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TRIVIAL LAGRANGIANS ON CONNECTIONS AND INVARIANCE UNDER AUTOMORPHISMS

M. CASTRILLÓN LÓPEZ, J. MUÑOZ MASQUÉ, AND T.S. RATIU

Abstract. Two results are presented: The characterization of first order Lagrangian densities on connections invariant under the full Lie algebra of principal infinitesimal automorphisms and the characterization of gauge-invariant null Lagrangians. Some explicit examples are given as well.

1. Introduction

Let $\pi\colon P \to M$ be an arbitrary principal bundle with structure group a Lie group G. The geometric formulation of gauge theories takes place on the bundle of connections $p\colon C \to M$ by considering variational problems defined by Lagrangians $\mathcal{L}\colon J^1C \to \mathbb{R}$ on the first jet bundle. Moreover, gauge symmetries play an important role in these theories. Due to this fact, the Lagrangians used in their formulation are required to be gauge invariant with respect to the natural representation of the gauge vector fields of P in the bundle of connections. In the fifties, Utiyama gave a complete description of these Lagrangians. Roughly speaking, this result says that a Lagrangian $\mathcal{L}$ is gauge invariant if and only if it factors through the curvature map (that is, $\mathcal{L} = \bar{\mathcal{L}} \circ \Omega$) by means of an adjoint invariant function $\bar{\mathcal{L}}$ (see below for a more precise statement).

Motivated by this geometric characterization, we present in this paper two results. First we give a characterization of those Lagrangians defined on J^1C which are invariant under the full algebra of infinitesimal automorphisms. Due to the interest of the Utiyama's theorem, it looks reasonable to ask for the invariance under the functorial group of morphisms of the category of principal bundles (*i.e.*, the automorphisms) and not only under the subgroup of vertical morphisms (the gauge transformations). It turns out that these Lagrangians provide variationally trivial Lagrangians whose action functionals are the characteristic numbers of π of degree $\dim M$. Partial results in this direction can be found for the special case $G = U(n)$ and polynomial Lagrangians in [4, Theorem 2.6.2].

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Secondly, we study whether every gauge-invariant variationally trivial Lagrangian density is as described above; that is, invariant under automorphisms and, therefore, related to a characteristic class. The answer is negative and in fact it turns out that these Lagrangians are characterized by de Rham cohomology of the base manifold M and the characteristic classes of P, now of arbitrary degree.

We finally illustrate these results with explicit examples for some classical structure groups in field theories.

2. Preliminaries

2.1. **Automorphisms.** An *automorphism* of a G-principal bundle $\pi\colon P \to M$ is a diffeomorphism $\Phi\colon P \to P$ which is equivariant with respect to the action of G on P, that is, $\Phi(u \cdot g) = \Phi(u) \cdot g$, $u \in P$, $g \in G$. Automorphisms are projectable; *i.e.*, given an automorphism Φ, there exists $\varphi \in \mathrm{Diff} M$ such that $\pi \circ \Phi = \varphi \circ \pi$. The automorphisms such that $\varphi = \mathrm{Id}_M$ are called *gauge transformations*. The set $\mathrm{Aut}P$ of all automorphisms is a Lie group under the composition and $\mathrm{Gau}P$, the subset of gauge transformation, is a normal subset. In fact $\mathrm{Aut}P$ and $\mathrm{Gau}P$ are infinitely dimensional Lie groups whose Lie algebras can be described as follows: $\mathrm{aut}P$ is the algebra of G-invariant vector fields of P and $\mathrm{gau}P$ is the subalgebra of vertical G-invariant vector fields. It is clear that the elements $\mathrm{aut}P$ are π-projectable. In fact we have the following short exact sequence of algebras

$$0 \hookrightarrow \mathrm{gau}P \hookrightarrow \mathrm{aut}P \xrightarrow{\pi_*} \mathfrak{X}(M) \to 0.$$

2.2. **The bundle of connections.** Given the principal bundle $\pi\colon P \to M$, let $p\colon C \to M$ be the bundle of connections of π. As it is well known, C is an affine bundle modelled over the vector bundle $T^*M \otimes \mathrm{ad}P$, where $\mathrm{ad}P$ denotes the adjoint bundle, that is, the associated bundle defined by the adjoint action of G on $\mathfrak{g}$. There is a bijective correspondence between connections Γ on P and sections σ_Γ of C. Giving an automorphism $\Phi \in \mathrm{Aut}P$, there exist an unique diffeomorphism $\Phi_C\colon C \to C$ such that for every connection Γ we have $\Phi_C \circ \sigma_\Gamma = \sigma_{\Phi(\Gamma)} \circ \varphi$, where $\Phi(\Gamma)$ is the image of Γ by Φ (*cf.* [5, II.6]). The mapping $\mathrm{Aut}P \to \mathrm{Diff}C$, $\Phi \mapsto \Phi_C$ is a homomorphism of Lie groups which gives, at the level of their Lie algebras, a Lie algebra homomorphism $\mathrm{aut}P \to \mathfrak{X}(M)$, $X \mapsto X_C$ which is called *the natural representation of* $\mathrm{aut}P$ *on* C.

Next, we give the local expression of this representation. Let (U, x^i), $i = 1, ..., n = \dim M$, an open coordinate domain of M such that P is trivializable over it, that is, $\pi^{-1}(U) \simeq U \times G$. The natural identification $\mathrm{ad}(U \times G) = U \times \mathfrak{g}$ and the choice of the trivial connection on $U \times G$ enables us to identify $p^{-1}(U) \subset C$ with $T^*U \otimes \mathfrak{g}$ in a natural way. If $\{B_1, ..., B_m\}$ is a basis of $\mathfrak{g}$, we endow $T^*U \otimes \mathfrak{g}$ (and therefore $p^{-1}(U)$) with the coordinates (x^i, A_i^α), $i = 1, ..., n$, $\alpha = 1, ..., m = \dim G$, defined by the condition

$$\omega = -A_i^\alpha(\omega) \mathrm{d}x^i \otimes B_\alpha, \quad \forall \omega \in T^*U \otimes \mathfrak{g}.$$

Similarly, as $TP|_U = TU \times TG$, for every $X \in \mathrm{aut}P$, we can write

$$X = f^i \frac{\partial}{\partial x^i} + g^\alpha \tilde{B}_\alpha, \quad f^i, g^\alpha \in C^\infty(U),$$

where $\tilde{B}$ is the infinitesimal generator of the flow $t \mapsto \exp(tB)g$, $B \in \mathfrak{g}$, in G. Note that $X \in \mathrm{gau}P$ if and only if $f^i = 0$, $i = 1, ..., n$. The local expression of the natural representation X_C of $X \in \mathrm{aut}P$ is

$$X_C = f^i \frac{\partial}{\partial x^i} - \left(\frac{\partial g^\alpha}{\partial x^i} - c^\alpha_{\beta\gamma} g^\beta A^\gamma_i + \frac{\partial f^h}{\partial x^i} A^\alpha_h \right) \frac{\partial}{\partial A^\alpha_i}.$$

2.3. **Gauge-invariant Lagrangians.** We are concerned with first order variational problems on connections defined by Lagrangians densities $\Lambda\colon J^1C \to \bigwedge^n T^*M$. From now on, we will consider that the manifold M is oriented by a fix volume form $v \in \Omega^n(M)$. Then we can write $\Lambda = \mathcal{L}v$ for certain function $\mathcal{L}\colon J^1C \to \mathbb{R}$ called the Lagrangian. We say that $\mathcal{L}$ is *gauge invariant* (also called *natural* for some authors; *e.g.* [6, XII]) if $X^{(1)}_C(\mathcal{L}) = 0$ for every $X \in \mathrm{gau}P$, where $X^{(1)}_C$ is the natural lifting of X_C to the jet bundle J^1C. Moreover, a Lagrangian density Λ is said to be $\mathrm{aut}P$-invariant if

$$L_{X^{(1)}_C}\Lambda = X^{(1)}_C(\mathcal{L})v + \mathcal{L}L_{X'}v = 0, \quad \forall X \in \mathrm{aut}P,$$

where $X' \in \mathfrak{X}(M)$ is the projection of X onto M. Taking into account that $X' = 0$ for $X \in \mathrm{gau}P$, we can recover the definition of gauge invariance from that of $\mathrm{aut}P$-invariance. For the geometric characterization of gauge-invariant Lagrangians we need to introduce the curvature mapping. We define $\Omega\colon J^1C \to \bigwedge^2 T^*M \otimes \mathrm{ad}P$ as $\Omega(j^1_x\sigma_\Gamma) = \Omega^\Gamma_x$, that is, Ω sends the 1-jet of a (local) connection Γ to the curvature of Γ seen as a two from on M with values in the adjoint bundle. Then we have

Utiyama's Theorem ([3], [8]). *A Lagrangian $\mathcal{L}\colon J^1C \to \mathbb{R}$ is gauge invariant if and only if $\mathcal{L} = \bar{\mathcal{L}} \circ \Omega$ by means of a function $\bar{\mathcal{L}}\colon \bigwedge^2 T^*M \otimes \mathrm{ad}P \to \mathbb{R}$ which in turn must be invariant under the adjoint action of $\mathfrak{g}$ in $\mathrm{ad}P$.*

3. $\mathrm{aut}P$-invariant Lagrangians

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We say that a multilinear map $f\colon \mathfrak{g} \oplus \overset{(k}{\ldots} \oplus \mathfrak{g} \to \mathbb{R}$ is *Weil polynomial of degree k* if it is invariant under the adjoint representation of G on $\mathfrak{g}$; *i.e.*, if

$$f(\mathrm{Ad}_g B_1, ..., \mathrm{Ad}_g B_k) = f(B_1, ..., B_k), \quad \forall B_1, ..., B_k \in \mathfrak{g}, \forall g \in G.$$

The set of all Weil polynomials of degree k is denoted by I^G_k. The space

$$I^G = \bigoplus_{k \geq 0} I^G_k$$

is a subalgebra of the symmetric algebra $S^\bullet(\mathfrak{g}^*)$ of the dual space $\mathfrak{g}^*$ (*cf.* [5, XII.§1]). Given $f \in I^G_k$, we define the function $\bar{f}\colon \mathrm{ad}P \oplus \overset{(k}{\ldots} \oplus \mathrm{ad}P \to \mathbb{R}$, by the condition

$$\bar{f}((u, B_1)_{\mathrm{ad}}, \ldots, (u, B_k)_{\mathrm{ad}}) = f(B_1, \ldots, B_k),$$

where $(u, B)_{\mathrm{ad}} \in (\mathrm{ad}P)_x$, $B \in \mathfrak{g}$, $x \in M$, $u \in \pi^{-1}(x)$ denotes the equivalence class of (u, B) in $\mathrm{ad}P = (P \times \mathfrak{g})/G$. We now suppose that $\dim M = n = 2k$. Let $f \in I_k^G$ be a Weil polynomial of degree k. We define the fibred map

$$\bar{\Lambda}_f \colon \bigwedge^2 T^*M \otimes \mathrm{ad}P \to \bigwedge^n T^*M,$$

by the condition

$$\bar{\Lambda}_f(w)(X_1, \dots, X_n) = \tfrac{1}{(2k)!} \sum_{\tau \in S_{2k}} \varepsilon(\tau) \bar{f}\left(w(X_{\tau(1)}, X_{\tau(2)}), \dots, w(X_{\tau(2k-1)}, X_{\tau(2k)})\right),$$

for all $w \in (\bigwedge^2 T^*M \otimes \mathrm{ad}P)_x$, and $X_1, \dots, X_n \in T_xM$, $x \in M$, where $\varepsilon(\tau)$ is the signature of the permutation τ. The composition of $\bar{\Lambda}_f$ with the curvature mapping gives a fibered morphism $\Lambda_f \colon J^1P \to \bigwedge^n T^*M$ called *the Lagrangian density associated to* f. Similarly, *the Lagrangian associated to* f is the unique function $\mathcal{L}_f \in C^\infty(J^1C)$ such that $\Lambda_f = \mathcal{L}_f v$. These Lagrangian densities enjoy the following property

Proposition 1. *The Lagrangians densities on the bundle of connections associated to Weil polynomials are* $\mathrm{aut}P$*-invariant.*

In fact, the next result shows that these Lagrangian densities are the only ones which are $\mathrm{aut}P$-invariant. More precisely:

Theorem 1. *Let* $\pi \colon P \to M$ *be a principal* G*-bundle with* G *connected and* M *connected and orientable. Then we have:*

(1) *If* $\dim M$ *is odd, the only* $\mathrm{aut}P$*-invariant Lagrangian density on connections is the zero density.*
(2) *If* $\dim M$ *is even (say* $\dim M = 2k$*), all* $\mathrm{aut}P$*-invariant Lagrangian densities on connections are of the form* Λ_f*, with* $f \in I_k^G$*.*

Assume now that M is compact. From the theory of the Weil homomorphism and the construction of the characteristic classes from the curvature (*cf.*[5, XIII]), we know that given any section $\sigma_\Gamma \colon M \to P$ of the bundle of connections, the form $\Lambda_f \circ j^1\sigma_\Gamma$ is a closed form whose cohomology class (the characteristic class of f) does not depend on Γ. Hence the action funtional $\sigma_\Gamma \mapsto \int_M \Lambda_f \circ j^1\sigma_\Gamma$ is constant. We have thus proved the following

Corollary 1. *Every* $\mathrm{aut}P$*-invariant Lagrangian density on connections is variationally trivial.*

4. Variationally trivial Lagrangians

Once we have the characterization of the $\mathrm{aut}P$-invariant Lagrangians, one could ask if these are all possible natural Lagrangian densities which are variationally trivial at the same time. In fact, variationally trivial natural Lagrangians gives topological information of the bundle $\pi \colon P \to M$. The $\mathrm{aut}P$-invariant densities exclusively give the characteristic classes of degree $(\dim M)/2$ (if $\dim M$ is even). In Theorem 2 below we will see that other trivial natural Lagrangians can be found and, in fact, they give the rest of the characteristic classes as well, as a part of the de Rham complex of M. For the complete characterization we first need some notation.

The bundle of connections $p\colon C \to M$ is endowed with a canonical $p^*\mathrm{ad}P$-valued 2-form Ω_2 called the universal curvature (for example, see [2], [3]). This form is a generalization to an arbitrary fiber bundle P of the canonical $\mathfrak{g}$-valued symplectic form on $C = T^*M \otimes \mathfrak{g}$ in the special case $P = M \times G$ and enjoys the following property: If $\sigma_\Gamma\colon M \to C$ is the section of the bundle of connections defined by a connection Γ on P, then we have that $\Omega_\Gamma = \sigma_\Gamma^*\Omega_2$, where Ω^Γ is the curvature of Γ seen as a 2-form on M taking valued on $\mathrm{ad}P$.

Given a Weil polynomial $f \in I^G$ of arbitrary degree d, we define the $2d$-form $f(\Omega_2)$ on C by setting

$$f(\Omega_2)(Y_1,\dots,Y_{2d}) = \tfrac{1}{(2d)!}\sum_{\tau\in S_{2d}} \varepsilon(\tau)\bar{f}\left(\Omega_2(Y_{\tau(1)},Y_{\tau(2)}),\dots,\Omega_2(Y_{\tau(2d-1)},Y_{\tau(2d)})\right),$$

for all $Y_1,...,Y_d \in T_{\Gamma_x}C$, $\Gamma_x \in C$. The form $f(\Omega_2)$ is called *the characteristic form associated to* f. As C is an affine bundle over M, we have that $H^\bullet(M;\mathbb{R}) \simeq H^\bullet(C;\mathbb{R})$ and it is not difficult to see that the cohomology class of $f(\Omega_2)$ coincides with the characteristic class of P defined by f.

Finally, given an arbitrary n-form Θ on the bundle of connections, we can define a Lagrangian density $\Lambda_\Theta = \Theta^h\colon J^1C \to \bigwedge^n T^*M$ as follows: $\Theta^h(j^1_x\sigma_\Gamma) = \sigma_\Gamma^*\Theta$. This construction is also called *horizontalization* and it can be defined as well for forms of other degrees on arbitrary fibered manifolds.

Theorem 2. *Let $\pi\colon P \to M$ be a G-principal fiber bundle with G connected and M orientable. Then, every gauge-invariant variationally trivial Lagrangian density on J^1C is of the form $\Lambda = \Theta^h$, where Θ is a form on C of the type*

$$\Theta = \sum_{\deg f \le n/2} p^*\omega_f \wedge f(\Omega_2),$$

and ω_f are forms on M of degree $n - 2\deg f$ which, in turn, must be closed.

Remark 1. There is a general result yielding a description of variational triviality on arbitrary bundles: A Lagrangian density Λ is variationally trivial if and only if Λ is the horizontalization of a closed form (*cf.* [7]). Theorem 2 above agrees with this result but it shows that the additional condition of naturality gives a richer structure on this Lagrangians as it provides the characteristic classes of $\pi\colon P \to M$ of arbitrary degree and the de Rham classes (defined by the forms ω_f) of M with the same parity of $\dim M$.

5. Examples

5.1. **The group $U(1)$.** Let $\pi\colon P \to M$ be a principal fiber bundle with structure group $G = U(1)$. Making the identification $\mathfrak{u}(1) = \mathbb{R}$, the algebra of Weil polynomials is generated over $\mathbb{R}$ by the monomial $\mathrm{Id}(t) = t$. If $\dim M = n = 2k$, the only Weil polynomial of degree k is, up to a constant, $f(t) = t^k$. The Lagrangian density associated to f is $\Lambda_f \colon J^1(C(P)) \to \bigwedge^n T^*M$, $\Lambda_f = \mathrm{Pf} \circ \Omega$, where

$$\mathrm{Pf}\colon \bigwedge^2 T^*M \to \bigwedge^n T^*M, \quad \mathrm{Pf}(w_2) = w_2 \wedge \overset{(k}{\cdots} \wedge w_2.$$

Locally, in a coordinate system (x^i, A_i) on C such that $v = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, we have

$$\mathcal{L}_f \cdot v = \frac{1}{n!} \sum_{\tau \in S^n} \varepsilon(\tau) R_{\tau(1)\tau(2)} \dots R_{\tau(n-1)\tau(n)} \cdot v,$$

where $R_{ij} = A_{i,j} - A_{j,i}$. This expression is nothing but the Pfaffian of $\pi\colon P \to M$ as a function of the curvature (see for example [5, XII]). The cohomology class of the Pfaffian is the Euler class of π and the action defined by $\mathcal{L}_f v$ yields the Euler characteristic of π. Similarly, every natural trivial Lagrangian is of the form $\Lambda = \Theta^h$ with $\Theta = \sum_d \omega_{n-2d} \wedge \Omega_2 \wedge \overset{(d}{\cdots} \wedge \Omega_2$, where Ω_2 is the canonical symplectic form of C and $\omega_{n-2d} \in \Omega^{n-2d}(M)$ are closed. For a deeper study of these results for $G = U(1)$, see [2].

5.2. **The group** $SU(2)$**.** The algebra of Weil polynomial of $\mathfrak{su}(2)$ is generated by the determinant, that is $I^{SU(2)} = \mathbb{R}[\det]$. We remark that the determinant is a polynomial of degree two and as a consequence every Weil polynomial must have even degree. From Theorem 1, if $\dim M = 2k$, we conclude that C will have aut$P$-invariant Lagrangian densities only for k even. Hence the first dimension where we can find a non-zero aut$P$-invariant Lagrangian density Λ, is four. In this case the local expression of Λ takes the form (up to a constant)

$$\Lambda_{\det} = \mathfrak{S}_{123} \sum_{\tau \in S_4} \frac{1}{4} \varepsilon(\tau) (A^1_{\tau(1),\tau(2)} A^1_{\tau(3),\tau(4)} - 2A^1_{\tau(1),\tau(2)} A^2_{\tau(3)} A^3_{\tau(4)}) \cdot v,$$

where the basis of $\mathfrak{g}$ is assumed to be the Pauli matrices $\{B_1, B_2, B_3\}$ and $\mathfrak{S}_{123}$ denotes cyclic summation over the indices $1, 2, 3$. This density provides the second Chern class of π.

Natural trivial Lagrangians are of the form $\Lambda = \Theta^h$ with

$$\Theta = \sum_d \omega_{n-4d} \wedge \det(\Omega_2) \wedge \overset{(d}{\cdots} \wedge \det(\Omega_2),$$

where $\omega_{n-4d} \in \Omega^{n-4d}(M)$ are closed.

5.3. **The group** $U(2)$**.** In this case, the algebra of Weil polynomials on $\mathfrak{u}(2)$ is generated by the trace and determinant, that is, $I^{U(2)} = \mathbb{R}[\mathrm{tr}, \det]$. For every even dimension of M there are non-vanishing aut$P$-invariant Lagrangians. For example, for $\dim M = 4$ we have basically two different invariant densities: One associated to the polynomial $\mathrm{tr} \cdot \mathrm{tr}$ and the other associated to det. Locally, these densities have the following expression

$$\Lambda_{\mathrm{tr}\cdot\mathrm{tr}} = \sum_{\tau \in S_4} \varepsilon(\tau) A^0_{\tau(1),\tau(2)} A^0_{\tau(3),\tau(4)} v,$$

and

$$\Lambda_{\det} = -\frac{1}{4}\mathcal{L}_{\mathrm{tr}\cdot\mathrm{tr}} + \mathfrak{S}_{123}\sum_{\tau\in S_4}\frac{1}{4}\varepsilon(\tau)(A^1_{\tau(1),\tau(2)}A^1_{\tau(3),\tau(4)} - 2A^1_{\tau(1),\tau(2)}A^2_{\tau(3)}A^3_{\tau(4)})v,$$

where the chosen basis $\{B_0, B_1, B_2, B_3\}$ for $\mathfrak{u}(2)$ are the Pauli matrices B_1, B_2, B_3 and the matrix $B_0 = i\,\mathrm{Id}$. In this case, natural trivial Lagrangians are of the form $\Lambda = \Theta^h$ with

$$\Theta = \sum_{l,d}\omega_{n-2l-4d}\wedge \mathrm{tr}(\Omega_2)^l \wedge \det(\Omega_2)^d,$$

where $\omega_{n-2l-4d} \in \Omega^{n-2l-4d}(M)$ are closed.

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Département de Mathématiques, École Polytechnique Fédérale de Lausanne, Switzerland

E-mail address: marco.castrillon@epfl.ch, mcastri@mat.ucm.es

CSIC, Instituto de Física Aplicada, C/ Serrano 144, 28006-Madrid, Spain

E-mail address: jaime@iec.csic.es

Département de Mathématiques, École Polytechnique Fédérale de Lausanne, Switzerland

E-mail address: tudor.ratiu@epfl.ch