001.)

METHODOLOGY OF THE REPORTED RESEARCH

The research reported here is the first cycle in such a research program. Fifteen students at a Teachers' College took a one-semester course in linear algebra. Of the 15 students, 12 were interviewed shortly after completing the course (the rest did not show up for the interview). The interviews were conducted individually. They consisted of a structured questionnaire, which each student solved in front of the interviewer and discussed his work with her. Each interview lasted about 45 minutes, and was video-recorded. Question 1 was constructed to investigate the concept of solution (see appendix 1).

The purpose of this part of the interview was two-fold: Getting to know students' ideas about *solution*, and getting started with a first version of a GD for this concept.

We were not interested in statistical data about the occurrence of the different reactions, as the group of interviewees was not sampled and hence not representative.

FINDINGS

What we discovered after interviewing the students was that our questionnaire was not adequate for providing sufficient insight into our research questions. The responses to this questionnaire gave us little information about the constructions that students have made in their understanding of the concept *solution of an equation*. This basically occurred because only students who had constructed a solution of an equation as an Object could answer the questions in a meaningful way. Also, this provided little possibility of distinguishing between "no understanding" and "partial understanding" (between the Action or Process levels).

In the first part of this report we describe some of the responses obtained. Their analysis leads to suggesting an improved protocol for the interview, an initial version of a GD for *solution*, and a proposed teaching sequence.

Response type 1: *What does it mean "What does a solution look like"*

Some students at first tended not to reply to the question *What does a solution look like?*. Some explained they did not understand the question:

Interviewer: We now deal with question 1:A.
Hersch: What does it mean "What does a solution look like"
Interviewer: What do you think it will look like?
Hersch: The solution here is a number?
Interviewer: What number for instance? Can you give an example?
Hersch: No, because I don't understand the question.

Upon further probing Hersch concluded:

> "If it's a solution of such a thing, there are four elements here. ...So we should also get four solutions for such an equation."

Hersch really could only think of a solution as a number. Consequently, questions regarding sums of solutions and number of solutions did not provide insight into Hersch's thinking.

Response type 2: Memorized rules about the sum of two solutions.

Lin relied on a memorized rule rather than reason in answering questions about solutions.

Interviewer: Okay suppose we had two solutions u and v. Is u+v also a solution?
Lin: Yes by the rule that the sum of two solutions is also a solution.

The interviewer questioned him as to why this rule was true and all Lin could do was repeat the rule.

There is another indication of the fact that the rule students quoted was memorized without understanding. Some of them applied it both to homogeneous and to non-homogeneous systems. Example:

Earlier in the interview, Tania was certain that the sum of two solutions of a homogeneous system is also a solution. Now she is asked about a non-homogeneous system:

Interviewer: Here is an equation. Is it homogeneous?
Tania: No.
Interviewer: Why not?
Tania: It is not equal to zero.
Interviewer: If we took two solutions of these infinitely many, and added them the way one adds vectors, will it also be a solution?… We'll take an actual sum. Will the sum also be a solution?
Tania: Yes.

We suspect that students who provided responses of types 1 and 2 have certainly not developed *solution* into the Object level. The deficiencies of our questionnaire prevented us from tracing any lower levels of knowledge, if such existed.

Response type 3

Some students confused a solution of an equation (or system), with the constant "Right Hand Wing" of the equation (or system). This might be related to findings about the concepts associated by college (as well as k-12) students with the equality sign. Research shows that students of different ages tend to interpret the equality sign to mean: "the result is", rather than symbolizing equivalence (such as the equivalence accomplished when substituting a solution into both sides of the equations). See for example Kieran, 1981.

Response type 4: *Solution* as *solving*

Several students in response to the question of what a solution looked like, proceeded to solve the equation. Tania provides an example of this. She correctly described the procedure for finding the solutions of the equation. She did not think of substitution to verify equality being a defining property of being a solution. It was apparent that most such students were confusing the concepts of *solution* and *solving*.

Our explanation of responses of type 4 is that for these students the concept of *solution* developed out of the action of solving the equation (or system of equations), rather than the action of substitution. By this we mean the solving methods (algorithms) they used (such as Gaussian elimination, or any other). Such algorithms are difficult to interiorize, and do not make it easy for the student to predict the form of the outcome, the solution, without actually calculating it. Using APOS terms, we suspect that for these students, *solution* as *solving* is at the Action level of development, and we know that when a concept is still at that level of its development, the student can only perform the action one step at a time. Hence their tendency to start solving when asked about the solutions. Another characteristic of this level is that the student has no ability to predict the outcome without actually performing the action. Here - the students could not predict the mathematical form of the solution, of the outcome of the solving procedure, before they actually carried it out. We predict that using the action *substitution* as basis for the development of the concept *solution* will end up with easier interiorization of the action and its transforming into process.

AN INITIAL GENETIC DECOMPOSITION FOR *SOLUTION*

We will sum up this discussion with a proposal of an initial GD for a *solution of an equation*:

An equation is an ordered pair of functions (f, g) with a common domain and a common co-domain. A *solution* of an equation is an element s of the domain for which f(s)=g(s). The *solution set* of an equation is the set of all solutions.

Note: In linear algebra we are usually interested in linear functions from F^n to F^m , and the function g is constant. The pair (f, g) can be represented by means of a system of linear equations, matrices or linear transformations.

SUGGESTED LEARNING SEQUENCE

Action level of the concept *equation*, including *solution*.

We propose to start by helping students construct the Action level of the concept of *equation*, including the ability to identify the two functions, their common domain and co-domain, and *solution* in the sense of an element of the domain, the substitution of which produces a true equality. Here we propose to have them substitute elements of the domain into the two functions and learn to identify solutions and non-solutions.

Process level of *equation* (including *solution*)

Students should be taught to identify the functions and their domains and co-domains for various forms of equation, without being given examples of elements for substitution. They might also be asked to describe the format of possible solutions and non-solutions.

Object level of *solution*

Here we recommend working on finite fields. Students might be asked to design a computer program that receives an equation as its input and produces the solution set of the equation as its output. The program does this by substituting and checking all the elements of the finite $\mathbf{Z_p}^{\mathbf{n}}$ for equality. The programming language ISETL was found to be adequate for that purpose (see Asiala et al., 1996). Later, when we give students a system of equations over an infinite field they will face a need for other methods, as the previous method has now become useless for both computers and humans. Learning to solve algorithms will now include the understanding of what the algorithm does: It produces only substitutions that are truth-valued, and all such substitutions.

IMPROVED QUESTIONNAIRE

In appendix 2 we presented our improved questionnaire. In the first interview (Appendix 1) most of the questions required an object level understanding of *solution* in order to give any answer at all to the questions. Consequently, we did not get any sense of the level of cognitive development regarding the concept. So in the second round we tried to probe more fundamental constructions regarding the *solution*. For example, in Question 1 we give the student a specific ordered pair and ask if it is a solution, rather than asking what *a solution* would look like. This would indicate at least an Action conception if the student substitutes into the equation.

Further on, in Question 2, checking by substitution whether a matrix is a solution demands some tedious calculations. If the students have reached process conception of solution, they might realize without actually substituting, that the 3x2 matrix (b) is non-substitutionable. Thus we can identify a process conception of solution.

CONCLUSION

In the present research cycle we learned a little about what students think of *solution.* We also recognized the deficiencies of our research tools. As a result, we constructed an improved questionnaire, an initial version of GD for *solution*, and a suggestion for a teaching sequence resulting from that GD. We are now ready for the next cycle of our research.

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Appendix 1: The questionnaire of the reported research

A. What is a solution of this equation (what does it look like)? $\mathbf{3x_1+2x_2-x_3+x_4=5}$

How many solutions does the equation have?

Is the sum of two solutions also a solution? What about a scalar multiplication?

B. What does a solution of this equation look like? $x_1\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}+x_2\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}+x_3\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}+x_4\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}$

Which of the following might be a solution? **a**. $\begin{pmatrix}1&0\\2&1\end{pmatrix}$ **b**. **7** **c**. **(1, 0, 1, 7)** **d**. $\begin{pmatrix}2\\0\\1.5\\7\end{pmatrix}$

How would you check whether it is a solution?

C. Here is a homogenous system of equations Ax=0. Suppose each of the vectors u and v is a solution of this system. What do you think of the vector u+v? Is it a solution of the system or not?

(If no answer) Would you like to use an example?

(If no answer) Would you like me to present an example to you?

Here is an example: $\begin{pmatrix}1&2&3&0\\0&1&1&2\\1&0&1&1\\-2&-3&-5&-3\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ What would be a solution of this system?

Which of the following vectors is a solution $\begin{pmatrix}1\\2\\0\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\\2\\0\end{pmatrix}$, $\begin{pmatrix}8\\8\\-8\\0\end{pmatrix}$? How can we check?

Is the sum of two vectors which are solutions, also a solution?

D. What about a non-homogenous system? How does it differ from a homogenous system?

Here is a non-homogenous system: $\begin{pmatrix}1&2&3&0\\0&1&1&2\\1&0&1&1\\-2&-3&-5&-3\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$

Suppose each of the vectors u and v is a solution of this system. What do you think of the vector u+v? Is it a solution of the system as well? How can we check/prove?

E. A and B are nxn matrices of the same order. What would be a solution of such an equation: AX=B

Appendix 2: The new questionnaire

1. Consider the equation $2x_1 + 3x_2 = 6$

(a) Explain why **[6,-2]** is a solution. (b) Find another solution.

(c) What is the sum of the solution in (a) and your solution in (b)?

(d) Is the sum you found in (c) also a solution? Why or why not?

(e) Is a scalar multiple of a solution also a solution? Why or why not?

(f) How many solutions does this equation have? Explain.

2. Consider the equation: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 7 & 0 \\ 3 & 0 \\ 9 & 0 \end{bmatrix}$

(a) Is $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ a solution? Why or why not? (b) Is $\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 6 & 2 \end{bmatrix}$ a solution? Why or why not?

3. Consider the system of equations: $3x_1 + 2x_2 - x_3 = 0 \quad x_1 - x_2 + x_3 = 0$

(a) Is [2,-3,0] a solution? Why or why not? (b) Is [3,-2,-5] a solution? Why or why not?

(c) Does the system have more than one solution? Explain.

(d) Find the solution set of the system.

(e) Is the sum of two solutions also a solution? Why or why not?

(f) Is a scalar multiple of a solution also a solution? Why or why not?

4. Consider the equation: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

(a) Is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ a solution to this equation? Why of why not? (b) Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ a solution to this equation? Why or why not?