THE DEVELOPMENT OF STRUCTURE IN THE NUMBER SYSTEM

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A cross-sectional study of 132 Australian rural children from grades K-6 assessed children's understanding of the number system. Task-based interview data exhibited lack of understanding of the base ten system, with little progress made during Grades 5 and 6. Few Grade 6 children used holistic strategies or generalised the structure of the number system. Grouping strategies were not well linked to formation of multiunits and addititve rather than multiplicative relations dominated the interpretation of multidigit numbers.

Our numeration system is a consistent and infinitely extendable base ten system that facilitates mental and written notational forms of number for both whole number and decimal fractions. The numeration system allows us to allocate words for numbers in a pattern that follows a base ten system where the multiunit conceptual structures used involve the powers of ten. These multiplicative units, ie 1, 10, 100, … form the basis for conceptualising ever increasing numerical quantities and enable an infinitely extendable number system. The notational number system, known as the Hindu-Arabic system, records these numbers and together with the mental system evolved through centuries of human thought and use.

This paper explores a part of the study on children's understanding of the number system. The focus is on children's understanding of structure. A critical problem is that children do not recognise that the numbers they use are part of a system, and thus they do not have the multiunit structures to understand how the numbers are regrouped in mental and written algorithms. Further, understanding of the use of powers of ten is needed in order to construct the multiunit conceptual structures for multidigit numbers. Understanding the multiplicative nature of the base 10 system is critical to the development of numeration, place value and number sense.

RESEARCH ON CHILDREN'S CONSTRUCTION OF MULTIUNIT CONCEPTUAL STRUCTURES

From the developing body of research on numeration and place value we know that a child's understanding of the numeration system is complex, is not necessarily lock-step, and develops over many years. There is also a fundamental change in the way a child understands number from the early notion of number as a counting unit, to the construction of composite units (Steffe, 1994) and the reinitialising of units (Confrey,

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1994). The process that starts with treating a collection as a whole and then develops as a system that is built on the iteration of grouping collections, requires significant cognitive reorientations.

Research on how children extend their early number understanding and skills to cater for the expanding system of generating number names and symbols has been far less comprehensive than the work on early number. Despite much research on counting and place value in the 1980's, by the 1990's researchers could not assert any firm explanations about why children fail to grasp the structure of the number system. Sinclair, Garin, & Tieche-Christinat (1992) make the crucial point that:

> Understanding place value is not a matter of simply 'cracking' an arbitrary written code following adult explanation or some degree of exposure to computation. It is indissolubly linked to understanding the number system itself. Grasping it implies understanding a multiplicative recursive structure. (p. 93)

Children need structural flexibility in counting and grouping in order to operate meaningfully with the number system. The role of visualisation of the counting sequence was examined in view of children's representations of the numeration system (Thomas, Mulligan & Goldin, 2002). Thomas, et al (2002) suggested that the further developed the structure of a child's internal representational system for counting numbers the more coherent and well organised will be the child's externally produced representations and the wider will be his or her range of numerical understandings.

In numeration the notion of multiplicative units is necessary in order to understand the conceptual structure of multiunit quantities (Behr, Harel, Post, & Lesh, 1994). Construction of a unit occurs through internalising repeatable actions. The unitising action of assigning a number word to each object in a collection is the basis of counting. Multiplicative structures are important because the units in the numeration system are multiplicative. Children construct multiunits through the multiplicative relation. An infinite sequence of multiplicative units are created by grouping units with a particular multiplying number, treating the composite as a unit and iterating them to form further units. Units can also be created by splitting existing units in a similar way (Confrey & Smith, 1995).

METHOD

The study was designed as a broad exploratory investigation employing task-based interviews and quantitative and qualitative methods of analysis. A descriptive approach was used to provide evidence of qualitative differences in the way children use their strategies and relate key elements of the numeration system.

Sample: A cross-sectional sample of 132 children from Grades K to 6 was randomly selected from six Government schools in rural Australia. Five of the schools were from three large regional towns and one was the only school in a small rural town. The sample was representative of a wide range of socio-economic backgrounds.

Interview tasks: A total of eighty-nine tasks were incorporated after trialling in a pilot study. The tasks were designed to probe understanding of numeration through: counting; grouping/partitioning; regrouping, place value; structure of numeration and number sense. Many of the tasks were refined from those used by previous researchers (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Davydov, 1982; Denvir & Brown, 1986; Mulligan, 1992; Ross, 1990; Steffe & Cobb, 1988; Wright, 1991). The tasks were graded by level of difficulty and different subsets of tasks were given to each grade cohort.

Analysis of Data: Item Response Analysis using Student-Problem curve theory (Harnish, 1983) and the Rasch model (Rasch, 1980) were used initially to obtain some overall measure of student performance in Grades 4 to 6. The main analysis of results involved coding responses for student performance and strategy use (which is reported partially in this paper) across tasks and grades. The coding of responses was trialled in a pilot study which was devised to indicate correct, incorrect or non-response to the tasks. Strategies used for both incorrect and correct responses to tasks were coded in order to classify the range of numeration skills and understandings. Re-coding was conducted by two independent coders for 20% of responses which established a high level of intercoder reliability (0.92).

RESULTS

The structure tasks in the study aimed to assess the children's ability to identify structure in the number sequence, such as using and extending grouping systems (based on tens and other grouping numbers), using arrays to quantify a large number of items, and calculating with powers of ten. There is evident from the results that there was a diverse range of strategy use across tasks but performance generally increased through the grades with some leveling off (and even decline) in the upper primary grades.

Recognition of Place Value Structure

Figure 1 shows children’s performance on the various tasks which tested their recognition of place value structure. These tasks simply asked children to recognise or represent a number or to use place value structure in counting. No calculation was involved.

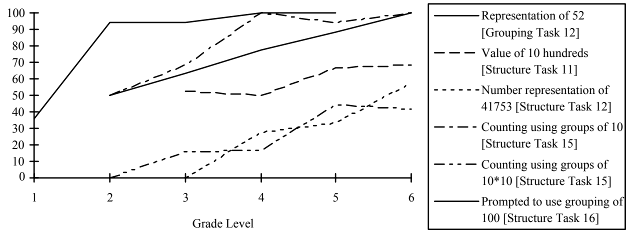

Figure 1. Percentage of sample correctly performing place value recognition tasks

By the end of Grade 2, most children could represent the 52 shells using the pregrouped material [Task 12]. A surprisingly large number (increasing from 50% at Grade 3 to 68% at Grade 6) recognised that a box of lollies (containing 10 bags of 10 rolls of 10 lollies) held 1000 lollies [Task 11]. The number who could recognise the number of lollies in a collection when they were also packed in cases of 10 boxes [Task 12] grew steadily from none in Grade 3 to 58% in Grade 6.

The number of children who recognised and used the fact that the red circle enclosed 100 marks [part of Task 15] was much smaller, and consistently smaller than the number of students who recognised the structure of 1000 [Task 11]. When prompted to consider the red circle [Task 16] the number of children who used the enclosure of 100 marks was consistently below the number of those who used tens groupings in Grouping Task 12 or Structure Task 15. One explanation for these discrepancies is that students have learned to interpret certain concrete materials (bags, blocks, bundles, etc.) representing the number system but have not reached the general level of understanding needed to interpret unfamiliar groupings (circled marks) in the same way.

Significance of recursive grouping by tens

Several tasks sought to find if children would spontaneously group by tens recursively in order to make counting easier. These tasks are to be distinguished from the tasks discussed so far, where a grouping by tens was given by the interviewer. The results are shown in Figure 2.

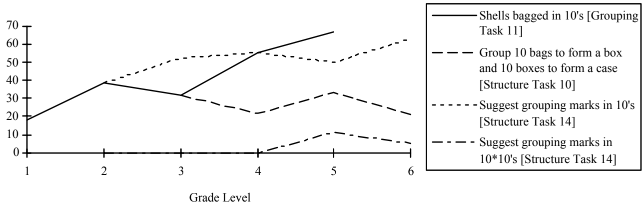

Figure 2. Percentage of sample suggesting grouping by tens

The numbers suggesting grouping by tens for counting shells [Task 11] or marks [Task 14] show a gradual increase from about 20% in Grade 2 to about 60% in Grade 6. A range of grouping numbers were considered appropriate. Most of those who suggested grouping by tens could not offer a reason for their choice. A notable exception was Andrew (Grade 1), who responded, ‘About ten in each ... then we only have to put 10 tens to make 100’. Given the use in schools of a variety of concrete materials to model the place value system - all of which are, of course, based on grouping in tens - this is indeed a surprising result. It suggests that many students are not aware, even at the most basic level, of the purpose or usefulness of our place value system.

The numbers suggesting recursive grouping by tens for packing lollies [Task 14] or counting marks Task 14] was even smaller. It is again surprising that so few students are aware of a further fundamental characteristic of our numeration system.

Understanding the meaning of multiplication

The action of grouping by tens, so basic to the place value system, is closely associated with the operation of multiplication. Recursive grouping by tens is linked with repeated multiplication or exponentiation. Three tasks related to students’ understanding of multiplication. The results are shown in Figure 3.

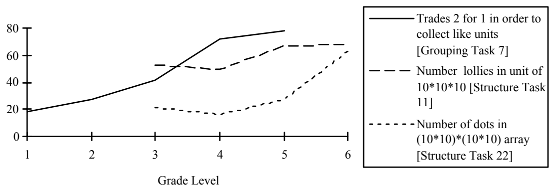

Figure 3. Percentage of sample correctly performing multiplication tasks.

Children showed an increasing success at calculating the result of trading two stickers for one [Task 7] - a multiplicative task with the semantic structure of a ratio (Mulligan, 1992). This task is similar to that of relating the values of successive places in a numeral, and the shape of the developmental curve resembles that of several of the place value tasks shown in Figures 1 and 2.

The task of calculating the number of lollies in 10 bags of 10 rolls of 10 lollies [Task 11] has already been mentioned (see Figure 1): There is relatively little improvement between Grade 3 and Grade 6. Structure Task 22 involved a further recursion: The pattern could be regarded as made up of 10 rows of 10 groups of 10 rows of 10 dots. Successful students invented several strategies. For example, after they had determined that there were 100 dots in each square, some counting by 100s, 100 times; some counted by 100s to find that there were 1000 in the first row of squares and then counted in 1000s, 10 times; and some determined that there were 100 squares and multiplied 100 by 100. Most of the children in Grades 4 to 6 (89%, 89% and 100% respectively) recognised the pattern of 100s but many (72%, 61% and 37%) could not complete the calculation, that is, they were unable to cope with the recursion. In these three grades, about one third of the successful students used the most sophisticated strategy of multiplying 100 by 100.

Performance on Task 22 vividly illustrates the difficulties children experience relating recursive grouping to repeated multiplication. It may be conjectured that children have little experience with arrays or with repeated multiplication. It is no wonder that they also have increasing difficulties coping with the place value system as the numbers get exponentially larger.

DISCUSSION

It appears from this study that many children in Grades 1-6 are familiar with concrete materials used to represent grouping of numbers, but still rely on unitary counting. They may show good performance on 2-digit calculations, but generally use poor methods and cannot extend their success to numbers with larger numbers of digits. There is in general a weak awareness of structure and, in particular, of the multiplicative nature of this structure. Nevertheless, some children acquire a good understanding of place value and develop their own efficient strategies spontaneously.

The results emphasise the importance of units and multiunits (units of more than one) in understanding the structure of the numeration system. The way that children deal with the units of one and ten influences their understanding of larger numbers (Cobb & Wheatley, 1988; Steffe & Cobb, 1988). A child who uses ten as a singleton unit might be able to recite the decade numbers (i.e., skip count in tens) but makes no sense of the increments of ten. The units of one and ten co-exists but are not coordinated. Only children who can coordinate the units and various multiunits can use these units in mental strategies for operations on larger numbers. There were also a substantial number of students in Grades 2 and 3 who were not successful in recognising and using groupings of ten to quantify a collection of objects.

Children do not realise that multiunits are related and can be exchanged if they do not understand the abstract properties of quantity (Davidov, 1982), one of the conceptual underpinnings of multiplication. Clark and Kamii (1996) reported that although some children develop multiplicative thinking as early as Grade 2, most children still cannot demonstrate consistent thinking in Grade 5. The present study confirms these results. A substantial minority (about 20%) of the Grade 3 students had developed such an intuitive understanding of powers of ten that they could use the recursive multiplicative structure of the array of 10,000 dots to count the number of dots successfully. But by Grade 6, there were still a significant number who could not count 10 groups of 10 groups of 10.

CONCLUSIONS

This study showed that understanding of numeration developed slowly over the Kindergarten to Grade 6 period and that very few children were able to generalise the multiplicative structure of the system. Although there was good performance on using grouping in quantifying and building grouped material, there were indications that children did not understand the significance of ten in the number system. This understanding is critical to their further development of understanding and use of the numeration system.

The study highlights the difficulties that primary school children have in understanding the complex nature of the number system. Children did not understand the multiplicative relationships within the system that are the basis of place value structure and the patterns in the counting sequence. Children could count and group in tens but did not relate these processes to a base ten structure.

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