# MATEMATICAL LANGUAGE AND ADVANCED MATHEMATICS LEARNING

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*This paper is concerned with the role of language in mathematics learning at college level. Its main aim is to provide a perspective on mathematical language appropriate to effectively interpret students' linguistic behaviors in mathematics and to suggest new teaching ideas. Examples are given to show that the explanation of students' behaviors requires to take into account the role of context. Some ideas from functional linguistics are outlined and some features of the texts usually produced by students are discussed and compared to the corresponding features of standard mathematical texts. Some teaching implications are discussed as well.*

## INTRODUCTION

The role of language in mathematics learning is a critical topic, and it is usually dealt with from a variety of theoretical perspectives. A controversial issue is the relationships between communication processes and the development of thinking. In the opinion of some researchers[1] thinking and communication are closely linked, whereas others[2] regard them as quite independent processes. The language of mathematics itself is interpreted from a variety of perspectives. In the opinion of a good share of mathematicians the specific features of mathematical language chiefly reside in mathematical formalism. On the other hand, verbal language is widely used in mathematical activities (including research), and language-related troubles are not confined to the symbolic component at all. I presume that most mathematics educators, no matter the theoretical frame they adopt, would agree that linguistic problems may undermine any further intervention, for students might misunderstand what they are told or they read, or be unable to express what they mean. It should be widely acknowledged too that this issue grows even more important if groups of language minority students are involved[3]. Moreover, if one assumes too that *"learning mathematics may now be defined as an initiation to mathematical discourse, ..."*[4], and languages are regarded not as carriers of pre-existing meanings, but as builders of the meanings themselves, then the linguistic means adopted in communicating mathematics are crucial also in the development of mathematical thinking. So, poor linguistic resources would produce poor development of thinking.

The main aim of this paper is to provide a perspective on mathematical language appropriate to effectively interpret students' linguistic behaviors in mathematics and

[1] For example, Sfard (2001). Also Duval (1995) underlines the cognitive functions of languages in mathematics.

[2] For example, Dubinsky (2000).

[3] This topic is widely discussed in the book edited by Cocking & Mestre (1988).

[4] Sfard (2001, p.28)

to suggest new teaching ideas. To achieve this goal some ideas borrowed from pragmatics (which is the subfield of linguistics dealing with the interplay between text and context) and functional linguistics (which is a theoretical stance within the field of pragmatics) are introduced. The colloquial way of using language (which is often the one adopted by students) is compared to the mathematical one through examples. The application of ideas from functional linguistics to mathematics has been carried out by a number of researchers such as Pimm (1987), Morgan (1996, 1998), Burton & Morgan (2000). Also Sfard's focal analysis (2000) might be related to standard topics of functional linguistics. Ferrari (2001, 2002) used the same ideas to interpret some empirical findings. This paper focuses on the theoretical aspects.

## MATHEMATICAL LANGUAGE AND ITS USE

Through the paper I mostly refer to Italian Science freshman students and their learning problems in mathematics. At college level, students' troubles are customarily ascribed to the lack of specific contents in their high school curricula. On the contrary, my claim is that students' competence in ordinary language and in the specific languages used in mathematics are other sources of trouble.

To point out some aspects of this topic, I give a couple of examples. The following problem has been given to a wide range of samples, from grade 7 to college.[5]

Example 1

| | |
|---|---|
| Link each sentence on the left to the sentence or the sentences on the right with the same meaning, if any. | |
| a) Not all the workers of the factory are Italian. | a') All the workers of the factory are foreigners |
| b) No worker of the factory is Italian | b') Some workers of the factory are Italian. |
| c) Not all the workers of the factory are not Italian. | c') All the workers of the factory are Italian |
| | d') Some workers of the factory are foreigners |

In all the samples (including college students) although most of the subjects properly treated sentence b), a good share connected a) to both b') and d'), and the same happened for c). The more suitable treatment (from the mathematical standpoint) of sentence b) compared to sentences a) and c) is a common feature of all the samples.

Sentence a') is equivalent to b) from the viewpoint of both everyday-life and mathematics. As regards sentences a) and c), the state of affairs is not so simple. From the mathematical viewpoint, d') is equivalent to a). From the same perspective,

5 The translation into English of a text written in another language may affect some linguistic properties of the original text. Here, the text is simple enough to be translated without substantially changing the features I am taking into account.

b') is not equivalent to a) at all. Nevertheless, it is a *conversational implicature* of a). In the frame of pragmatics, a conversational implicature of a text[6] is the portion of the information provided by the text that follows from the assumption that it is adequate to the context rather than from its propositional content. b') does not follow from the content of a), but from the assumption that a) is appropriate. If b') were false, then a) might be still true, but it would prove inadequate, as a sentence like a') would be much more cooperative. Therefore, the link students recognize between a) and b') does not reside in the propositional content of a), but in the assumption that it is a cooperative contribution to the exchange. It goes without saying that mathematical language[7] is customarily forced to break cooperative criteria, which means that some implicatures cannot be drawn.

Example 2

| | | | |
|---|---|---|---|
| (2A) | Find a real polynomial $p$ such that: | (a) | the degree of $p$ is 2; |
| | (b) $p$ has at least one real root; | (c) | $p$ has at least two integer roots. |

This example is taken from Ferrari (2001). Problem (2A) is easily solved by almost all Science freshman students, after a short unit on real polynomials. The only source of trouble is the interpretation of 'real' and 'integer', which requires some accuracy, as the adoption of the mathematical use (according to which an integer is a real as well) rather than the everyday-life one (according to which the combined use of the two words may suggest the implicature that 'real' should mean 'non-integer'). If the items (b), (c) are included in a more complex context such as problem (2B), students' behaviors are quite different.

| | | |
|---|---|---|
| (2B) | Find a real polynomial $p$ such that: | |
| | (a) the degree of $p$ is 4; | (b) $p$ has at least one real root; |
| | (c) $p$ has at least two integer roots. | (d) $p$ has at least one complex non-real root. |

Problems like be are usually solved by less than 60% of each sample of freshman students. Once more, the main obstacle resides in students' failure in recognizing that any integer root is a real root as well. A good share of students who can apply this property to (2A) seem unable to apply it to (2B). Behaviors like these can be hardly ascribed to the lack of knowledge on integers and reals. More likely, in (2B) students focus on condition (d), who asks for the application of a theorem[8] they regard as important and difficult. This condition is interpreted quite accurately, as most students are not misled by conversational schemes and realize that, though only one non-real root is mentioned, two of them are to be considered. The interpretation of

6 By 'text' I mean any written or spoken instantiation of language of any length, not necessarily a book.

7 Through the paper, by 'mathematical language' I refer to the language customarily used in doing and communicating mathematics at undergraduate level, including verbal and symbolic expressions. In this paper visual representations are not explicitly discussed, although they play a major role in communicating mathematics.

8 "For any real polynomial p, if a complex number z is a root of p, then its conjugate $\bar{z}$ is a root of p too."

'integer' and 'real' is not taken as a focal point of the problem and is performed according to conversational schemes. Notice that, as often happens in mathematical language, there are only few discourse markers to help the reader to recognize the global organization of the text, including focal points, goals and so on. Most likely, in problem (2A) the lack of an easily recognizable focal point, and the relative shortness of the text, induces students to interpret all of the condition according to mathematical uses. It goes without saying too that these behaviors are common, and usually effective, in everyday-life contexts.

**Theoretical implications**

The above examples provide us a number of hints I am going to list.

First, they corroborate the claim that troubles heavily involve the verbal component. Second, they point out that the interpretation of a text is hardly a plain translation (based on vocabulary and grammar), but involves the context the text is produced within (including participants and goals). Third, they suggest that the interpretation of texts is a cooperative enterprise which requires the readers (or hearers) to play an active role, performing some inferences, recognizing some part of the text as essential and focusing on them. Fourth, they show that the investigation of single expressions can hardly provide significant insights, but whole texts are to be taken into account; example 2 shows that some expressions may prove more or less troublesome according to the text they occur within. Finally, they suggest that everyday-life and mathematical language are considerably divergent as to use, and that this may prove a severe obstacle to learning.

The last point implies that, as the goal of just preventing students from adopting conversational schemes is of course neither a reasonable nor a viable one, they need to be able to recognize the two ways of using language and to switch between them. This requires some metalinguistic awareness that most often has to be built, as not all students have developed it. All these hints suggest that we need a theoretical frame apt to spot the use-related differences between mathematical language and ordinary one which are relevant to mathematics learning. For these reason we need to borrow ideas and constructions from pragmatics, which fulfils all the above requirements.

## A FUNCTIONAL PERSPECTIVE

More precisely, I adopt the frame of functional linguistics, which focuses on functions of language rather than on its forms[9]. The emphasis on functions is quite appropriate because the gap between ordinary language and mathematical one mainly resides in the difference of the functions they play. Mathematical language is not shaped so as to promote interpersonal communication, but rather to provide an effective, well-organized picture of mathematical knowledge and to support the application of algorithms. Anyway, mathematicians, mathematics educators and

9 The main sources in functional linguistics adopted in this paper are Halliday (1985) and Leckie-Tarry (1995).

students must communicate. This may result in using the same words and constructions with different meanings, according to the goal of the text. As the examples above show, the conflict between the interpersonal function of language and the logical, ideational one may hinder students' interpretations processes.

The construction linking texts to contexts is *register*[10]. A register is defined as a linguistic variety based on use. It is a construction linking the situation to both the text, the linguistic and the social system. Each individual can use a register by selecting his or her own linguistic resources. Through the paper, the registers adopted in everyday-life are referred to as 'colloquial', whereas those adopted in academic communication and most books are referred to as 'literate'. Colloquial registers are mostly adopted in spoken communication, although they may be used in writing too, as in informal notes, e-mail or sms messages, whereas literate ones are mostly adopted in written texts, though they may be used in spoken form too, as in academic lectures or some talks between educated people. Literate registers are not necessarily associated with advanced topics nor with high-level linguistic resources nor with the writers or speakers' age. For example, a group of 2nd-graders writing down a report of some complex activity might actually use a literate register.

One of the main claims of this paper is that the registers customarily adopted in advanced mathematics share a number of features with literate registers and may be regarded as extreme forms of them. Some specific features of mathematical registers, such as the violation of cooperation principles, the unfeasibility of most implicatures and the lack of discourse markers have been mentioned above. The example below points out some other aspects.

Example 3

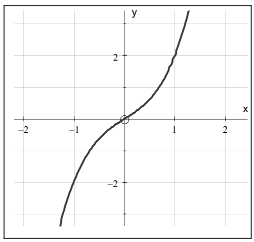
A group of freshman students were required to recognize (and explain) which equation, out of the following

(a) $y = x^3+1$

(b) $y = x^3+x$

(c) $y = x^2+ x$

might match the graph of the function $f$ on the right.




To explain her (right) answer, a student wrote the following text (translated into English verbatim).

> "The graph is increasing and decreasing and passes through 0. I see that $x$ and $y$ >0 and $x$ and $y$<0. So the graph corresponding to $f$ is the equation (b)."

Texts of this kind are quite common among freshman students. This one is quite inaccurate: the graph is described as 'increasing and decreasing' (which is inconsistent),

[10] Here I adopt Halliday's definition of register, which has been thoroughly discussed by Leckie-Tarry (1995).

it is claimed that it ‘passes through 0’ (in place of (0,0)) and the second occurrence of ‘graph’ is used to mean ‘equation’. The claim that the graph is ‘increasing and decreasing’ might be related to the student’s way of exploring the graph starting from the origin and moving rightwards or leftwards. The expression ‘$x$ and $y$ >0 and $x$ and $y$<0’ is quite obscure as well. There are two interpretations available. Maybe the first and the third occurrence of ‘and’ are intended to express some logical relationship (such as ‘if $x$>0 then $y$>0’). On the other hand, the student when reading ‘$x$ and $y$ >0’ pointed her forefinger to the right side of the diagram, and when reading ‘$x$ and $y$<0’ pointed to the right; maybe she meant to describe the two sides of the diagram separately, but in writing failed to make this reference explicit through words. In all cases, we are dealing with behaviors common in spoken colloquial registers: relationships between statements are not made explicit through syntax, and a conjunction like ‘and’ is used to express a variety of meanings; references to the context are not made explicit, maybe because in spoken communication the act of pointing or other gestures may get the same goal; words are used quite inaccurately, as often happens in spoken communication, where the addressee can ask for explanation if the meaning is not clear enough; the expressions explicitly defined in mathematical setting (such as ‘increasing function’) are used according to ordinary meaning rather than to the definition; little attention is paid to inconsistency.

All the features of the text suggest that the student in question cannot use literate registers, or, if she can, some reason prevented her from actually using them. As a matter of fact, in literate written registers syntax is a basic way to express meanings, any reference to the context is made explicit through words, words are used accurately, and texts are to be consistent. These features of literate written registers are imposed by a variety of reasons including the need for communicating with people not sharing the same context the text has been produced within, and the need for representing a great amount of complex data with complex relations. In mathematical language, the above-mentioned features of literate registers occur in an extreme form, and, especially if the symbolic notations are involved, there are fewer opportunities of expressing meanings and organizing discourse. The role of syntax, for example, is crucial, as far as often it is the only way to express some meanings. The need for making any reference to the context explicit is even more acute; moreover, there is plenty of words whose meaning has been redefined and that are to be used accurately. On the other hand, despite all the criticism, it is undeniable that the text in question was somewhat effective, as the instructor understood its meaning after all. He had to be much cooperative, and most likely he was expected to be such, as students know that instructors know mathematics quite well. In general, in most teaching contexts, students expect instructors to be cooperative. If communication fails or if the instructor claims that the text is inappropriate, the student might ascribe failure not to his or her product, but to the lack of cooperation by the addressee.

## TEACHING IMPLICATIONS

In the previous sections I tried to show that mathematical language shares a number of properties with written literate registers of ordinary language. This means that being familiar with literate registers and their use, which is not a 'natural' condition but has to be built, is a good starting point, if not a prerequisite, to learn to use mathematical language. This raises the problem of the methods more suitable to help students to learn to use literate registers. Of course this cannot be done just at undergraduate level, but a long-term work is needed which should start in primary school. Teaching methods based on grammatical patterns do not work anymore. On the other hand, in standard learning situation students are hardly required to deal with genuine communicative problems. Most often they are required to communicate mathematics to people who already knows it, and whose only task is to evaluate their performance. So we need to design learning situations requiring students to develop suitable linguistic or metalinguistic resources not in conformity to prearranged patterns, but as answers to shared communicative and representational constraints[11].

At college level there are few opportunities to put into practice long term activities aimed at improving linguistic skills. Requiring high degrees of correctness to students with a poor linguistic background just means inducing them to learn by heart or to use stereotyped expressions with no understanding. On the other hand, some linguistic accuracy seems essential in doing and communicating mathematics, and must be developed anyway. To get this, verbal language is to be exploited as a tool to describe and justify procedures, and to gain a better control on performances.

In this frame, discussions between students, at any age level, play a major role, as they provide some of the simplest teaching situations satisfying the conditions stated above. Of course, discussions alone do not produce mathematical knowledge, but nevertheless they may help students to develop linguistic skills that are essential to understand and communicate mathematics, if not to develop mathematical thinking. This requires a shift of emphasis from 'solutions' to verbal explanations and may involve students' and teachers' beliefs and attitudes towards mathematics and mathematics education.

Information technology, if properly exploited, provides a variety of semiotic systems (verbal language, graphs, formulas, tables, …) which allow instructors to design activities requiring interpretation, comparison, conversion and treatment of representations, related to goals explicitly shared by students. Technology provides constraints (e.g., on the format of the data) that are often taken by students more easily than the ones put by the instructors, as they appear as objective requirements rather than decisions subject to the whims and moods of an individual.

11 Ferrari (2002) has shown an example of an activity like that at middle school level.

## FURTHER DEVELOPMENTS

The ongoing research on this topic is aimed at refining the comparison between colloquial registers and mathematical ones. This investigation should provide hints on the most appropriate ways of organizing texts intended for students as well as teaching ideas aimed at the improvement of linguistic skills and metalinguistic awareness through the design of teaching methods apt to develop linguistic resources matching the needs of scientific thought without generating needless obstacles. The full exploitation of the opportunities provided by information technology (including the availability of visual representations) is a necessary step to achieve all this. Last but not least there is the goal of making clear the interplay between the use of language and students' beliefs and attitudes towards mathematics and languages.

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