USES OF EXAMPLE OBJECTS IN PROVING

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This paper builds upon discussions of the importance of semantic or intuitive work in proving by identifying three ways in which experienced mathematicians use example objects in constructing and evaluating proofs. It observes that students often do not use objects in these ways, and discusses the pedagogical question of how we might teach students to be more effective in proving by designing instruction to focus their attention on relevant objects. Data are drawn from interviews with five mathematicians experienced in teaching an introductory proofs course.

INTRODUCTION

It is well-recognized that students sometimes attempt to prove a general statement by empirical means, checking a number of examples to give evidence of its truth, rather searching for a deductive proof (e.g. Harel & Sowder, 1998). This is considered to be an inappropriate approach, and students are warned not to "prove by example". On the other hand, it is also noted that semantic or intuitive considerations can be very important in the work of successful mathematicians (e.g. Thurston, 1995). This paper builds on these considerations by offering more precise characterizations of the ways in which successful mathematicians use example objects to aid in proof construction and evaluation.

RESEARCH CONTEXT

The characterizations to be given below are derived from analysis of interviews with mathematicians experienced in teaching a course called "Introduction to Mathematical Reasoning", which is designed to provide students with a grounding in proving before they take courses in real analysis and abstract algebra. This is taught at a large state university in the USA. Classes typically have between 20 and 25 students, so that the professors are in relatively close contact with individuals and become familiar with their work during the 14-week course.

Five participants were involved in this exploratory study, which set out to address a gap in mathematics education research on proof and proving by drawing on the experience held by mathematicians who teach such courses, and seeking to formalize this into knowledge that can be more readily discussed and applied. Each participant was interviewed up to three times during a year. The first interview asked the participant to describe their experience in teaching this course, to give their views on important things that students should learn, and to describe common student mistakes and misunderstandings and their pedagogical strategies.

These interviews were transcribed and analyzed following Glaser (1992). First, conceptual descriptions were added to the transcript, and summarized in memos. Concurrently, further memos were made about questions arising from this data, typically of a need for clarification, or of a possible theoretical link between comments. At this stage the analysis rotated regularly from one participants' interview to another, in order to facilitate synthesis of the ideas raised and to avoid becoming focused on the opinions of a single participant. Next, the memos were sorted according to their main substantive and/or theoretical content, producing a system of categories. Subsequent interviews asked more specific questions designed provide increasing saturation of the categories.

One outcome of this analysis was the identification of three uses of example mathematical objects in the mathematicians' reasoning, with a frequent lack of such use on the part of students. These are: (1) understanding a statement, (2) generating an argument, and (3) checking an argument. They are described and illustrated below.

UNDERSTANDING A STATEMENT

Our first point is that mathematicians view the instantiation of objects as important in reaching a meaningful understanding of a mathematical statement. In these excerpts, Professor 1 remarks upon this as a natural first step in understanding a definition.

> P1: …So one of the things, again, that's second nature to me but it's not to them [the students], is that if I see a definition, I immediately instantiate it. You know, just try some examples of this definition, and try to fit it in.

> P1: …what happens is…that you describe a new definition, you say "let f be a function, let x be a real number, we say that…" and then "some relationship between f and x holds if…blah, blah, blah." So then what they have to do, they have to realize that this definition only makes sense in the context of, I have to have a function in mind and I have to have a [number] in mind…

He notes, however, that such instantiation in response to a definition is not typical behavior for students in his classes.

> P1: And what they'll do is typically if you have a sequence, you know, if I have a sequence definition to use in the rest of the problem, and they don't understand the definition, they'll just skip that sentence and go on. I will – they will come in for help on a problem, and five or ten minutes into the discussion I'll realize that, that they never bothered to process this particular definition. They have no idea what this means.

In response to this phenomenon he has invented task sequences involving the construction of example objects that satisfy various combinations of properties associated with certain definitions.

> P1: So, what I've been trying to do is to have these exercises where the whole purpose of the exercise is just for them to process a mathematical definition. […] I have one where I, where I just define what it means for a – well, what a partition of a set means. I define it formally, so it has these two conditions, a collection of subsets, such that the empty set is not one of the subsets, for every element of the underlying set there is a subset that contains it, for any two sets in the partition the intersection is empty […] And then I just ask okay, construct three examples of a partition on the set {1,2,3,4,5}. And then, okay, construct an example of a collection of sets on {1,2,3,4,5} which satisfies the first two properties but not the third. The first and the third properties but not the second, the second and the third properties but not the first.

This task sequence resembles those suggested by Watson and Mason (2002), who report that requests for examples satisfying various constraints can encourage students to extend their thinking beyond "typical" examples. Such example generation is also recommended by Dahlberg and Housman (1997) on the strength of its effectiveness as a learning strategy when faced with a new definition.

GENERATING AN ARGUMENT

In the second use of example objects the mathematician either builds objects or instantiates known ones with the goal of generating a proof. The professors spoke of one way in which this might be achieved directly, and another less obvious heuristic that uses an informal version of the indirect argumentation used in proof by contradiction. The direct use involves trying to show that a result is true in a specific case, in the hope that the same argument or manipulations will work in general.

> P1: It's just to get them…if they have to prove, "for all *n*, something", when they come to the induction step, and the induction step is not completely trivial, so it actually involves…actually think about it and you have to come up with an idea. And so, well how do you go about finding this idea? And I, I try to convey to them, that the first thing you do, is that…suppose I'm trying to prove it for *n* equals 10, how can I use that it's true for *n* equals 9? So, do something very specific, do a concrete example and try to reason from that.

> P5: See if you can get from 2 to 3, if you can't get from *n* to *n* plus 1.

The second, indirect case reveals a less obvious way of thinking about proving universal statements. In the excerpt below, Professor 1 talks about generating a proof for such a statement by searching for a reason why one could not find a counterexample to this statement.

> P1: …the way I often think about a proof is that, you know you imagine this as, try to beat this. Meaning, try to find a counterexample. […] If you think about, if you think

about the reason why you were failing to find a counterexample, okay, then, that sometimes gives you a clue, to why the thing is true.

When the interviewer commented that it seemed non-obvious that one would try to prove a universal statement by thinking about why there could never be cases for which it did not hold, he remarked that in fact he considered this a natural approach, and gave the following explanation.

P1: The natural…the sort of natural thing that our brains can do, is sort of build examples and check them. Okay, and…all you, you know if one thinks of universal statements as saying that it's really a statement of impossibility, it's the negation, right? It's a statement that you can't…do something. […] And the way you understand that you can't do it is by thinking about doing it.

Professor 2 describes this indirect strategy very concisely, and in doing so highlights its relationship to a straightforward way of proving an existential statement.

P2: …if it's an existential statement I look to see whether I can produce an example. And if it's a universal statement I probably try to show that I can't find a counterexample.

Professor 1's comments resemble the ideas of "mental models" theories in which human beings generate and evaluate deductions by instantiating a model of the situation under consideration, evaluating a statement relative to that model, then varying the model in a search for counterexamples (Johnson-Laird & Byrne, 1991). The difference here is that one is not looking for a counterexample, but for *a reason why one cannot build one*, which will then form the basis for an argument.

CHECKING AN ARGUMENT

The third way in which professors routinely use objects in proving is in checking the correctness of individual deductions. Professor 1 describes this process in the abstract as follows

P1: …there is a locality principle about proofs, about every proof, and that is that we somehow recognize that even though you're proving some very specific thing, that there are portions in the argument. Each portion in the argument is actually doing something more general. […] And therefore I can do a local check on this part of the argument by thinking about that more general situation and doing, examples within that more general situation […] so that's one thing that, you know, is just completely second nature to me and to most mathematicians, is that you're constantly doing those kinds of checks.

A good illustration of this process is provided by Professor 3 in the following comments on a student proof attempt.

P3: For instance, problem: "express the number 30 as the difference of two squares. Or show that it cannot be done". Answer: "It cannot be done because 30 is divisible by 6 and a number that is divisible by 6 cannot be written as the difference of two squares." Well, 12 is 16 minus 4. Ah…take any number that's a difference of two squares, multiply it by 36, you'll get a number that's the difference of two squares and is divisible by 6...

In this case, although the proof is about the number 30, a general claim is made about numbers that are divisible by 6, and it is this claim that is shown to be incorrect by considering examples in the "general situation" of numbers that are divisible by 6. Once again, the study indicated that students appear not to engage in this process to a degree that their professors would like. In this excerpt, professor 3 expresses frustration at the fact that students regularly write statements that are "obviously wrong", in the sense that they could readily be refuted using such checks.

> P3: …I don't have a clue as to…what gets them…to, to say things like that. In other words I would say, things that are obviously false. To a normal person with a little bit of mathematical education it would seem obvious that you could never say such a thing because it's so obvious that it's false. Take any example that you want, you see clearly that it's false.

DISCUSSION

Summary

This paper contributes to our understanding of the semantic aspects of proving by identifying three specific ways in which thinking about example objects can assist in this process:

1. Instantiating examples in order to understand a statement or definition.
2. Generating an argument for a universal statement, by (directly) arguing about or manipulating a specific example and translating this to a general case or (indirectly) trying to construct a counterexample and attending to why this is impossible.
3. Considering possible counterexamples to general claims in a proof.

Each of these involves the consideration of example objects in a crucial way, as opposed to algebraic manipulations or deductions based only on the form of a statement. Each is also considered natural and even "second nature" by the mathematicians who took part in this study; they regularly commented upon their surprise when students made errors that could have been avoided by their use.

There are at least two possible reasons why students may not use example objects effectively in the construction and evaluation of proofs. One is that a key difference between novice mathematics students and their teachers is simply that the teachers have access to a great deal more experience with examples (Moore, 1994). A second is that students may not be accustomed to thinking in about the objects to which statements apply, instead thinking of mathematics (including proof) as a procedural enterprise in which algebraic statements are laid out and manipulated according to certain rules or a standard format (Hoyles, 1997).

In either case, it seems that it would be helpful for students to improve their knowledge of example objects. However, experience in itself may be insufficient to support successful proof attempts; we do not wish students to offer examples in place of deductive proof. I suggest that the specific uses of example objects outlined above can help us to think about teaching students to use such objects more effectively.

Pedagogy: understanding a statement

Dahlberg and Housman (1997), Mason and Watson (2002) and Professor 1 all suggest setting tasks that require students to generate examples with given combinations of properties. Another possible design involves providing both objects and properties and asking students to decide which properties apply to which objects. I have used such a task in an introductory real analysis class. Students were presented with this list of subsets of the reals, and definitions of the given topological properties:

$\varnothing$, $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$, $\{0,1\}$, $(0,1)$, $[0,1]$, $[0,1)$, $(0,\infty)$, $\{1/n | n \in \mathbf{N}\}$, the Cantor set

Open, limit point, isolated point, closed, bounded, compact

They were asked to work in pairs and decide (without proof) which sets (or which points) satisfied which properties. Such a task design does not ask for example construction, but may be useful in cases where the objects to be considered are unfamiliar or where one wishes students to engage with non-standard examples. Certainly it goes beyond the typical lecture or book presentation of one or two "standard" examples for each definition, thus arguably encouraging the idea that we can ask about the applicability of properties more widely, and discouraging reliance on "prototypical" concept images (Vinner, 1991).

Pegagogy: generating an argument

In this study, the professors' descriptions of the indirect heuristic seem particularly interesting, since mathematicians generally find it difficult to describe the origin of the "key ideas" (in the sense of Raman, 2003) that lead to a proof. This heuristic is one way of systematically seeking such ideas, and could be articulated as part of a broad strategy for proving universal statements in which one first tries to prove the statement directly, and, if this fails, tries instead to construct counterexamples to the statement, articulating what prevents this from being accomplished.

Of course, it is often not easy to clearly articulate a mathematical claim. However, the heuristic seems no less teachable than that of trying particular examples in the hope of finding a generalizable argument. Its initial introduction may be facilitated by experience with certain tasks – Antonini suggests that questions of the form "Given A, what can you deduce?" are likely to lead to indirect arguments (Antonini, 2003). Once in place its regular use by a professor may help students to overcome the feeling that they do not know how to proceed when faced with a statement that seems "obviously true" or otherwise difficult to prove.

Pedagogy: checking an argument

This study suggested that the idea of checking an argument by considering possible counterexamples is already part of the culture of the participants' classrooms. They informally model the process when explaining their own reasoning, and give tasks that ask students to identify false statements and give counterexamples. Here I would like to suggest that the use of these strategies could be strengthened in two ways.

First, by highlighting the fact that counterexample-production tasks have the same structure as that of making a "local check" of a deduction in a proof, but that this is often disguised by the fact that instead of one line saying "For all A, B" (or "If A then B"), we have two lines of the form:

A.

Therefore/so/hence B.

Although this seems a trivial distinction to a mathematician, it may not to a student who is struggling to coordinate their understanding of the form and content of an argument.

Second, by giving specific consideration to how to decide which objects to check. The subtleties involved in this process are highlighted by the following example of a student proof attempt given by professor 3.

> P3: …for example, I can show you a homework problem in which a student is trying to prove that the number 1007 is prime, and he said "well 7 is prime, and adding 1000 doesn't change anything". End of proof.

Faced with such a claim, the mathematician infers a general statement about adding primes to other numbers, and searches for a counterexample to this. However, in this case there are at least three possible general statements, and the central claim is not couched in language that makes its logical structure clear. Indeed, more acceptable mathematical writing also contains such "suppressed quantifiers" (Selden & Selden, 1995). Thus, being able to re-frame a statement in more appropriate language is closely connected with deciding which objects to check. Emphasizing this when teaching appropriate "mathematical language" may help to explain the need for clarity as well as facilitating checking.

Further research

Having identified these uses of example objects as an important feature of mathematicians' thinking, and as one that is often lacking in students, two questions arise: (1) Does this lack account for the failure of students in introductory proof courses? (2) If so, can teaching that focuses more on the underlying mathematical objects help students to be more successful? The first of these questions is being investigated now in a study comparing explanations of proof attempts by students in the Introduction to Mathematical Reasoning course.

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