CHILDREN'S DEVELOPMENT OF STRUCTURE IN EARLY MATHEMATICS

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In a descriptive study of 103 Grade 1 children, including 16 longitudinal case studies, we investigated the use of mathematical structure across a range of mathematical tasks. Children's structural development was inferred from their imagistic representations, supported by analysis of problem-solving behavior during videotaped task-based clinical interviews. We describe with examples how internal representational systems change through four stages of structural development.

In this paper we describe an analysis of mathematical structure present in children's representations as they solve five mathematical tasks during clinical interviews: time, *triangular pattern, area, length* and *picture graph*. Goldin (2002) emphasises that individual representational configurations, whether external or internal, cannot be understood in isolation. Rather they occur within representational systems. These are not mere collections of representations, but have complex structures that in practice may be ambiguously defined or context-dependent. In this study we build upon previous analyses (De Windt-King & Goldin, 2001; Goldin & Passantino, 1996; Mulligan, 2002; Mulligan, Mitchelmore, Outhred & Russell, 1997; Thomas, Mulligan & Goldin, 2002), and extend the mathematical content domains with the aim of making as explicit as possible the bases for children's structural development.

BACKGROUND

In a longitudinal study of Grade 2 to 3 children's intuitive models for multiplication and division, Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular word problem did not necessarily reflect any specific problem feature but rather *the mathematical structure that the student was able to impose on it*. Students acquired increasingly sophisticated strategies based on an equal-groups structure and their calculation procedures reflected this structure.

Another 2-year longitudinal study of Grade 2 to 3 children's representations focused on how they imposed structure, or lack thereof, on numerical situations (Mulligan et al., 1997). Low achievers were more likely to produce poorly organised, pictorial and iconic representations lacking in structure. These children lacked flexibility in their thinking; they were barely able to replicate models of groups, arrays, or patterns. Low achievement was attributed to children's dependence on unitary counting and weak visualisation skills. A follow-up study of 24 of these children tracked to Grade 5 indicated that low achievers consistently lacked mathematical structure; pictorial and iconic representations dominated responses (Mulligan, 2002). High achievers, however, used abstract notational representations with well-developed structures from the outset in Grade 2.

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Further studies investigating children's mental images of number concepts have identified structural elements such as grouping, regrouping, partitioning and patterning found within the recordings of the numbers 1 to 100 (Thomas et al., 2002). Mathematically gifted children's representations showed recognisable mathematical structure and dynamic imagery, whereas low achieving children's representations showed no signs of underlying structure, and use of static imagery.

Our interest in mathematical structure has also drawn upon research on imagery and spatial visualisation (Battista, 1999). Imagery appears to be a central influence in structural development of mathematical ideas (Pirie & Martin, 2000). Battista (1999) defines *spatial structuring* as:

> the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object. (p. 418)

Children's spatial structuring has been highlighted in studies of two and three-dimensional situations such as arrays of squares in rectangles, and cubes in rectangular boxes (Battista, 1999; Battista, Clements, Arnoff, Battista, & Borrow, 1998). Most children learn to construct the row-by-column structure of rectangular arrays and also acquire the equal-groups structure required for counting rows and layers in multiples (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996).

The following conclusions are drawn as common findings from these studies:

- The more that the child's internal representational system has developed structurally, the more coherent, well-organised, and stable in its structural aspects will be their external representations, and the more mathematically competent the child will be.

- A child's external imagery reflects the structural features of his or her internal representations, and this provides a view of the child's conceptual understanding.

- There are common structural features evident in children's representations across mathematical content domains; spatial structuring is a critical feature.

THE PURPOSE OF THIS STUDY

Our analyses of structural development has not documented early stages sufficiently, and has relied on specific number tasks. We therefore posed the research question:

- Do young children (children aged from 5 years 6 months) develop and use structure consistently across different mathematical content domains and contexts?

METHOD

The sample comprised 103 Grade 1 children, 55 girls and 48 boys, ranging from 5.5 to 6.7 years of age who were administered task-based interviews. This broadly representative sample was drawn from nine state schools in Sydney. A sub-sample of

sixteen children representing extremes in mathematical ability was selected for in-depth case study over two years on the basis of initial interview data.

Thirty tasks were developed to explore common elements of children's use of mathematical and spatial structure within number, measurement, space and data. Each task required children to use or represent elements of mathematical structure such as equal groups or units, spatial structure such as rows or columns, or numerical and geometrical patterns. Number tasks included subitizing, counting in multiples, fractions and partitioning, combinations and sharing. Space and data tasks included a triangular pattern, visualising and filling a box, and completing a picture graph. Measurement tasks investigated units of length, area, volume, mass and time. This paper reports an analysis of children's responses to the *time, triangular pattern, area, length,* and *picture graph* tasks (see Table 1).

Table 1: Selected interview tasks

Time: clockface	Draw a clock and show 8 o'clock on it.
Triangular pattern	(Show flash card with triangular pattern of six dots.) Draw from memory exactly what you saw.
Area: unitizing	(Show 3x4 rectangle with squares drawn along two adjacent sides.). Finish drawing the squares, exactly like these, to cover all of this shape.
Length: ruler	(Show drawing of long thin rectangle.) Imagine this strip is a ruler. Draw as many things as you can remember about a ruler.
Picture graph	(Show table of data.) There are 7 dogs, 5 cats, 3 birds. Finish drawing the graph to show the number of animals.

Children were asked to visualise, then draw and explain their mental images. They were given plenty of time and opportunity to provide alternate solutions and reproduce or modify their drawings. Operational definitions and a refined coding system were formulated from the range of responses elicited in pilot interviews and in accordance with coding employed in previous studies. A coding system was complied from analysis of videotapes with an inter-rater reliability of 94%.

Children's representations were considered with respect to three dimensions: (i) the type of component signs in the representation (pictorial, iconic or notational); (ii) the stage of structural development evidenced by features such as equal units; and (iii) the static or dynamic nature of the image represented. Additionally, we considered (iv) the verbal explanations describing the representation.

RESULTS

Children's responses indicated four broad stages of structural development:

1. In an initial *pre-structural* stage, representations lacked any evidence of mathematical or spatial structure; most examples showed idiosyncratic features.

2. This is followed by an *emergent* inventive-semiotic stage where representations show some elements of structure and in which characters or configurations are first given meaning in relation to previously constructed representations.

3. The next stage shows evidence of *partial structure*: Some aspects of mathematical notation and/or spatial features such as grids or arrays are found.

4. The following stage is a *stage of structural development*, where the representations clearly integrate mathematical and spatial structural features.

In order to illustrate levels of structural development, we discuss representative examples below. (Codes such as C1, T2 refer to individual drawings, not to stages.) We focus on the child's imagistic representations and how these conform to mathematical aspects such as the correct numerical quantity, use of formal notation, uniformity in size of units, and spatial organization of units.

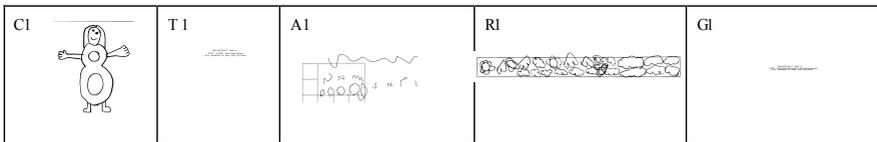**Pre-structural stage of development**

Figure 1: Stage of pre-structural development

Figure 1 shows representations that reflect some features of the mathematical tasks, such as a dog in G1, but there is no evidence of mathematical or spatial structure. Idiosyncratic features dominate, for example, depiction of a clock (C1) as a girl - although there is an attempt to represent the numeral 8. In the triangular pattern (T1), the row of circles bears no relationship to the triangular shape, numerical pattern, or quantity of the circles in the stimulus picture.

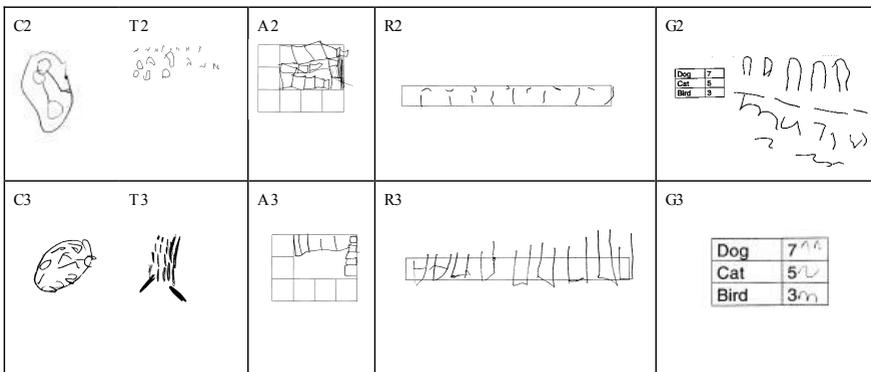**Emergent, inventive-semiotic stage of development**

Figure 2: Stage of emergent structural development

Figure 2 shows evidence of emergent, inventive-semiotic representations. These include iconic, pictorial, and notational signs with some structural feature but focus on the aspect of the task representation most significant to the child. Figure T2 shows squiggles that are unrelated to the triangular pattern of dots but there is at least a representation of the correct quantity of circles. Figure T3 shows a triangular form drawn as a 'Christmas tree' and an attempt to draw the pattern as vertical rows of five dots. There is little awareness of the structure or number of items in the pattern; there is some indication of spatial structure with equally spaced marks. We found evidence of meaning assigned to idiosyncratic characters where attempts are made to use icons (dashes, dots, lines, objects), and symbols to represent numeral quantity, equal units or 'spatial structure'.
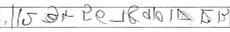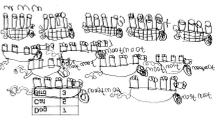
Stage of partial structural development

Figure 3: Stage of partial structural development

At this stage children depict partial structure in their representations, with at least one structural element; recordings were more organised but independent of idiosyncratic features. For example, drawings G4 and G5 show correct quantities, but there is no attempt to draw rows or align items with the grid provided. Representations of animals dominate, but these images are directly related to the task content. Representations for the area task show an awareness of structure, consistent with the findings of Outhred & Mitchelmore (2000). Drawings A4 and A5 show partial structure as a correct number and size of squares in a border pattern, and correct number and alignment of individual squares respectively. These examples do not, however, show any indication that rows and columns are coordinated – although the equal groups structure of multiplication may be emerging.
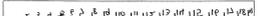
Stage of structural development

Figure 4: Stage of structural development

Responses showing well-developed features of mathematical and spatial structure are depicted in all the examples in Figure 4. These show clear evidence of an internal system that uses common elements of mathematical structure (e.g. the use of equal sized units in the clock face, area and ruler examples; the integration of spatial and numerical pattern in the triangular pattern task; and one-to-one matching and horizontal alignment in the graphs task). In the ruler task, structural features such as equal-size units were often symbolised as centimetres with subdivision, indicating conceptual understanding of linear measurement prior to formal instruction (Bragg & Outhred, 2000; Nuhrenborger, 2002).

The above results show that children's responses to the tasks can be reliably coded for the presence of structural features to obtain a developmental sequence. We questioned whether this structural development is consistent across tasks. Responses to all 30 tasks were coded for all 103 children and the matrix examined for patterns. It was found that the children could be unambiguously sorted into four groups. In Group 1, 95% or more of their 30 responses fell in the pre-structural stage. In Group 2, 80% or more of the responses fell in the stage of emergent structural development, and in Group 3, 80% or more of the responses fell in the stage of partial structural development. In Group 4, 95% or more of the responses fell in the stage of structural development. These results indicate an astonishing consistency of structural representation across tasks.

Case studies of low achieving children indicated that the key difficulty was not necessarily the comprehension of tasks or their ability to draw or to count, but the child's lack of perception of structure. All the low achieving children in fact fell into Group 1. Conversely, the high achieving children all fell into Group 4 and readily expressed the mathematical structure in all or almost all of the tasks.

DISCUSSION

This study has produced two extremely important findings:

- Young children's perception and representation of mathematical structure generalises across a wide variety of mathematical tasks.
- Early school mathematics achievement is strongly correlated with the child's development and perception of mathematical structure.

The first result supports what many have said about the importance of structure in mathematical understanding (Gray, Pitta & Tall, 2000). This study, however, advances our understanding by showing that stages of structural development can be measured and described across a wide variety of tasks.

We must clearly ask why it is that some young children do not develop structure in their representations of critical mathematical concepts. Teaching young children to focus on structure in early mathematics may require professionals to pay more attention to children's representations and to help them focus more clearly on the mathematical structure inherent in the various situations encountered. In individual studies, a structural approach has already shown benefits in terms of children's understanding of length and area. Consistent application of a structural approach (e.g. assisting children to visualise and record simple spatial patterns accurately) could lead to much broader improvements in children's mathematical understanding. We plan to undertake longitudinal investigations (using multiple case studies) to track the structural development of low and high achieving children from school entry through to primary level. We aim to identify classroom influences that tend to promote or impede the development of structure in children's representations.

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