DISCIPLINED CALCULATORS OR FLEXIBLE PROBLEM SOLVERS?

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Efforts to develop a mathematics curriculum that meets the needs of a modern society are reflected in reform recommendations across the developed world. A common requirement is for students to understand the calculation procedures they are taught and to develop 'number sense'. This paper will analyse students' strategies for calculating in the USA, England and the Netherlands and consider the way these relate to curriculum priorities.

Traditional approaches have emphasised a place value approach to calculations, often modelled on base ten materials, with students taught a standard vertical algorithm. Recent developments emphasise a more thinking approach based on 'number sense'. In the US Standards 'understanding number and operations, developing number sense, and gaining fluency in arithmetic computation form the core of mathematics' in the elementary grades' (NCTM, 2003:1). The National Numeracy Strategy in England (DfEE, 1998) proposes more emphasis on mental strategies with delayed introduction of standard algorithms. Students are expected 'to understand' the four operations and relationships among them and to 'use mental methods if the calculations are suitable' (DfEE, 1999:69). In the Netherlands, the Realistic Mathematics approach emphasises the development of 'models' rooted in concrete situations. Written methods are developed with progressively increasing efficiency using unpartitioned numbers (van den Heuvel Panhuizen, 2001). Implementing change is not straightforward and national proposals are meet with different responses by teachers, educationalists, politicians and the public at large. In the USA, 'math wars' reflect controversies in attempts to change priorities. England and the Netherlands are also subject to different initiatives and although aims are compatible the routes to change involve contrasting practices (Beishuizen and Anghileri, 1998, Anghileri, 2001).

The operation of division

Two distinct procedures for written calculations relate to the *partitive* and *quotitive* models for division (Greer, 1992): repeated subtraction of the divisor (becoming more efficient by judicious choice of 'chunks' that are multiples of the divisor) and sharing, based on a place value partitioning of the number to be divided (used efficiently in the traditional algorithm). The traditional algorithm takes two forms: 'short division' in which the calculation is completed in a single line and 'long division' involving sub-procedures recorded in a vertical format. In England, written division is initially restricted to a single digit divisor with 'informal methods of dividing by a two-digit divisor' (DfEE, 1999). In the Netherlands larger numbers (both divisor and dividend) are introduced to justify the need for a written strategy and a standard procedure based on repeated subtraction is taught with efficiency gained through the subtraction of larger ‘chunks’ (Beishuizen and Anghileri, 1998).. In the USA, the traditional algorithm is introduced for one- and two-digit divisors.

Students’ Strategies in England, the Netherlands and the USA

Students written strategies for ten division problems were collected in three countries. English and Dutch cohorts were tested in June of year 5/group 6 when ages were similar (English: mean = 10.21 yrs, s.d. = 0.28; Dutch: mean = 10.32 yrs, s.d. = 0.44). In the USA testing took place later when the mean age of students was 10.75 yrs with s.d. 0.43. These age distributions reflect national policies; in England students’ ages determine their class and it is rare to find any variation (Prais 1997). In the Netherlands the age range in many classes will be wider, reflecting a national policy for accelerating able students and repeating years for those who do not reach the required standard. In the US a policy of repeating years also operates.

The study involved students (n=647) from 23 schools in and around small university cities: 10 high achieving English (n=275) schools, 10 Dutch schools implementing curriculum change (n=259) and 3 schools in the USA (n=113). Time constrained the sample to six classes in one state in the USA in one private school, one selective and one non-selective public school. Solutions were collected in individual workbooks using five word problems that varied in their numerical content and their semantic structure, together with five parallel ‘bare’ problems. The same protocol was used in all classes. Using the students written records, codes for the strategies were established (Anghileri, Beishuizen, & van Putten, 2002, van Putten 2002).

RESULTS

Solutions were predominantly those taught in each country but the frequency of use varied. In England the short division algorithm was used in 53% of all attempted questions, in the US the long division algorithm was used in 81% of all attempts and in the Netherlands the repeated subtraction procedure was used in 60%. The US and English algorithms allowed for no flexibility but the Dutch repeated subtraction method allowed students to choose the number facts to use and could be completed at different levels of efficiency. Other strategies used were generally low-level approaches such as tallying or repeated addition or subtraction.

Single digit divisors

A pair of items involved exact division of a two-digit number (one context and one bare) and another pair involved a four-digit dividend and a remainder:

- q1: 98 flowers are bundled in bunches of 7. How many bunches can be made?
- q6: 96÷6
- q5: 1542 apples are divided among 5 shopkeepers. How many apples will each shopkeeper get? How many apples will be left?
- q10: 1256÷6

Use of the taught algorithms was highest in these questions and percentage use (whether correct or incorrect) is given in Table 1.

algorithm	short division				repeated subtraction				long division			
	q1	q6	q5	q10	q1	q6	q5	q10	q1	q6	q5	q10
England	66	66	70	68								
Netherland					76	74	72	72				
USA									92	91	92	87

Table 1: Percentage use of the different algorithms

The English cohort showed more diversity in the strategies used for these questions with about a quarter of attempts using informal approaches including tallying and chunking using known number facts. The US students used the algorithm almost exclusively. Facilities for these four questions are shown in Table 2.

	q1	q6	q5	q10
England	53	53	35	23
Netherlands	84	81	63	56
USA	96	85	74	69

Table 2: Overall success rates for each question (percentage).

English students did least well with particular difficulty with 4-digit dividends. The Dutch, with their repeated subtraction procedure were more successful. The US success in these four questions reflects extremely high performances in the private school (99% correct overall in these items) and the selective school (91% correct overall in these 4 items) while in the non-selective public school students' success was more modest at 89%, 47%, 47% and 42% respectively for these questions.

Two digit divisors

Two pairs of questions involved two-digit divisors with numbers chosen to encourage informal approaches:

- q2: 64 pencils have to be packed in boxes of 16. How many boxes will be needed?
- q7: 84÷14
- q3: 432 children have to be transported by 15 seater buses. How many buses will be needed?
- q8: 538÷15

Informal strategies were evident in some English and Dutch solutions but rarely in the American students' work. The nationally taught algorithms (English short division; Dutch repeated subtraction algorithm; US long division algorithm) were again most commonly attempted and the following table (Table 3) shows the percentage of items correctly solved compared with the percentage correctly solved using the algorithms in brackets.

	q2	q7	q3	q8
England	48 (8)	55 (14)	23 (15)	13 (8)
Netherlands	75 (50)	76 (40)	53 (45)	53 (43)
USA	73 (55)	81 (68)	36 (35)	64 (64)

Table 3: Facilities (percentage) for each question (success using national algorithm).

Success on these four questions was most limited in the English classes, not least because the short division algorithm is not easily adapted for 2-digit divisors and informal methods were widely used although many students (9%) omitted these problems. In question 3 many US students gave as their answer the result of their calculation and not the number of coaches required.

Due to the nature of the sample, results for the US classes are interesting when comparison is made between the highest and lowest scoring classes (Table 4).

	q2	q7	q3	q8
non selective public school (n=19)	26	11	0 (*10)	11
selective private school (n=19)	100	100	74 (*89)	89

*includes correct calculation but wrong answer

Table 4: Facilities of two US classes for questions involving two-digit divisors

Many wrong answers in the non-selective public school class were due to choice of the wrong operation in the context questions (40%).

Division by ten

Two of the items involved division by ten with a remainder:

- q4: 604 blocks are laid down in rows of 10. How many rows will there be?
- q9: $802 \div 10$

The English students used a variety of strategies with mental methods (an answer given but no working shown) being the most common (34% and 37% respectively). The context question 4 was tackled by 28% using the algorithm while the non-context question was tackled this way by 40%. The Dutch students also used a variety of methods with less difference between the context and non-context questions (54% and 50% use respectively) in the use of the algorithm. A bigger difference occurred with a mental strategy used by 28% and 33% of the students.

As in the other items the US students predominantly used the algorithm in 72% of attempts at question 4 (54% correct) and in 81% of attempts at question 9 (72% correct) and a mental strategy was used by only 8%. None of the US students curtailed the algorithm in any way and full working was shown throughout.

Discussion

It has been proposed that arithmetic instruction is not about designing ways for students to develop facility in calculation, albeit meaningfully, it is about fostering students' underlying arithmetical conceptions (Steffe and Kieren, 1994). Findings of this study suggest this objective is not greatly evident in the written methods for division, and students' approaches in the different countries are starkly contrasted. The US students gained the highest scores overall (72% correct) but since the cohorts from different countries in this study are not directly comparable it is not possible to conclude that this provides the key to successful division computation. Success rates for the non-selective class (Table 4) suggests that the taught algorithm presents considerable difficulties for many students. Facilities are comparable with the English cohort who used the algorithm in 49% of all attempts (Table 5).

	q1	q2	q3	q4	q5	q6	q7	q8	q9	q10
US non selective	89	26	0	26	47	47	11	11	47	42
mean England	76	48	23	46	38	69	54	13	45	22

Table 5: Comparison of facilities (%) in one US class with all English students

The US students are well disciplined with the algorithm used extensively but there was no flexibility (for example no curtailed procedures for division by 10) and there was little evidence of number sense (for example few mental approaches and lack of reference back to a meaningful solution in context). The US students found many bare question, for example division by ten, easier than the context problem, suggesting that they focus on formal calculations more than problem solving.

Where English and Dutch classes can be more readily compared the results show superior mastery with the Dutch approach (see Anghileri et al., 2002). Dutch students predominantly (60%) used 'chunking' with flexibility in the degree of efficiency involved. This algorithm allows individuals to make choices about the number facts they use, thus retaining some ownership of the method rather than replicating a standard procedure. It is suggested that by the time Dutch students are the age of the US students they will have achieved equal or greater success rates although this can only be speculation.

English students used a greater diversity of methods and this fits with the objective of introducing flexibility but appears to be at the expense of competence in calculating. English students used a mental method most (36%) of all three countries for division by ten and were most effective (22% correct) in use of this strategy for these questions. Correct solutions to other questions sometimes (7%) showed inventive strategies for division by a 2-digit divisor which had not been taught, for example for $432 \div 15$ a solution given was $30 \times 15 = 450$ $450 - 15 = 435$ Answer 28r12. Overall English students arrived at correct solutions to 25% of all items using the algorithm but a further 19% using other methods. More diversity in approach can lead to some good strategies but many students are unable to develop their own informal methods for problems such as 64÷16. At the other extreme it may be questioned whether it is desirable for US students to persist in a rigorous and completed procedure where a less formal strategy would be more efficient and reflect the number sense that is desired. The progressive nature of the Dutch method involves flexibility as it allows students to use the number facts they know without being constrained to the unique steps in the traditional algorithms. With the desire to encourage number sense it is important to question the priority that is given to teaching the traditional algorithm for division, but competence in calculating does not appear to develop where students are left to develop their own methods. A balance needs to be established between flexibility with use of number sense and accuracy in computation. The Dutch approach appears to go furthest in developing an approach that combines both.

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