EXTENDING LINEAR MODELS TO NON-LINEAR CONTEXTS: AN IN-DEPTH STUDY ABOUT TWO UNIVERSITY STUDENTS' MATHEMATICAL PRODUCTIONS[1]

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This research report presents a study of the work of agronomy majors in which an extension of linear models to non-linear contexts can be observed. By linear models we mean the model y=a.x+b, some particular representations of direct proportionality and the diagram for the rule of three. Its presence and persistence in different types of problems and teaching contexts have drawn us to search for alternative explanations; we employ a qualitative methodology using individual interviews and students' mathematical written tasks. The data allowed us to make an in-depth descriptive analysis of students' strategies when solving non-linear problems and their reasons to decide the model to be applied.

THEORETICAL AND EMPIRICAL BACKGROUND

This study is related to the mathematical understanding of agronomy majors from the University of Córdoba (Argentina). It extends a previous research (Esteley, Villarreal and Alagia, 2001) in which we focus on the documentation, description and analysis of a phenomenon that occurs among these students, which we denominate *extension of lineal models to non-linear contexts* or *overgeneralization of linear models*. By linear models we mean the model $y=a.x+b$, some particular representations of direct proportionality and the diagram for the rule of three. Such a phenomenon occurs when the resolution of certain mathematical questions relating two variables, is solved applying linear models, even though the situation, from the teacher's point of view, is obviously a non-linear one. The presence of this phenomenon doesn't necessarily imply that the students are conscious that they are applying linear models in non-linear contexts.

This phenomenon has been studied with students of the elementary school, with focus on the particular representation of direct proportionality and it is known as *linear misconception*, *illusion of proportionality* or *linearity* and also *proportionality trap* (Behr, Hare, Post & Lesh, 1992). The tendency of overgeneralising the use of linear models beyond its range of validity is also present in secondary school pupils. The extensive studies of De Bock, Von Doorem, Janssens & Verschaffel (2002), De Bock, Von Doorem, Verschaffel & Janssens (2001) and De Bock, Verschaffel & Janssens (1998), carried out with 12-16-year old students, reveal a strong tendency to apply linear models to solve proportional and non-proportional word problems about the relationship between lengths and perimeters/areas/volumes of similar figures or solids. These authors carried out research based on written tests and some in-depth interviews. The explanation for the illusion of linearity has been investigated mainly from the perspective of the error as a learners' deficiency, and an intuitive approach towards mathematical problems, inadaptative beliefs and attitudes or poor uses of heuristics are indicated as factors of the student's unwarranted proportional reasoning (Van Doren, De Bock, De Bolle, Janssens & Verschaffel, 2003). The literature is extensive reporting on primary and secondary school students' illusion of linearity and there exists agreement in describing the phenomenon as persistent and resistant to change. Nevertheless studies on this phenomenon among university students are not frequent, even though its presence and persistence have been frequently observed at that level within diverse types of problems and contexts. That situation led us to carry out an exploratory study (Esteley et al, 2001) to document, describe and analyze the presence of the phenomenon of overgeneralization of linear models among Argentinean 18-20-year students, which studied Agronomy in the University of Córdoba. We analyzed, through the students' written productions, the types of problems that were solved by extension of a linear model, the strategies followed by the students and the difficulties of interpretation that could be associated with the statements of some problems

At this point, we decided to yield explanations beyond the notion of the error as students' deficiency or fault. Therefore, in our studies, the errors were assumed as symptoms of the conceptions underlying the students' mathematical activities, in the sense of Ginsburg (1977) or Brousseau (in Balacheff, 1984). After our exploratory study, briefly described above, we decided to deepen our investigation, performing interviews with those students that, in our previous studies, had applied linear models to solve non-linear problems. In this sense, the studies of Confrey (1991, 1994) and Confrey & Smith (1994) addressing questions about the epistemological value of the students' mathematical constructions are paramount for us. Confrey (1991) argues that to understand the students' actions implies to be introduced in their perspective and not to presuppose that it coincides with that of the teacher/researcher. The students' answers that stray from the expectation of the teacher/researcher, can be legitimate as alternative or valid and effective in other contexts. To encourage the student to show their points of view implies for the teacher/researcher an opportunity to glimpse at the students' perspectives and to question her/his own, examining them through the ideas of the students.

RESEARCH METHODOLOGY

The research methodology was qualitative (Lincoln & Guba, 1985) since we aimed to get in-depth understanding of the students' thinking processes when they extend linear models to non-linear contexts. Individual semi-structured interviews were performed with 18-20-year old agronomy majors from the University of Córdoba (UNC) that were attending a calculus course. These students had shown the appearance of the phenomenon of interest in our previous studies. We carried out a single an-hour-long interview with each student. The interviewer was not the students' teacher. All interviews were audio-taped and paper, pencil, and a scientific calculator were at the interviewees' disposal.

The interviews were structured around the activities and aims we describe next. Activity 1) Ask the student to explain the way she/he had solved the Problem A (see Figure 1) in our previous study (Villarreal et al, in press) with the aim of elicit the student's strategies and reasons to apply a linear model in a non-linear context. Activity 2) Propose to solve Problem B (see Fig. 1) with the aim of elicit the student's strategies while solving a new non-linear word problem. Activity 3) Ask to calculate the height of the plant in Problem B at any time with the aim of challenge the student to produce a general model and verify the consistency of the student's strategy. In all the activities we asked the students to think aloud (Ginsburg, Kossan, Schwartz & Swanson, 1982) while solving the problems. The selected problems are typical in the introductory mathematics course for the agronomy majors from UNC.

We did an inductive/constructive analysis (Lincoln & Guba, 1985), since we didn't raise a priori hypothesis, but rather, we generate conjectures from the gathered data. The analysis of the students' strategies and solutions to the problems was not carried out in terms of "right or wrong". Although conceptions not accepted as correct by the mathematicians can be indicated, the emphasis was on students' thinking processes, without making comparisons, but trying to listen closely (Confrey, 1994).

Problem A) *Say if the following statement is true or false and justify your answer. An insect, that weighs 30 gr. when being born, increases its weight at 20% monthly. Then, its weight after two months is 43,2 gr.*

Problem B) *If a plant measures, at the beginning of an experiment, 30 cm and every month its height increases 50% of the height of the previous month, how much will it measure after 3 months?*

Figure 1: Problems A and B

RESULTS AND ANALYSIS

In this report we decided to present the results related to the interviews performed with Santiago and Clelia. In the selected excerpts, we underlined some of the students' assertions to indicate the words we believe support our analysis; we include, between brackets, explanations to better understand the students' expressions or words that give continuity to the text; [...] indicates long pauses, meanwhile short pauses are represented with simple dots.

Santiago

Santiago had solved Problem A as follows: he wrote y(t) = 30 + 6.t, he calculated y(2) and finally he answered that the statement was false because "*42 gr ≠ 43.2gr*"

After looking at his written solution, during activity 1, Santiago said: *"I wanted to do it using a function ... and not with a rule of three* [...] *Then, I realized that it wasn't a linear function because that* [he refers to the insect] *hasn't an unlimited growth, it was an exponential or something like that"*. When Santiago rejected his linear solution and gave his justification, it became apparent to us that he was using biological reasons instead of mathematical ones to select a new model. Although Santiago realized that the insect growth was not linear but exponential, he stated: *"When I did realized that it was an exponential, I also realized that I wasn't able how to do it, because, I don't know* [...] *it seems to me that I don't have the tools yet"*.

Santiago considers the rule of three as a mechanical procedure with a low mathematical status, at least, for university students and that is why he decided to use the linear function $y(x) = a.x + b$. We can also point out that the student use biological reasons to reject his initial linear solution and try a different approach, although he considered conditions not given in the problem statement. We could also recognized these aspects when Santiago solve Problem B), showing a strong consistency.

During activity 2) and, after reading Problem B), the following dialogue occurred:

Santiago: the height of the plant is 30 cm at the beginning of the experiment, so, that is the "base", and each month it grows up 50% of the height it had the previous month …

Interviewer: yes

Santiago: well, I do it with the rul… well in a sort of mechanical way, the first month it would be 30... plus the 50% of 30. [he used the calculator and wrote the first line in Fig. 2] In the second month I start at 45 plus 50% of 45, that would be... [he wrote the second line in Fig. 2] and in the third month I start at 67.5 and I do the same [he wrote the third line in Fig. 2]

30 cm	$\rightarrow 30 + 15 = 45$	*1 mes*	[1st month]
45 cm	$\rightarrow 45 + 22.5 = 67.5$	*2 mes*	[2nd month]
67.5 cm	$\rightarrow 67.5 + 33.75 = 101.25$	*3 mes*	[3rd month]

Figure 2: Santiago's written solution of Problem B

When the interviewer asked how to calculate the height of the plant after twelve months or at any time *t*, the next dialogue took place:

Santiago: I should do it with a function, to make it easier [...] it has to be an exponential and... because the growth has to be like this [he makes a gesture with his hand indicating an S- like line], it cannot grow up indefinitely… besides, because the variable is changing.

Interviewer: what do you mean when you say "the variable is changing"?

Santiago: because when I get to the first month, I see that it changes, I stop working with 30 and I start working with 45 and then, I change from 45 to 67… It is like that it is always moving beyond. I should see if there is any ..., no, I don't know if it is possible to have a relationship of growth, no, no, no ... if the 45 has the same increment proportion with this… but not… that from 30 to 45 it jumps the same, no, neither here, no, I don't know.

Interviewer: what do you mean with: “it jumps”?

Santiago explained that he was trying to find a relationship between the height of the plant for each month. He talked about a "parameter" that would show the growth. In order to get it he drew a line as the one sketched in Fig. 3. The student started searching for that "parameter" calculating the following differences: *45 - 30; 67.5 - 45* and *101.25 - 67.5*. Santiago wrote those differences in the second row of numbers in the Fig. 3. He indicated that the increment from 30 to 45 was not the same as that from 45 to 67.5 and from 67.5 to 101.25. After that, he continued searching for the "parameter" but, this time, doing the differences between the values in the second row. He calculates *22.5 - 15* and wrote *7.5* on third row (see Fig.3). While working with the calculator, he realized that adding *7.5* to *22.5* he wouldn't obtain *33.75*. After a while, he gave up this strategy and said he should have to look for information in a textbook where a solution of a similar problem could appear.

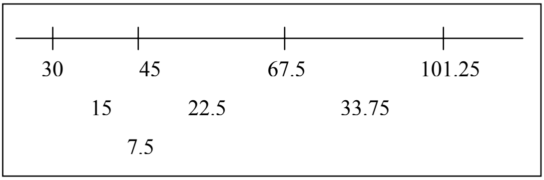

Figure 3: Sketch of Santiago's line

Therefore he tried with other models. Firstly, he drew an upward pointing parabola and immediately rejected it because of its *"unlimited increasing"* and *"time couldn't be negative"*. Then, he proposed $y = a^x + b$ and when the interviewer asked him for the value of b, he said it would be *"the initial 30 cm"* but, he finally said: *"That one* [he referred to $y = a^x + b$] *doesn't help me, either, since it also has an unlimited growth, it must be something tending to a number... something that bends down* [he drew an increasing graph with an asymptote].

The graphical representations of a limited growth that Santiago considered are consistent with the biological conditions he added to the problems. We should point out that his option for searching an additive constant (the "parameter") to model the variation of growth finally became an obstacle for him.

Clelia

Clelia solved problem A as follows: she wrote the initial weight of the insect as p_i=*30gr*, she calculated the 20% of p_i using a rule of three (see Fig. 4) and continued working as it is shown in Fig. 4.

30 gr.----------100% p_i *6gr. = x------- 20% p_i* *1 mes pesa 30 + 20 % de 30 = 3* *2 meses pesa = (30 + 20 /100) 2 + 30 gr* *12 gr. + 30 = 42 gr*	[In 1 month weights...] [In 2 months weights...]

Figure 4: Clelia's written solution of Problem A

Finally, she concluded that the statement was false since after two months the insect would weigh 42 gr. From Clelia's written solution we can infer that, for her, *(30 + 20/100)* is equivalent to *6*, and that the implicit model for the insect weight, behind her calculations, is $w(t)= 6.t + 30$. After observing her written solution, Clelia said: *"I thought on it and then, talking with my classmates, we realized that we had to calculate the 20% of the weight, month by month, while the weight was increasing, <u>not to the initial weight</u>... we added in that way".* Clelia referred to the fact that she was assuming that the insect always grows 6 gr. every month.

During activity 2), after Clelia had read Problem B), the following dialogue occurred:

Clelia:	After a month it will measures 30 [...] plus 50% .... of 30... This is for one month, and for three months [...] <u>all this times three</u> [she laughed]
Interviewer:	please, continue
Clelia:	I should calculate one by one to make it easier... 50% of 30 [she made a pause while whispering something inaudible and stopped]
Interviewer:	you have just said "all this times three", what do you mean by "all this"?
Clelia:	no, it just seemed to me
Interviewer:	it doesn't matter, when you said "all this", what were you referring to?
Clelia:	To the 50% of 30 plus 30

At this point, Clelia gave up her approach, probably because it was the same that she had used to solve Problem A, and she knew that it wasn't right. After a while, she started using a calculator and finding a value for each month as it is shown in Fig. 5.

1er. Mes $30 + \frac{50}{100} 30 = 45$	[1st month]
2do. Mes $30 + \frac{50}{100} 45 = 52,5$	[2nd month]
3er. Mes $30 + \frac{50}{100} 52,5 = 56, 25$	[3rd month]

Figure 5: Clelia's written solution of Problem B

When the interviewer asked Clelia to calculate the height of the plant after twelve months she answered: *"I don't know. We know that plants don't grow unlimited, at* *some moment they stop growing... Well, I would calculate the 50% of the height from the previous month and add* [such percentage] *to it* [referring to the previous height of the plant]. After talking about this particular case, Clelia wrote the following formulae as a model for the growth of the plant:

$$h = 30 + \left(\frac{50}{100}\,x\right) \qquad x = \textit{altura del mes anterior} \text{ [height of previous month]}$$

In this way, Clelia wrote down an expression in which the height of the plant for each month was calculated from its height in the previous month. Finally she couldn't get a height-time expression.

The interviewer asked to Clelia what she was thinking about when she spontaneously took the decision of always adding 30 cm to the percentages she had been calculating while solving Problem B (see Fig. 5), and she answered:

Clelia: The existence of a constant [with emphasis in her voice]. It begins here, and <u>to this one... I have to add</u>

Interviewer: Is that what you were thinking on?

Clelia: Yes, I always notice that [with strong emphasis in her voice]

We would like to point out the strong and consistent presence of an initial value (30 in both problems) to which the student always adds the variation in and weight and height.

CONCLUSIONS AND DISCUSSION

The interviews provided us with relevant information about the strategies and thinking processes followed by the interviewees while working with non-linear problems. In that sense we point out that Santiago and Clelia share some important aspects. Both students chose a linear model to solve problems A, but while Santiago explicitly said that he had applied a linear function, Clelia didn't say anything about the kind of model she had applied. When the interviewer challenged them to find a general model both students regarded the growth (for the insect or the plant) as an "additive model" in the form of 30 + (something variable in time). We called it additive model since the initial value (either the weight or the height) is always added to the next variation. For example, in the case of Santiago he considered $y(t) = 30 + 6.t$ in Problem A or $y(t) = a^t + 30$ in Problem B. In the case of Clelia she proposed a recursive relation that enables her to calculate the height of the plant (of Problem B) at time t adding to 30 cm its height at time $t-1$. Both students introduced biological constraints (unlimited increasing), external to the statements of the problems and proposed graphical models that were consistent with those biological constrains. The additive model and the external conditions finally became obstacles for obtaining the general model that accounts for the situation posed in activity 3. We must recognize that the students were able to establish connections with reality, which is considered a positive habit for agronomy students. Finally we want to point out that although the students had the mathematical tools necessary to construct a general algebraic model, they couldn't get it. This last observation with the results reported suggest that the overgeneralization problem is not a trivial one, goes beyond school levels and needs a thorough investigation.

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