INSIGHT INTO PUPILS' UNDERSTANDING OF INFINITY IN A GEOMETRICAL CONTEXT

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Abstract. *School students' intuitive concepts of infinity are gained from personal experiences and in many cases the tacit models built up by them are inconsistent. The paper describes and analyses a series of tasks which were developed to enable the researchers to look into the mental processes used by students when they are thinking about infinity and to help the students to clarify their thoughts on the topic.*

INTRODUCTION

Students' understanding of infinity is determined by life experiences, on the base of which they have developed tacit models of infinity in their minds. These tacit models (Fischbein, 2001) deal with repeated, everlasting processes, such as going on forever, continued sub-division etc. which are considered mathematically as potential infinity, or by the consideration of a sequence Boero et al (2003) called sequential infinity. Basic concepts of mathematical analysis are usually introduced through limiting processes, which also reinforces the idea of infinity as a process. However, the symbol ∞ is used as a number when tasks about limits and some other concepts of mathematical analysis are solved.

When we consider the historical development of infinity, philosophers and mathematicians were dealing with this concept only in its potential form for over 1500 years. Moreno and Waldegg (1991) noted how a grammatical analysis of the use of infinity could be applied to Greek culture. The Greeks used the word 'infinity' in a mathematical context only as an adverb, indicating an infinitely continuing action, which is an expression of potential infinity and it was not until 1877that actual infinity, expressed as a noun in mathematics, was defined by Cantor. His ideas were only accepted years later by the general community of mathematicians, following the initial strong denial of its existence. Nevertheless secondary and university students usually go from their intuitive ideas on infinity, which are not purposefully developed during their primary and secondary school years, to lectures which require their understanding of actual infinity. This lack of development must contribute to the difficulties the students find with this concept. Actual infinity contradicts many of their intuitive ideas of infinity. The historical development of the understanding of the concept of infinity gives us the clear answer to the question 'What intuition of infinity is more natural for pupils and why do students have such problems with accepting actual infinity?'

THEORETICAL FRAMEWORK

Most previous research in the area of infinity has been with older students, upper secondary and university students (Tirosh & Tsamir, 1994; Monaghan, 1986; Tsamir, 2002). Most researches have looked for difficulties students have in understanding actual infinity and consequently difficulties with the understanding of such mathematical concepts as infinitesimal calculus, series, limits, continuity (Tirosh, Fischbein & Dor, 1985, Nuñez, 1993).

Monaghan (2001) argues that pupils mainly think of infinity as a process and that even when pupils use the phrase 'Going towards infinity' they are not necessarily thinking of infinity as an object which could be interpreted as actual infinity. In our earlier research (Jirotková & Littler, 2003) we have found it difficult to determine whether or not the students who use the noun infinity have any idea of the difference between actual and potential infinity; we believe they do not. They use the words infinity and infinitely often in the same statement with no real differentiation between the two.

We believe that initially pupils get their main information about infinity away from the classroom environment. They live in a space exploration age and hear about satellites going into deep space 'which is never ending'. It is such experiences which set up the tacit models, which Fischbein (2001) states are not consciously controlled and which may lead to wrong interpretations and contradictions later. We believe that these tacit models themselves are often internally contradictory. The pupils live in a real world, which is perceived as finite and there is little opportunity for them to relate to or discuss about infinity in school. We are not aware of any secondary school syllabus in the Czech Republic or the UK in which infinity is given as part of the curriculum.

Boero, et al (2003) undertook research with fifth grade classes and one of their findings was that more than a third of the students undertaking the research changed their understanding from a finite position to an infinite position or visa-versa when discussing infinity related to number problems. This would indicate that the pupils' knowledge is not firm at this stage and that listening to arguments within a group can easily alter their perceptions. This finding supports our aim of challenging teachers to make use of any opportunity to discuss the concept of infinity with their pupils of any age. The concept of infinity is difficult; the ontogenesis of this concept gives evidence for this. It is important that the phylogenesis of the concept should correspond to the ontogenesis.

In our previous paper on this topic (Jirotková & Littler, 2003) we gave the results of analysing the written definitions of the concept of a straight line given by 72 teacher-students. The method of analysis, which we developed during the initial stage of our research, was based on taking simple mental units, which involved the word infinity and any words derived from it out of students' responses, and classifying them on three levels. The criterion for the second level of classification was the grammatical use of infinity – noun, adjective or adverb, and indirect expression of infinity (Moreno & Waldegg, 1991). This classification proved to be a useful tool to help us to identify phenomena which characterised and described students' understanding of infinity. We were aware that our analysis was dependent on our interpretation of the student's writings and that the students themselves might not have the communication skills to express what was in their minds succinctly (Jirotková &

Littler, 2002). This is why we tried to find as many plausible interpretations as possible, developed additional tasks to diagnose the most likely interpretation and went on with our research.

METHODOLOGY

Our aims for the follow-up research were 1) to use the initial task given to university students with school students, and do a comparative analysis of the results of Czech and English students; 2) to make use of the tasks we had developed as a research tool to help us to look at and analyse the mental processes which school students used when writing about infinity.

Pilot Study

In March 2003 we asked 27 Czech and 35 English school students, of 11-12 years of age, to write down their definition of a straight line. No mention was made of the word infinity because we wanted the students to use the term spontaneously in their responses. In most research in this field almost all the experiments have been carried out on a one-to-one basis with the researcher interacting with the pupil. Boero et al (2003) used class discussion to counteract this phenomenon, whereas we used the written form of response to avoid distorting the results by an adult.

The English students did not provide any phrases which included the word infinity which meant that we were unable to do any comparative analysis. We looked for the reason for this unusual and surprising phenomenon. The main reason found was in the differences in geometrical syllabuses for the two countries. The Czech geometrical syllabuses since 1976 are based on a deductive approach starting with concepts of point, which was considered as the simplest concept from the point of view of the axiomatic building of geometry, line segments, ray and straight line and so on. This approach required the pupils to speak about extending bounded objects into infinity (line segments into straight line etc.). Moreover the word infinity is used in common language in Czech. The English pupils had received a much more Euclidean approach to straight line, in which the difference between straight line and line segment is not stressed which corresponds to Euclid's concept 'euthea' used in his work Stoichea.

Main Study

The pilot study resulted in the development of a series of 7 tasks in which students' ideas from our previous research were used. We gave them to 44 Czech and 54 English pupils of 11-15 years of age in June 2003. The series involved some tasks which we devised following the analysis of the student-teachers' responses from our previous research (Jirotková & Littler, 2003) for those students who provided phrases in their definitions, which contradicted each other or phrases which could be interpreted in different ways. These tasks helped us to clarify our interpretations of students' phrases and provided us with good evidence about the respondents' intuitive ideas of infinity. We used a geometrical context in all but one of the tasks because we believe that the geometrical world provides unique and exceptional possibilities to develop the understanding of the concept of infinity and infinite processes (Vopěnka, 1989).

The seven tasks were set out on paper so that the pupils could record their answers on the same sheet. The tasks were set during their normal mathematics lessons and there was no time limit. Generally the time needed was between 20 and 40 minutes.

Task 1. Two boys are talking. Decide which of them is correct.
Adam: *A straight line has two 'infinities'. If I go in one direction I'll reach infinity. If I go in the opposite direction I'll also reach infinity.*
Boris: *Those two 'infinities' are the same, so there is only one infinity on a straight line. It is the place where both ends join together like a circle.*
Write the name of the boy who you think is correct. Explain why.

Task 2. Three girls are talking. Decide which of them is correct.
Cecilie: *Two parallel rays end in two different infinite points.*
Dana: *I do not agree. They end in the same infinite point.*
Eva: *Neither of you is correct. A ray goes on and on and never ends.*
Write the name of the girl who you think is correct. Explain why.

Task 3. Given a line segment *AB*. We extend it twice, three times, …, one thousand times, …, infinitely many times. Describe the resulting object and give its name

Task 4. Given a line segment *AB* whose end points *A* and *B* have been cut off. We extend it twice, three times, …, one thousand times, …, infinitely many times. Describe the resulting object and give its name.
Are the resulting objects from the above tasks 3 and 4 the same? If not, explain why not.

Task 5. Frank states that he can write down the smallest positive decimal number. Is this possible?
If YES, write down the number. If NO, give reasons.

Task 6. Given a semicircle with diameter *AB* [diagrams were drawn]. Consider all the possible triangles *ABC* having the vertex *C* on the circumference of the semicircle. Draw two triangles *ABC* such that the height *h* (from *C* to *AB*) is (a) the greatest possible, (b) the smallest possible.
Describe as precisely as possible, the position of point *C* in each case.

Task 7. Given a straight line *b* and a point *A* not lying on *b*. Consider all squares *ABCD* whose vertex *B* is on the straight line *b*. Draw square *ABCD* with **(a)** the smallest possible area, (b) the greatest possible area. Describe position of the point *B*. Draw the diagonals *AC* and *BD* and mark the centre *S* of the square *ABCD*. If you do not have enough room on your paper to mark any of the points, draw arrows to indicate the direction in which they lie. [Diagrams were provided.]

Task 1 gives two statements which argue for either one or two infinities. We are aware that both ideas are supported by some mathematical theory so that no answer is unambiguously correct. First we will describe what each child's hypothetical statements in tasks expresses.

Three ideas about a straight line occur in the five responses given in tasks 1 and 2. A straight line is like:
1) A line segment extended in both directions so that its end points are put into infinity. It is homomorphic to a closed line segment. This idea is expressed by Adam.
2) A circle which is created by extending a line segment into infinity where the two end points are joined. It is homomorphic to a projective straight line. This idea is explicitly expressed in Boris's statement.
3) A line segment without its end points infinitely extended in both directions. It is homomorphic to an affine straight line. This idea is expressed in Eva's statement.

Eva's understanding of a straight line is potentially infinite whereas Boris's and Adam's are quite clearly actually infinite. For Adam infinity exists even at both ends of a straight line, and it is reachable by man although it is very far away. Saying *I'll reach infinity* he finalises the process and by that he actualises it. Infinity is placed before the horizon and it is possible to handle it as if it is a final phenomenon. For Boris, Cecilie and Dana infinity also exists but they do not mention the possibility of being able to reach it. Only Eva refuses the existence of infinity for a ray and is the only one who supports potential infinity. Boris expresses his interpretation of Adams's statement – a straight line has two different infinities. He does not refute anything that Adam said but just gives his own interpretation. It is possible to derive Cecilie's understanding of two infinite points very naturally: Let us consider a rectangle *ABCD*. Fix the side *AB* and take the side *CD* and go on and on into infinity. Then infinite points *C* and *D* are still different. Dana states that the infinite end points are placed at the horizon. When we look at two parallel rays we can see that when they are reaching the horizon they are closer and closer to each other and when they arrive at the horizon and we lose them from our sight they reach infinity where they join together.

We did not expect students to express anything new about infinity in tasks 3 and 4. These tasks as well as task 7 were included to provide respondents with other contexts in which to express their ideas about an infinite object such as a straight line. The different contexts for the same mathematical idea were a rich source for inconsistencies in the students' answers.

In our experience most pupils felt more comfortable speaking about infinity even if it is infinitesimally small, in the context of numbers. Task 5 and 6b) were included to enable us to compare the responses to the same mathematical problem, task 5 in the context of numbers and task 6 in a geometrical context.

Tasks 6a) and 7a) played the role of warming up exercises so that the students could start to think about the task in the way they normally thought in geometry. The core of the tasks are the speculative parts b). In 6b) the smaller the height of triangle *ABC*, the shorter the side *AC* is. So the task is transferred to a simpler one, that is to find such a point on the semicircle that is as close as possible to the point *A* or *B*. The solution to this task is connected with the solvers understanding of an infinitesimally small line segment which is impossible to get just by using our perception. We will call this type of infinity, *infinitesimal infinity.*

In task 7 the area of the square is dependent on the length of the line segment *AB*. The problem is thus transferred into looking for the shortest or longest line segment *AB*, where *B* is an element of the given line *b*. The line segment *AB* can extend without limit when the point *B* goes further and further in any direction. The problem is to think and then communicate about the process and its finalisation.

ANALYSIS AND DISCUSSION OF TASKS 1 AND 2

We have used the method of analysis for the present data which we developed in our previous research (Jirotková & Littler, 2003) and partly comparative analysis as well.

It was surprising that other than two boys (aged 15) out of all the respondents everyone else accepted one of the offered answers as being correct. We feel that this is the consequence of the teaching approaches and textbooks used, which do not offer open-ended tasks to pupils and do not challenge them to think speculatively.

In both countries more students considered the idea of two infinities correct, 75 % of Czech students and 59 % of English students choosing Adam. In spite of clear indications that Adam's answer was right we found inconsistencies between their choice of this sentence and the explanation for their choice. For instance a student who chose Adam then wrote "When I go in any direction I will reach the infinity" and another "... because a straight line goes on for infinity". Because the word infinity was used in the task, students from both countries used it, more Czech pupils, 50 % compared with 41 % of English pupils.

Task 2 produced an interesting result in that most students of both countries chose the sentence spoken by Eva, 61 % of Czech and 63 % of English students. This would indicate that these students were happier thinking of infinity as a process which never ends, as a potential infinity. Choosing Eva's statement as correct, the respondents contradicted any choice they made in task 1. Let us consider each possible combination of answers from the two aspects – potential/actual infinity, affine/projective straight line.

Adam, Eva (38 %) – These students do not perceive any contradiction between actual and potential infinity. We can say that their understanding of infinity is not clear.
Boris, Eva (24 %) – A similar situation to the one above, the only difference being the students' understanding of straight line. The idea of a projective straight line is closer to these students.
Adam, Cecilie (16 %), Adam, Dana (10 %) and Boris, Dana (8 %) – These pairs of answers are consistent. There is no contradiction in the two observed aspects.
Boris, Cecilie (4 %) – a contradiction exists only in the understanding of straight line. This contradiction comes probably from the students seeing the distance between two parallel lines as a constant number which they project to the distance between the two infinities. Quality such as distance does not change itself when approaching infinity.

The experiences of these students are linked just to finite or bounded objects. They do not have any experience with phenomena beyond their perception. That is why they speak about these phenomena as if they were in the 'illuminated' world reachable by our senses. All who answered Dana accept the change in the quality distance when it comes to infinity, and all who answered Cecilie do not accept any change. They are ready to extend the rule *distance between two parallel lines does not change* from perceivable world to infinity. Consideration about the change of quality of phenomenon when coming from finite world to infinity is very difficult for students. We have evidence about finite phenomena, about perceivable objects and we project these to our consideration about phenomena beyond the boundary of the real world. Tall & Tirosh (2001) note the 'epistemological distinctions and contradictory aspects from extending finite experiences into the infinite case'.

COMMENTS ON TASKS 3 TO 7

57 % of the Czech and 64% of the English students said that Frank could write the smallest positive decimal citing answers such as $0,\overline{0}1$ or $0,0000000000000....1$ with the comment "the number includes an infinity of zeros". Those who said it was not possible gave reasons such as "this number can have an infinite number of zeros". As can be seen from these answers the problem from smallest object was transferred to one of how they could write infinitely many zeros. Only 3 % of students did not attempt a solution to task 5 comparing 35 % in task 6, which looked at infinitesimally small object in a geometrical context. The most frequently given solution in task 6 was "*C* is the point closely neighbouring to the point *A* or *B*".

We can divide the images about the furthest point of a straight line which we met in the students' solutions in task 7 into three groups:

1. Any solution to the task is accepted. There are three cases of this represented by students' authentic statements:
   a) "Point *B* does not exist, nor does the square *ABCD.*"
   b) "It is not possible to determine point *B*, nor the square *ABCD*."
   c) "I cannot describe/find the point *B*, nor the square *ABCD*."
2. Point *B* is situated beyond the horizon (Vopěnka, 1996). It means that the point *B* is unreachable to our perceptions and also the square *ABCD*. It is an object that exists but our perceptors cannot grasp it. All we can perceive is the tendency of the point *B* to go far away when the point *B* is still in front of the horizon. The tendency is expressed by:
   a) the verb denoting a movement, eg leaving, going, running, ….
   b) a phrase - as far as possible, the biggest possible, the furthest possible.
3. The point *B* lies on the fixed locality of the line *b***,** marked by ∞, which we call infinity.

CONCLUSIONS

We analysed the students' responses from two didactical points of view:

- to get our explanation of the mechanism of the creation of their images,

- to find a suitable tool to further develop the images.

We prefer to use the phrase "further develop' rather than 're-educate' because we do not agree with teachers' frequent opinion that if the students have a different approach to some concept, it is wrong. It is not possible for this demanding and abstract concept of infinity to be explained to students within a short time; the understanding has to develop over a long period in the cognitive net of a student and the more abstract the concept is, the wider the spectrum of experiences an understanding requires. There is research evidence (Monaghan, 2001) to show that a student's view about infinity differs with the context in which it is expressed, therefore we cannot say that some students' understanding is incorrect. The students are only at a particular developmental stage in their understanding. This implies that we have to think about what other experiences we should provide to the cognitive net of these students to raise their quality of understanding.

Fischbein, Tirosh & Hess (1979) found that the intuition of infinity was relatively stable from 12 onwards. However, the pupils we tested did not show the stability of intuition of infinity when the contexts were changed.

We believe that investigating the process of the understanding of the concept of infinity in different contexts enabled us to approach the world of very special and qualitatively different mental activities, which are related to non existing phenomena which cannot be founded on direct empirical experiences. The important point is the concept building process of the concept of infinity. This is based on introducing the concept in many different contexts and solving the contradictions which come to our minds as a consequence of it.

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