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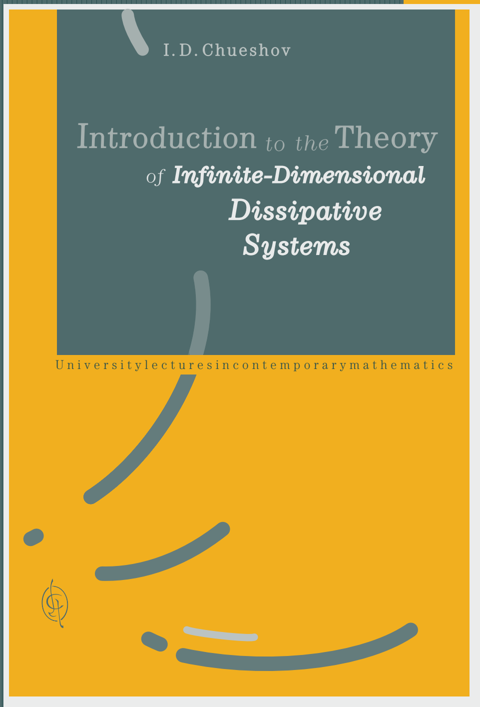

This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

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Chapter 4

The Problem on Nonlinear Oscillations of a Plate in a Supersonic Gas Flow

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In this chapter we use the ideas and the results of Chapters 1 and 3 to study in details the asymptotic behaviour of a class of problems arising in the nonlinear theory of oscillations of distributed parameter systems. The main object is the following second order in time equation in a separable Hilbert space H:

$$\frac{d^2}{dt^2}u + \gamma \frac{d}{dt}u + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu = p(t)\,, \tag{0.1}$$

$$u\big|_{t=0} = u_0\,, \quad \frac{du}{dt}\bigg|_{t=0} = u_1\,, \tag{0.2}$$

where A is a positive operator with discrete spectrum in H, $M(s)$ is a real function (its properties are described below), L is a linear operator in H, $p(t)$ is a given bounded function with the values in H, and γ is a nonnegative parameter. The problem of type (0.1) and (0.2) arises in the study of nonlinear oscillations of a plate in the supersonic flow of gas. For example, in Berger's approach (see [1, 2]), the dynamics of a plate can be described by the following quasilinear partial differential equation:

$$\partial_t^2 u + \gamma\partial_t u + \Delta^2 u + \left(\Gamma - \int_\Omega |\nabla u|^2\, dx\right)\Delta u + \rho\partial_{x_1} u = p(x,\, t)\,, \tag{0.3}$$

$$x \in (x_1,\, x_2) \subset \Omega \subset \mathbb{R}^2\,, \qquad t > 0$$

with boundary and initial conditions of the form

$$u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0\,, \qquad u\big|_{t=0} = u_0(x)\,, \qquad \partial_t u\big|_{t=0} = u_1(x)\,. \tag{0.4}$$

Here Δ is the Laplace operator in the domain Ω; $\gamma > 0$, $\rho \ge 0$, and Γ are constants; and $p(x,\, t)$, $u_0(x)$, and $u_1(x)$ are given functions. Equations (0.3)–(0.4) describe nonlinear oscillations of a plate occupying the domain Ω on a plane which is located in a supersonic gas flow moving along the x_1-axis. The aerodynamic pressure on the plate is taken into account according to Ilyushin's "piston" theory (see, e. g., [3]) and is described by the term $\rho\partial_{x_1}u$. The parameter ρ is determined by the velocity of the flow. The function $u(x,\, t)$ measures the plate deflection at the point x and the moment t. The boundary conditions imply that the edges of the plate are hinged. The function $p(x,\, t)$ describes the transverse load on the plate. The parameter Γ is proportional to the value of compressive force acting in the plane of the plate. The value γ takes into account the environment resistance.

Our choice of problem (0.1) and (0.2) as the base example is conditioned by the following circumstances. First, using this model we can avoid significant technical difficulties to demonstrate the main steps of reasoning required to construct a solution and to prove the existence of a global attractor for a nonlinear evolutionary second order in time partial differential equation. Second, a study of the limit regimes of system (0.3)–(0.4) is of practical interest. The point is that the most important (from the point of view of applications) type of instability which can be found in the system under consideration is the flutter, i.e. autooscillations of a plate subjected to aerodynamical loads. The modern look on the flutter instability of a plate is the following: there arises the Andronov-Hopf bifurcation leading to the appearance of a stable limit cycle in the system. However, there are experimental and numerical data that enable us to conjecture that an increase in flow velocity may result in the complication of the dynamics and appearance of chaotic fluctuations [4]. Therefore, the study of the existence and properties of the attractor of the given problem enables us to better understand the mechanism of appearance of a nonlinear flutter.

§ 1 Spaces

As above (see Chapter 2), we use the scale of spaces $\mathcal{F}_s$ generated by a positive operator A with discrete spectrum acting in a separable Hilbert space H. We remind (see Section 2.1) that the space $\mathcal{F}_s$ is defined by the equation

$$\mathcal{F}_s \equiv D(A^s) = \left\{ v : \sum_{k=1}^{\infty} c_k e_k : \sum_{k=1}^{\infty} c_k^2 \lambda_k^{2s} < \infty \right\},$$

where $\{e_k\}$ is the orthonormal basis of the eigenelements of the operator A in H, $\lambda_1 \le \lambda_2 \le \ldots$ are the corresponding eigenvalues and s is a real parameter (for $s = 0$ we have $\mathcal{F}_s = H$ and for $s < 0$ the space $\mathcal{F}_s$ should be treated as a class of formal series). The norm in $\mathcal{F}_s$ is given by the equality

$$\|v\|_s^2 = \sum_{k=1}^{\infty} c_k \lambda_k^{2s} \quad \text{for} \quad v = \sum_{k=1}^{\infty} c_k e_k .$$

Further we use the notation $L^2(0, T; \mathcal{F}_s)$ for the set of measurable functions on the segment $[0, T]$ with the values in the space $\mathcal{F}_s$ such that the norm

$$\|v\|_{L^2(0, T; \mathcal{F}_s)} = \left(\int_0^T \|v(t)\|_s^2 \, dt \right)^{1/2}$$

is finite. The notation $L^p(0, T; X)$ has a similar meaning for $1 \le p \le \infty$.

We remind that a function $u(t)$ with the values in a separable Hilbert space H is said to be ***Bochner measurable*** on a segment $[0, T]$ if it is a limit of a sequence of functions

$$u_N(t) = \sum_{k=1}^{N} u_{N,k} \, \chi_{N,k}(t)$$

for almost all $t \in [0, T]$, where $u_{N,k}$ are elements of H and $\chi_{N,k}(t)$ are the characteristic functions of the pairwise disjoint Lebesgue measurable sets $A_{N,k}$. One can prove (see, e.g., the book by K. Yosida [5]) that for separable Hilbert spaces under consideration a function $u(t)$ is measurable if and only if the scalar function $(u(t), h)_H$ is measurable for every $h \in H$. Furthermore, a function $u(t)$ is said to be ***Bochner integrable*** over $[0, T]$ if

$$\int_0^T \|u(t) - u_N(t)\|_H^2 \, dt \to 0, \qquad N \to \infty,$$

where $\{u_N(t)\}$ is a sequence of simple functions defined above. The integral of the function $u(t)$ over a measurable set $S \subset [0, T]$ is defined by the equation

$$\int_S u(t)\, dt = \lim_{N \to \infty} \int_0^T \chi_S(\tau) u_N(\tau)\, d\tau,$$

where $\chi_S(\tau)$ is the characteristic function of the set S and the integral of a simple function in the right-hand side of the equality is defined in an obvious way.

For the function with the values in Hilbert spaces most facts of the ordinary Lebesgue integration theory remain true.

— Exercise *1.1* Let $u(t)$ be a function on $[0, T]$ with the values in a separable Hilbert space H. If there exists a sequence of measurable functions $u_n(t)$ such that $u_n(t) \to u(t)$ almost everywhere, then $u(t)$ is also measurable.

— Exercise *1.2* Show that a measurable function $u(t)$ with the values in H is integrable if and only if $\|u(t)\| \in L^1(0, T)$. Therewith

$$\left\| \int_B u(t)\, d\tau \right\| \le \int_B \|u(t)\|\, d\tau$$

for any measurable set $B \subset [0, T]$.

— Exercise *1.3* Let a function $u(t)$ be integrable over $[0, T]$ and let B be a measurable set from $[0, T]$. Show that

$$\int_B (u(\tau), h)_H \, d\tau = \left(\int_B u(\tau)\, d\tau, h \right)_H$$

for any $h \in H$.

— Exercise *1.4* Show that the space $L^2(0, T; \mathcal{F}_s)$ can be described as a set of series

$$h(t) = \sum_{k=1}^{\infty} c_k(t) e_k,$$

where $c_k(t)$ are scalar functions that are square-integrable over $[0, T]$ and such that

$$\sum_{k=1}^{\infty} \lambda_k^{2s} \int_0^T [c_k(t)]^2 \, dt < \infty . \tag{1.1}$$

Below we also use the space $C(0, T; \mathcal{F}_s)$ of strongly continuous functions on $[0, T]$ with the values in $\mathcal{F}_s$ and the norm

$$|v|_{C(0, T; \mathcal{F}_s)} = \max_{t \in [0, T]} \|v(t)\|_s .$$

Exercise *1.5* Let $u(t)$ be a function with the values in $\mathcal{F}_s$ integrable over $[0, T]$. Show that the function

$$v(t) = \int_0^t u(\tau) \, d\tau$$

lies in $C(0, T; \mathcal{F}_s)$. Moreover, $v(t)$ is an absolutely continuous function with the values in $\mathcal{F}_s$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any collection of disjoint segments $[\alpha_k, \beta_k] \subset [0, T]$ the condition $\sum_k (\beta_k - \alpha_k) < \delta$ implies that

$$\sum_k \|v(\beta_k) - v(\alpha_k)\|_s < \varepsilon .$$

Exercise *1.6* Show that for any absolutely continuous function $v(t)$ on $[0, T]$ with the values in $\mathcal{F}_s$ there exists a function $u(t)$ with the values in $\mathcal{F}_s$ such that it is integrable over $[0, T]$ and

$$v(t) = v(0) + \int_0^t u(\tau) \, d\tau , \qquad t \in [0, T] .$$

(*Hint:* use the one-dimensional variant of this assertion).

The space

$$W_T = \left\{ v(t): \ v(t) \in L^2(0, T; \mathcal{F}_1), \quad \dot{v}(t) \in L^2(0, T; H) \right\} \tag{1.2}$$

with the norm

$$|v|_{W_T} = \left(\|v\|^2_{L^2(0, T; \mathcal{F}_1)} + \|\dot{v}\|^2_{L^2(0, T; H)} \right)^{1/2}$$

plays an important role below. Hereinafter the derivative $\dot{v}(t) = dv/dt$ stands for a function integrable over $[0, T]$ and such that

$$v(t) = h + \int_0^t \dot{v}(\tau)\,d\tau$$

almost everywhere for some $h \in H$ (see Exercises 1.5 and 1.6). Evidently, the space W_T is continuously embedded into $C(0, T; H)$, i.e. every function $u(t)$ from W_T lies in $C(0, T; H)$ and

$$\max_{t \in [0, T]} \|u(t)\| \le C|u|_{W_T},$$

where C is a constant. This fact is strengthened in the series of exercises given below.

— Exercise *1.7* Let p_m be the projector onto the span of the set $\{e_k : k = 1, \ldots, m\}$ and let $v(t) \in W_T$. Show that $p_m v(t)$ is absolutely continuous and possesses the property

$$\frac{d}{dt}(p_m v(t)) = p_m \dot{v}(t) \in L^2(0, T; p_m \mathcal{F}_1).$$

— Exercise *1.8* The equations

$$\|p_m v(t)\|_{1/2}^2 = \|p_m v(s)\|_{1/2}^2 + 2\int_s^t (p_m v(\tau),\ p_m \dot{v}(\tau))_{1/2}\,d\tau \quad (1.3a)$$

and

$$(t-s)\|p_m v(t)\|_{1/2}^2 =$$

$$= \int_s^t \left(\|p_m v(\tau)\|_{1/2}^2 + 2(\tau - s)(p_m v(\tau),\ p_m \dot{v}(\tau))_{1/2}\right)d\tau \quad (1.3b)$$

are valid for any $0 \le s \le t \le T$ and $v(t) \in W_T$.

— Exercise *1.9* Use (1.3) to prove that

$$\sup_{t \in [0, T]} \|p_m v(t)\|_{1/2} \le C_T |v|_{W_T} \quad (1.4a)$$

and

$$\sup_{t \in [0, T]} \|(p_m - p_k) v(t)\|_{1/2} \le C_T |(p_m - p_k) v|_{W_T}. \quad (1.4b)$$

— Exercise *1.10* Use (1.4) to prove that W_T is continuously embedded into $C(0, T; \mathcal{F}_{1/2})$ and

$$\max_{t \in [0, T]} \|v(t)\|_{1/2} \le C_T |v|_{W_T}.$$

The following three exercises result in a particular case of Dubinskii's theorem (see Exercise 1.13).

— Exercise *1.11* Let $\{h_k(t)\}_{k=0}^{\infty}$ be an orthonormal basis in $L^2(0, T; \mathbb{R})$ consisting of the trigonometric functions

$$h_0(t) = \frac{1}{\sqrt{T}}, \quad h_{2k-1}(t) = \sqrt{\frac{2}{T}} \sin\frac{2\pi k}{T}x, \quad h_{2k}(t) = \sqrt{\frac{2}{T}} \cos\frac{2\pi k}{T}x,$$

$k = 1, 2, \ldots$. Show that $f(t) \in L^2(0, T; \mathcal{F}_s)$ if and only if

$$f(t) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} c_{kj} h_k(t) e_j \tag{1.5}$$

and

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \lambda_j^{2s} |c_{kj}|^2 < \infty .$$

— Exercise *1.12* Show that the space W_T can be described as a set of series of the form (1.5) such that

$$\sum_{k,\, j=1}^{\infty} \left(k^2 + \lambda_j^2\right) |c_{kj}|^2 < \infty .$$

— Exercise *1.13* Use the method of the proof of Theorem 2.1.1 to show that W_T is compactly embedded into the space $L^2(0, T; \mathcal{F}_s)$ for any $s < 1$.

— Exercise *1.14* Show that W_T is compactly embedded into $C(0, T; H)$. *Hint:* use Exercise 1.10 and the reasoning which is usually applied to prove the Arzelà theorem on the compactness of a collection of scalar continuous functions.

§ 2 *Auxiliary Linear Problem*

In this section we study the properties of a solution to the following linear problem:

$$\begin{cases} \dfrac{d^2u}{dt^2} + \gamma \dfrac{du}{dt} + A^2 u + b(t) A u = h(t) , & (2.1) \\[2ex] u\big|_{t=0} = u_0, \quad \left.\dfrac{du}{dt}\right|_{t=0} = u_1 . & (2.2) \end{cases}$$

Here A is a positive operator with discrete spectrum. The vectors $h(t)$, u_0, u_1 as well as the scalar function $b(t)$ are given (for the corresponding hypotheses see the assertion of Theorem 2.1).

The main results of this section are the proof of the theorem on the existence and uniqueness of weak solutions to problem (2.1) and (2.2) and the construction of the evolutionary operator for the system when $h(t) \equiv 0$. In fact, the approach we use here is well-known (see, e.g., [6] and [7]).

A ***weak solution*** to problem (2.1) and (2.2) on a segment $[0, T]$ is a function $u(t) \in W_T$ such that $u(0) = u_0$ and the equation

$$-\int_0^T (\dot{u}(t) + \gamma u(t),\ \dot{v}(t))\,\mathrm{d}t + \int_0^T (Au(t) + b(t)u(t),\ Av(t))\,\mathrm{d}t =$$

$$= (u_1 + \gamma u_0,\ v(0)) + \int_0^T (h(t),\ v(t))\,\mathrm{d}t \tag{2.3}$$

holds for any function $v(t) \in W_T$ such that $v(T) = 0$. As above, $\dot{u}$ stands for the derivative of u with respect to t.

Exercise *2.1* Prove that if a weak solution $u(t)$ exists, then it satisfies the equation

$$(\dot{u}(t) + \gamma u(t),\ w) = (u_1 + \gamma u_0,\ w) -$$

$$-\int_0^t (Au(\tau) + b(\tau)u(\tau),\ Aw)\,\mathrm{d}\tau + \int_0^t (h(\tau),\ w)\,\mathrm{d}\tau \tag{2.4}$$

for every $w \in \mathcal{F}_1$ (*Hint:* take $v(t) = \int_t^T \varphi(\tau)\,\mathrm{d}\tau \cdot w$ in (2.3), where $\varphi(t)$ is a scalar function from $C[0, T]$).

Theorem 2.1

Let $u_0 \in \mathcal{F}_1$, $u_1 \in \mathcal{F}_0$, ***and*** $\gamma \geq 0$. ***We also assume that*** $b(t)$ ***is a bounded continuous function on*** $[0, T]$ ***and*** $h(t) \in L^\infty(0, T;\ \mathcal{F}_0)$, ***where*** T ***is a positive number. Then problem*** (2.1) ***and*** (2.2) ***has a unique weak solution*** $u(t)$ ***on the segment*** $[0, T]$. ***This solution possesses the properties***

$$u(t) \in C(0, T;\ \mathcal{F}_1), \qquad \dot{u}(t) \in C(0, T;\ \mathcal{F}_0) \tag{2.5}$$

and satisfies the energy equation

$$\frac{1}{2}\left(\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right) + \gamma\int_0^t \|\dot{u}(\tau)\|^2\,\mathrm{d}\tau + \int_0^t b(\tau)(Au(\tau),\ \dot{u}(\tau))\,\mathrm{d}\tau =$$

$$= \frac{1}{2}\left(\|u_1\|^2 + \|Au_0\|^2\right) + \int_0^t (h(\tau),\ \dot{u}(\tau))\,\mathrm{d}\tau\ . \tag{2.6}$$

Proof.

We use the compactness method to prove this theorem. At first we construct approximate solutions to problem (2.1) and (2.2). The approximate Galerkin solution (to this problem) of the order m with respect to the basis $\{e_k\}$ is considered to be the function

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k \tag{2.7}$$

satisfying the equations

$$(\ddot{u}_m + \gamma \dot{u}_m + A^2 u_m + b(t) A u_m - h(t),\ e_j) = 0\ , \tag{2.8}$$

$$(u_m(0),\ e_j) = (u_0,\ e_j)\ , \quad (\dot{u}_m(0),\ e_j) = (u_1,\ e_j)\ , \quad j = 1,\ 2,\ \ldots,\ m\ . \tag{2.9}$$

Here $g_k(t) \in C^1(0,\ T)$ and $\dot{g}_k(t)$ is absolutely continuous. Due to the orthogonality of the basis $\{e_k\}$ equations (2.8) and (2.9) can be rewritten as a system of ordinary differential equations:

$$\ddot{g}_k + \gamma \dot{g}_k + \lambda_k^2 g_k - b(t) \lambda_k g_k = h_k(t) \equiv (h(t),\ e_k)\, \mathrm{e},$$

$$g_k(0) = (u_0,\ e_k)\ , \quad \dot{g}_k(0) = (u_1,\ e_k)\ , \quad k = 1,\ 2,\ \ldots,\ m\ .$$

Lemma 2.1

Assume that $\gamma > 0$, $a \geq 0$, $b(t)$ is continuous, and $c(t)$ is a measurable bounded function. Then the Cauchy problem

$$\begin{cases} \ddot{g} + \gamma \dot{g} + a(a - b(t)) g = c(t), & t \in [0,\ T]\ , \\ g(0) = g_0\ , \quad \dot{g}(0) = g_1 \end{cases} \tag{2.10}$$

is uniquely solvable on any segment $[0,\ T]$. Its solution possesses the property

$$\dot{g}(t)^2 + a^2 g(t)^2 \leq \left(g_1^2 + a^2 g_0^2 + \frac{1}{2\gamma} \int_0^t c(\tau)^2 \, \mathrm{d}\tau \right) e^{b_0 t}\ , \quad t \in [0,\ T]\ , \tag{2.11}$$

where $b_0 = \max_t |b(t)|$. Moreover, if $b(t) \in C^1(0,\ T)$, $c(t) \equiv 0$ and for all $t \in [0,\ T]$ the conditions

$$-\frac{1}{2} a + \frac{\gamma^2}{a} \leq b(t) \leq \frac{1}{2} a\ , \quad \dot{b}(t) + \gamma\left(\frac{a}{4} - b(t)\right) \geq 0\ , \tag{2.12}$$

hold, then the following estimate is valid:

$$\dot{g}(t)^2 + a^2 g(t)^2 \leq 3(g_1^2 + a^2 g_0^2) \exp\left(-\frac{\gamma}{2} t\right)\ . \tag{2.13}$$

Proof.

Problem (2.10) is solvable at least locally, i.e. there exists $\bar{t}$ such that a solution exists on the half-interval $[0, \bar{t})$. Let us prove estimate (2.11) for the interval of existence of solution. To do that, we multiply equation (2.10) by $\dot{g}(t)$. As a result, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{g}^2 + a^2 g^2) + \gamma \dot{g}^2 = a\,b(t) g \dot{g} + c(t)\dot{g}\,.$$

We integrate this equality and use the equations

$$a|b(t) g \dot{g}| \le \frac{1}{2}\max_t |b(t)| \left(\dot{g}^2 + a^2 g^2\right), \qquad c\dot{g} \le \gamma \dot{g}^2 + \frac{1}{4\gamma}c^2\,,$$

to obtain that

$$\dot{g}(t)^2 + a^2 g(t) \le g_1^2 + a^2 g_0^2 + \frac{1}{2\gamma}\int_0^t c^2(\tau)\,\mathrm{d}\tau + b_0 \int_0^t (\dot{g}(\tau)^2 + a^2 g(\tau)^2)\,\mathrm{d}\tau\,.$$

This and Gronwall's lemma give us (2.11).

In particular, estimate (2.11) enables us to prove that the solution $g(t)$ can be extended on a segment $[0, T]$ of arbitrary length. Indeed, let us assume the contrary. Then there exists a point $\bar{t}$ such that the solution can not be extended through it. Therewith equation (2.11) implies that

$$\dot{g}(t)^2 + a^2 g(t)^2 \le C(T;\ g_0,\, g_1)\,, \qquad 0 < t < \bar{t} < T\,.$$

Therefore, (2.10) gives us that the derivative $\ddot{g}(t)$ is bounded on $[0, \bar{t})$. Hence, the values

$$\dot{g}(t) = g_0 + \int_0^t \ddot{g}(\tau)\,\mathrm{d}\tau\,, \qquad g(t) = g_0 + \int_0^t \dot{g}(\tau)\,\mathrm{d}\tau$$

are continuous up to the point $\bar{t}$. If we now apply the local theorem on existence to system (2.10) with the initial conditions at the point $\bar{t}$ that are equal to $g(\bar{t})$ and $\dot{g}(\bar{t})$, then we obtain that the solution can be extended through $\bar{t}$. This contradiction implies that the solution $g(t)$ exists on an arbitrary segment $[0, T]$.

Let us prove estimate (2.13). To do that, we consider the function

$$V(t) = \frac{1}{2}\left(\dot{g}^2 + a(a - b(t))\,g^2\right) + \frac{\gamma}{2}\left(g\dot{g} + \frac{\gamma}{2}g^2\right). \tag{2.14}$$

Using the inequality

$$-\frac{1}{2\gamma}\dot{g}^2 - \frac{\gamma}{2}g^2 \le g\dot{g} \le \frac{1}{2\gamma}\dot{g}^2 + \frac{\gamma}{2}g^2\,,$$

it is easy to find that the equation

$$\frac{1}{4}\left(\dot{g}^2 + a^2 g^2\right) \le V(t) \le \frac{3}{4}\left(\dot{g}^2 + a^2 g^2\right) \tag{2.15}$$

holds under the condition

$$\frac{1}{2}a - b(t) \geq 0\,, \qquad a\left[\frac{1}{2}a + b(t)\right] - \gamma^2 \geq 0\,.$$

Further we use (2.10) with $c(t) \equiv 0$ to obtain that

$$\frac{dV}{dt} = -\frac{\gamma}{2}\dot{g}^2 - \frac{1}{2}(a\dot{b} + a\gamma(a-b))g^2\,.$$

Consequently, with the help of (2.15) we get

$$\frac{dV}{dt} + \frac{\gamma}{2}V \leq 0$$

under conditions (2.12). This implies that

$$V(t) \leq V(0)\exp\left(-\frac{\gamma}{2}t\right).$$

We use (2.15) to obtain estimate (2.13). Thus, Lemma 2.1 is proved.

Exercise *2.2* Assume that $\gamma \leq 0$ in Lemma 2.1. Show that problem (2.10) is uniquely solvable on any segment $[0, T]$ and the estimate

$$\dot{g}(t) + a^2 g(t)^2 \leq \left(g_1^2 + a^2 g_0^2\right) e^{(b_0 + 2|\gamma| + \delta)t} + $$
$$+ \frac{1}{\delta}\int_0^t c(\tau)^2 e^{(b_0 + 2|\gamma| + \delta)(t-\tau)}\,d\tau$$

is valid for $g(t)$ and for any $\delta > 0$, where $b_0 = \max_t |b(t)|$.

Lemma 2.1 implies the existence of a sequence of approximate solutions $\{u_m(t)\}$ to problem (2.1) and (2.2) on any segment $[0, T]$.

Exercise *2.3* Show that every approximate solution u_m is a solution to problem (2.1) and (2.2) with $u_0 = u_{0m}$, $u_1 = u_{1m}$, and $h(x, t) = h_m(x, t)$, where

$$u_{im} = p_m u_i = \sum_{k=1}^{m} (u_i, e_k) e_k\,,$$
$$h_m = p_m h = \sum_{k=1}^{m} (h(t), e_k) e_k\,, \qquad (2.16)$$

and p_m is the orthoprojector onto the span of elements $\{e_k : k = 1, 2, \ldots, m\}$ in $\mathcal{F}_0 = H$.

Let us prove that the sequence of approximate solutions $\{u_m\}$ is convergent.

At first we note that

$$\Delta_{m,\,l}(t) = \|\dot{u}_m - \dot{u}_{m+l}\|^2 + \|A(u_m - u_{m+l})\|^2 = \sum_{k=m+1}^{m+l} (\dot{g}_k(t)^2 + \lambda_k^2 g_k(t)^2)$$

for every $t \in [0, T]$. Therefore, by virtue of Lemma 2.1 we have that

$$\Delta_{m,\,l}(t) \le e^{b_0 t} \sum_{k=m+1}^{m+l} \left[(u_1, e_k)^2 + \lambda_k^2 (u_0, e_k)^2 + \frac{1}{2\gamma} \int_0^T (h(t), e_k)^2 \,\mathrm{d}t \right]$$

for $\gamma > 0$. Moreover, in the case $\gamma = 0$, the result of Exercise 2.2 gives that

$$\Delta_{m,\,l}(t) \le e^{(b_0+1)t} \sum_{k=m+1}^{m+l} \left[(u_1, e_k)^2 + \lambda_k^2 (u_0, e_k)^2 + \int_0^T (h(t), e_k)^2 \,\mathrm{d}t \right].$$

These equations imply that the sequences $\{\dot{u}_m(t)\}$ and $\{Au_m(t)\}$ are the Cauchy sequences in the space $C(0, T; H)$ on any segment $[0, T]$. Consequently, there exists a function $u(t)$ such that

$$\dot{u}(t) \in C(0, T;\, H), \qquad u(t) \in C(0, T;\, \mathcal{F}_1),$$

$$\lim_{m \to \infty} \max_{[0,\,T]} \left(\|\dot{u}_m(t) - \dot{u}(t)\|^2 + \|u_m(t) - u(t)\|_1 \right) = 0. \tag{2.17}$$

Equations (2.8) and (2.9) further imply that

$$-\int_0^T \left(\dot{u}_m(t) + \gamma u_m(t),\ \dot{v}(t) \right) \mathrm{d}t + \int_0^T \left(A u_m(t) + b(t) u_m(t),\ A v(t) \right) \mathrm{d}t =$$

$$= (u_{1m} + \gamma u_{0m},\ v(0)) + \int_0^T (h_m(t),\ v(t))\,\mathrm{d}t$$

for all functions $v(t)$ from W_T such that $v(T) = 0$. Here u_{im}, and $h_m(t)$, $i = 0,\ 1$, are defined by (2.16). We use equation (2.17) to pass to the limit in this equation and to prove that the function $u(t)$ satisfies equality (2.3). Moreover, it follows from (2.17) that $u(0) = u_0$. Therefore, the function $u(t)$ is a weak solution to problem (2.1) and (2.2).

In order to prove the uniqueness of weak solutions we consider the function

$$v_s(t) = \begin{cases} -\displaystyle\int_t^s u(\tau)\,\mathrm{d}\tau, & t < s, \\ 0, \quad s \le t \le T, & \end{cases} \tag{2.18}$$

for $s \in [0, T]$. Here $u(t)$ is a weak solution to problem (2.1) and (2.2) for $h = 0$, $u_0 = 0$, and $u_1 = 0$. Evidently $v_s(t) \in W_T$. Therefore,

$$-\int_0^T (\dot{u}(t)+\gamma u(t),\ \dot{v}_s(t))\,\mathrm{d}t + \int_0^T (Au(t)+b(t)u(t),\ Av_s(t))\,\mathrm{d}t = 0\,.$$

Due to the structure of the function $v_s(t)$ we obtain that

$$\frac{1}{2}\Big(\|u(s)\|^2+\|Av_s(0)\|^2\Big)+\gamma\int_0^s \|u(t)\|^2\,\mathrm{d}t = J_s(u,\ v)\,, \tag{2.19}$$

where

$$J_s(u,\ v) = \int_0^s (b(t)u(t),\ Av_s(t))\,\mathrm{d}t\,.$$

It is evident that $Av_s(t) = Av_s(0)-Av_t(0)$ for $t \le s$. Therefore,

$$\big|J_s(u,\ v)\big| \le b_0\left(\|Av_s(0)\|\int_0^s \|u(t)\|\,\mathrm{d}t + \int_0^s \|u(t)\|\cdot\|Av_t(0)\|\,\mathrm{d}t\right) \le$$

$$\le \frac{1}{4}\|Av_s(0)\|^2 + s\,b_0^2\int_0^s \|u(t)\|^2\,\mathrm{d}t + \frac{b_0}{2}\int_0^s \Big(\|u(t)\|^2+\|Av_t(0)\|^2\Big)\,\mathrm{d}t\ .$$

If we substitute this estimate into equation (2.19), then it is easy to find that

$$\|u(s)\|^2+\|Av_s(0)\|^2 \le C_T\int_0^s \Big(\|u(t)\|^2+\|Av_t(0)\|^2\Big)\,\mathrm{d}t\ ,$$

where $s\in[0,\ T]$ and C_T is a positive constant depending on the length of the segment $[0,\ T]$. This and Gronwall's lemma imply that $u(t)\equiv 0$.

Let us prove the energy equation. If we multiply equation (2.8) by $\dot{g}_j(t)$ and summarize the result with respect to j, then we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\dot{u}_m\|^2+\|Au_m\|^2\Big)+\gamma\,\|\dot{u}_m\|^2+b(t)(Au_m,\ \dot{u}_m) = (h,\ \dot{u}_m)\,.$$

After integration with respect to t we use (2.17) to pass to the limit and obtain (2.6).

Theorem 2.1 is completely proved.

Exercise *2.4* Prove that the estimate

$$\|\dot{u}(t)\|^2+\|Au(t)\|^2 \le \left(\|u_1\|^2+\|Au_0\|^2+\frac{1}{2\gamma}\int_0^t \|h(\tau)\|^2\,\mathrm{d}\tau\right)e^{b_0 t} \tag{2.20}$$

is valid for a weak solution $u(t)$ to problem (2.1) and (2.2). Here $b_0 = \max\{|b(t)|:\ t\ge 0\}$ and $\gamma > 0$.

— Exercise *2.5* Let $u(t)$ be a weak solution to problem (2.1) and (2.2). Prove that $A^2u(t) \in C(0, T; \mathcal{F}_{-1})$ and

$$\dot{u}(t) + \gamma u(t) = u_1 + \gamma u_0 - \int_0^t (A^2 u(\tau) + b(\tau) A u(\tau) - h(\tau))\, d\tau .$$

Here we treat the equality as an equality of elements in $\mathcal{F}_{-1}$.
(*Hint:* use the results of Exercises 2.1 and 2.1.3).

— Exercise *2.6* Let $u(t)$ be a weak solution to problem (2.1) and (2.2) constructed in Theorem 2.1. Then the function $\dot{u}(t)$ is absolutely continuous as a vector-function with the values in $\mathcal{F}_{-1}$ while the derivative $\ddot{u}(t)$ belongs to the space $L^\infty(0, T; \mathcal{F}_{-1})$. Moreover, the function $u(t)$ satisfies equation (2.1) if we treat it as an equality of elements in $\mathcal{F}_{-1}$ for almost all $t \in [0, T]$.

In particular, the result of Exercise 2.6 shows that a weak solution satisfies equation (2.1) in a stronger sense then (2.4).

We also note that the assertions of Theorem 2.1 and Exercises 2.4–2.6 with the corresponding changes remain true if the initial condition is given at any other moment t_0 which is not equal to zero.

Now we consider the case $h(t) \equiv 0$ and construct the evolutionary operator of problem (2.1) and (2.2). To do that, let us consider the family of spaces

$$\mathcal{H}_\sigma = \mathcal{F}_{1+\sigma} \times \mathcal{F}_\sigma , \qquad \sigma \ge 0 .$$

Every space $\mathcal{H}_\sigma$ is a set of pairs $y = (u; v)$ such that $u \in \mathcal{F}_{1+\sigma}$ and $v \in \mathcal{F}_\sigma$. We define the inner product in $\mathcal{H}_\sigma$ by the formula

$$(y_1, y_2)_{\mathcal{H}_\sigma} = (u_1, u_2)_{1+\sigma} + (v_1, v_2)_\sigma .$$

— Exercise *2.7* Prove that $\mathcal{H}_{\sigma_1}$ is compactly embedded into $\mathcal{H}_\sigma$ for $\sigma_1 > \sigma$.

In the space $\mathcal{H}_0$ we define the evolutionary operator $U(t; t_0)$ of problem (2.1) and (2.2) for $h(t) \equiv 0$ by the equation

$$U(t; t_0)y = (u(t); \dot{u}(t)), \tag{2.21}$$

where $u(t)$ is a solution to (2.1) and (2.2) at the moment t with initial conditions that are equal to $y = (u_0; u_1)$ at the moment t_0.

The following assertion plays an important role in the study of asymptotic behaviour of solutions to problem (0.1) and (0.2).

Theorem 2.2

Assume that the function $b(t)$ *is continuously differentiable in* (2.1) *and such that*

$$b_0 = \sup_t |b(t)| < \infty, \qquad b_1 = \sup_t |\dot{b}(t)| < \infty .$$

Then the evolutionary operator $U(t;\ \tau)$ *of problem* (2.1) *and* (2.2) *for* $h(t) \equiv 0$ *is a linear bounded operator in each space* $\mathcal{H}_\sigma$ *for* $\sigma \geq 0$ *and it possesses the properties:*

a) $U(t;\ \tau)U(\tau;\ s) = U(t;\ s),\quad t \geq \tau \geq s,\quad U(t;\ t) = I;$

b) *for all* $\sigma \geq 0$ *the estimate*

$$\|U(t;\ \tau)y\|_{\mathcal{H}_\sigma} \leq \|y\|_{\mathcal{H}_\sigma} \exp\left(\frac{1}{2} b_0 (t-\tau)\right) \tag{2.22}$$

is valid;

c) *there exists a number* N_0 *depending on* γ, b_0, *and* b_1 *such that the equation*

$$\|(I - P_N)U(t;\ \tau)y\|_{\mathcal{H}_\sigma} \leq \sqrt{3}\,\|(I-P_N)y\|_{\mathcal{H}_\sigma} e^{-\frac{\gamma}{4}(t-\tau)}, \qquad t > \tau, \tag{2.23}$$

holds for all $N \geq N_0$, *where* P_N *is the orthoprojector onto the subspace*

$$L_N = \mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k):\ k = 1,\ 2,\ \ldots,\ N\}$$

in the space $\mathcal{H}_0$.

Proof.

Semigroup property a) follows from the uniqueness of a weak solution. The boundedness property of the operator $U(t;\ \tau)$ follows from (2.22). Let us prove relations (2.22) and (2.23). It is sufficient to consider the case $\tau = 0$. According to the definition of the evolutionary operator we have that

$$U(t;\ 0)y = (u(t);\ \dot{u}(t)), \qquad y = (u_0;\ u_1),$$

where $u(t)$ is the weak solution to problem (2.1) and (2.2) for $h(t) \equiv 0$. Due to (2.17) it can be represented as a convergent series of the form

$$u(t) = \sum_{k=1}^{\infty} g_k(t) e_k .$$

Moreover, Lemma 2.1 implies that

$$\dot{g}_k(t)^2 + \lambda_k^2 g_k(t)^2 \leq (\dot{g}_k(0)^2 + \lambda_k^2 g_k(0)^2)\, e^{b_0 t}. \tag{2.24}$$

Since

$$\|U(t;\ 0)y\|^2_{\mathcal{H}_\sigma} = \sum_{k=1}^{\infty} \left(\dot{g}_k(t)^2 + \lambda_k^2 g_k(t)^2\right) \lambda_k^{2\sigma},$$

equation (2.24) implies (2.22).

Further we use equation (2.13) to obtain that

$$\dot g_k(t)^2 + \lambda_k^2 g_k(t)^2 \le 3\left(g_k(0)^2 + \lambda_k^2 g_k(0)^2\right) e^{-\frac{\gamma}{2}t}, \tag{2.25}$$

provided the conditions (cf. (2.12))

$$-\frac{1}{2}\lambda_k + \frac{\gamma^2}{\lambda_k} \le b(t) \le \frac{1}{2}\lambda_k, \qquad \dot b(t) + \gamma\left(\frac{\lambda_k}{4} - b(t)\right) \ge 0,$$

are fulfilled. Evidently, these conditions hold if

$$\lambda_k \ge \frac{4b_1}{\gamma} + 4b_0 + \frac{\gamma^2}{b_0},$$

where $b_0 = \max_t |b(t)|$ and $b_1 = \max_t |\dot b(t)|$. Since

$$\|(I - P_N)U(t;\ 0)y\|^2_{\mathcal{H}_\sigma} = \sum_{k=N+1}^{\infty} \left(\dot g_k(t)^2 + k^4 g_k(t)^2\right) k^{2\sigma},$$

equation (2.25) gives us (2.23) for all $N \ge N_0 - 1$, where N_0 is the smallest natural number such that

$$\lambda_{N_0} \ge \frac{4b_1}{\gamma} + 4b_0 + \frac{\gamma^2}{b_0}. \tag{2.26}$$

Thus, **Theorem 2.2 is proved**.

— Exercise *2.8* Show that a weak solution $u(t)$ to problem (2.1) and (2.2) can be represented in the form

$$(u(t);\ \dot u(t)) = U(t;\ 0)y + \int_0^t U(t;\ \tau)(0;\ h(\tau))\, d\tau, \tag{2.27}$$

where $y = (u_0;\ u_1)$ and $U(t;\ \tau)$ is defined by (2.21).

— Exercise *2.9* Use the result of Exercise 2.2 to show that Theorem 2.1 and Theorem 2.2 (a, b) with another constant in (2.22) also remain true for $\gamma < 0$. Use this fact to prove that if the hypotheses of Theorems 2.1 and 2.2 hold on the whole time axis, then problem (2.1) and (2.2) is solvable in the class of functions

$$\mathcal{W} = C(\mathbb{R};\ \mathcal{F}_1) \cap C^1(\mathbb{R};\ \mathcal{F}_0)$$

with $\gamma \ge 0$.

— Exercise *2.10* Show that the evolutionary operator $U(t,\ \tau)$ has a bounded inverse operator in every space $\mathcal{H}_\sigma$ for $\sigma \ge 0$. How is the operator $[U(t,\ \tau)]^{-1}$ for $t > \tau$ related to the solution to equation (2.1) for $h(t) \equiv 0$? Define the operator $U(t,\ \tau)$ using the formula $U(t,\ \tau) = [U(\tau,\ t)]^{-1}$ for $t < \tau$ and prove assertion (a) of Theorem 2.2 for all $t, \tau \in \mathbb{R}$.

§ 3 Theorem on Existence and Uniqueness of Solutions

In this section we use the compactness method (see, e.g., [8]) to prove the theorem on the existence and uniqueness of weak solutions to problem (0.1) and (0.2) under the assumption that

$$u_0 \in \mathscr{F}_1, \qquad u_1 \in \mathscr{F}_0, \qquad p(t) \in L^\infty(0, T; \mathscr{F}_0); \tag{3.1}$$

$$M(z) \in C^1(\mathbb{R}_+), \qquad \mathscr{M}(z) \equiv \int_0^z M(\xi)\, d\xi \ \geq -az - b, \tag{3.2}$$

where $0 \le a < \lambda_1$, $b \in \mathbb{R}$, λ_1 is the first eigenvalue of the operator A, and the operator L is defined on $D(A)$ and satisfies the estimate

$$\|Lu\| \le C\|Au\|, \qquad u \in D(A). \tag{3.3}$$

Similarly to the linear problem (see Section 2), the function $u(t) \in W_T$ is said to be a ***weak solution*** to problem (0.1) and (0.2) on the segment $[0, T]$ if $u(0) = u_0$ and the equation

$$-\int_0^T (\dot{u}(t) + \gamma u(t),\ \dot{v}(t))\, dt + \int_0^T \Big(Au(t) + M\big(\|A^{1/2}u(t)\|^2\big)u(t),\ Av(t)\Big)\, dt \ +$$

$$+\int_0^T (Lu(t),\ v(t))\, dt \ = \ (u_1 + \gamma u_0,\ v(0)) + \int_0^T (p(t),\ v(t))\, dt \tag{3.4}$$

holds for any function $v(t) \in W_T$ such that $v(T) = 0$. Here the space W_T is defined by equation (1.2).

Exercise *3.1* Prove the analogue of formula (2.4) for weak solutions to problem (0.1) and (0.2).

The following assertion holds.

Theorem 3.1

Assume that conditions (3.1)–(3.3) ***hold. Then on every segment*** $[0, T]$ ***problem*** (0.1) ***and*** (0.2) ***has a weak solution*** $u(t)$. ***This solution is unique. It possesses the properties***

$$u(t) \in C(0, T; \mathscr{F}_1), \qquad \dot{u}(t) \in C(0, T; \mathscr{F}_0) \tag{3.5}$$

and satisfies the energy equality

$$E(u(t),\ \dot{u}(t)) = E(u_0,\ u_1) + \int_0^t \Big(-\gamma\|\dot{u}(\tau)\|^2 + (-Lu(t) + p(t),\ \dot{u}(\tau))\Big)\,d\tau\,, \quad (3.6)$$

where

$$E(u,\ v) = \frac{1}{2}\Big(\|v\|^2 + \|Au\|^2 + \mathscr{M}\Big(\|A^{1/2}u\|^2\Big)\Big). \quad (3.7)$$

We use the scheme from Section 2 to prove the theorem.

The Galerkin approximate solution of the order m to problem (0.1) and (0.2) with respect to the basis e_k is defined as a function of the form

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k$$

which satisfies the equations

$$(\ddot{u}_m(t) + \gamma \dot{u}_m(t),\ e_j) \ +$$

$$+\Big(Au_m(t) + M\Big(\|A^{1/2}u\|^2\Big)u_m(t),\ Ae_j\Big) + (Lu_m(t) - p(t),\ e_j) = 0 \quad (3.8)$$

for $j = 1,\ 2,\ \ldots,\ m$ with $t \in (0,\ T]$ and the initial conditions

$$(u_m(0),\ e_j) = (u_0,\ e_j),\quad (\dot{u}_m(0),\ e_j) = (u_1,\ e_j),\qquad j = 1,\ 2,\ \ldots,\ m. \quad (3.9)$$

Simple calculations show that the problem of determining of approximate solutions can be reduced to solving the following system of ordinary differential equations:

$$\ddot{g}_k + \gamma\dot{g}_k + \lambda_k^2 g_k + \lambda_k M\left(\sum_{j=1}^{m}\lambda_j g_j(t)^2\right)g_k + \sum_{j=1}^{m}(Le_j,\ e_k)g_j = p_k(t), \quad (3.10)$$

$$g_k(0) = g_{0k} = (u_0,\ e_k),\qquad \dot{g}_k(0) = g_{1k} = (u_1,\ e_k),\qquad k = 1,\ 2,\ \ldots,\ m. \quad (3.11)$$

The nonlinear terms of this system are continuously differentiable with respect to g_j. Therefore, it is solvable at least locally. The global solvability follows from the a priori estimate of a solution as in the linear problem. Let us prove this estimate.

We consider an approximate solution $u_m(t)$ to problem (0.1) and (0.2) on the solvability interval $(0,\ \bar{t})$. It satisfies equations (3.8) and (3.9) on the interval $(0,\ \bar{t})$. We multiply equation (3.8) by $\dot{g}_j(t)$ and summarize these equations with respect to j from 1 to m. Since

$$\frac{d}{dt}\mathscr{M}\Big(\|A^{1/2}u\|^2\Big) = 2M\Big(\|A^{1/2}u\|^2\Big)(Au(t),\ \dot{u}(t))\,,$$

we obtain

$$\frac{d}{dt}E(u_m(t),\ \dot{u}_m(t)) = -\gamma\,\|\dot{u}_m(t)\|^2 - (Lu_m(t) - p(t),\ \dot{u}_m(t)) \quad (3.12)$$

as a result, where $E(u, \dot{u})$ is defined by (3.7). Equation (3.3) implies that

$$\left|(Lu_m, \dot{u}_m)\right| \le C\|Au_m\|\|\dot{u}_m\| \le C\left(\|Au_m\|^2 + \|\dot{u}_m\|^2\right).$$

Condition (3.2) gives us the estimate

$$\frac{1}{2}\left(\|Au_m\|^2 + \|\dot{u}_m\|^2\right) \le C_1 + C_2 E(u_m, \dot{u}_m) \tag{3.13}$$

with the constants independent of m. Therefore, due to Gronwall's lemma equation (3.12) implies that

$$\|\dot{u}_m(t)\|^2 + \|Au_m(t)\|^2 \le \left(C_0 + C_1 E(u_m(0), \dot{u}_m(0))\right) e^{C_2 t}, \tag{3.14}$$

with the constants C_0, C_1, and C_2 depending on the problem parameters only.

— Exercise *3.2* Use equation (3.14) to prove the global solvability of Cauchy problem (3.10) and (3.11).

It is evident that

$$\|\dot{u}_m(0)\| \le \|u_1\| \quad \text{and} \quad \|A^{1/2}u_m(0)\| \le \frac{1}{\sqrt{\lambda_1}}\|Au_m(0)\| \le \frac{1}{\sqrt{\lambda_1}}\|u_0\|_1 .$$

Therefore,

$$E(u_m(0), \dot{u}_m(0)) \le \frac{1}{2}\left(\|u_1\|^2 + \|u_0\|_1^2 + C_M(\|u_0\|_1)\right),$$

where $C_M(\rho) = \max\{\mathcal{M}(z) : 0 \le z \le \rho^2/\lambda_1\}$. Consequently, equation (3.14) gives us that

$$\sup_{t \in [0, T]}\left(\|\dot{u}_m(t)\|^2 + \|Au_m(t)\|^2\right) \le C_T \tag{3.15}$$

for any $T > 0$, where C_T does not depend on m. Thus, the set of approximate solutions $\{u_m(t)\}$ is bounded in W_T for any $T > 0$. Hence, there exist an element $u(t) \in W_T$ and a sequence $\{m_k\}$ such that $u_{m_k}(t) \to u(t)$ weakly in W_T. Let us show that the weak limit point $u(t)$ possesses the property

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le C_T \tag{3.16}$$

for almost all $t \in [0, T]$. Indeed, the weak convergence of the sequence $\{u_{m_k}\}$ to the function u in W_T means that $\dot{u}_{m_k}$ and Au_{m_k} weakly (in $L^2(0, T; H)$) converge to $\dot{u}$ and Au respectively. Consequently, this convergence will also take place in $L^2(a, b; H)$ for any a and b from the segment $[0, T]$. Therefore, by virtue of the known property of the weak convergence we get

$$\int_a^b \left(\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right) dt \le \varliminf_{k \to \infty} \int_a^b \left(\|\dot{u}_{m_k}(t)\|^2 + \|Au_{m_k}(t)\|^2\right) dt .$$

With the help of (3.15) we find that

$$\int_a^b \left(\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right) \mathrm{d}t \;\le\; C_T(b-a)\,.$$

Therefore, due to the arbitrariness of a and b we obtain estimate (3.16).

Lemma 3.1

For any function $v(t) \in L^2(0,\, T;\; H)$

$$\lim_{k\to\infty} J_T(u_{m_k},\; v) = J_T(u,\; v)\;,$$

where

$$J_T(u,\; v) = \int_0^T M\left(\|A^{1/2}u(t)\|^2\right)(u(t),\; v(t))\,\mathrm{d}t\;.$$

Proof.

Since

$$\left|M\left(\left\|A^{1/2}u_{m_k}\right\|^2\right) - M\left(\|A^{1/2}u\|^2\right)\right| \;\le$$

$$\le \int_0^1 \left|\tilde{M}\left(\xi\left\|A^{1/2}u_{m_k}\right\| + (1-\xi)\|A^{1/2}u\|\right)\right| \mathrm{d}\xi \,\cdot\, \left\|A^{1/2}(u_{m_k} - u)\right\|\;,$$

where $\tilde{M}(z) = 2zM'(z^2) \in C(\mathbb{R}_+)$, due to (3.15) and (3.16) we have

$$\left|M\left(\left\|A^{1/2}u_{m_k}\right\|^2\right) - M\left(\|A^{1/2}u\|^2\right)\right| \;\le\; C_T^{(1)}\left\|A^{1/2}(u_{m_k}(t) - u(t))\right\|\,,$$

where the constant $C_T^{(1)}$ is the maximum of the function $\tilde{M}(z)$ on the sufficiently large segment $[0,\; a_T]$, determined by the constant C_T from inequalities (3.15) and (3.16). Hence,

$$\delta_k \equiv \int_0^T \left|M\left(\|A^{1/2}u_m(t)\|^2\right) - M\left(\|A^{1/2}u(t)\|^2\right)\right| \cdot \left|(u_{m_k}(t),\; v(t))\right| \mathrm{d}t \;\le$$

$$\le\; C_T\|v\|_{L^2(0,\,T;\;H)}\left(\int_0^T \left\|A^{1/2}(u_{m_k}(t) - u(t))\right\|^2 \mathrm{d}t\right)^{1/2}.$$

The compactness of the embedding of W_T into $L^2(0,\, T;\; \mathscr{F}_{1/2})$ (see Exercise 1.13) implies that $\delta_k \to 0$ as $k \to \infty$. It is evident that

$$\left|J_T(u_{m_k}, v) - J_T(u, v)\right| \le \delta_k + \left|\int_0^T (u_{m_k}(t) - u(t), v(t)) \cdot M\left(\|A^{1/2}u(t)\|^2\right) dt\right| .$$

Because of the weak convergence of u_{m_k} to u this gives us the assertion of the lemma.

Exercise 3.3 Prove that the functional

$$\mathcal{L}[u] = \int_0^T (Lu(t), v(t))\, dt$$

is continuous on W_T for any $v \in L^2(0, T; H)$.

Let us prove that the limit function $u(t)$ is a weak solution to problem (0.1) and (0.2).

Let p_l be the orthoprojector onto the span of elements e_k, $k = 1, 2, \ldots, l$ in the space H. We also assume that

$$\widetilde{W}_T = \{v \in W_T:\ v(T) = 0\}$$

and

$$\widetilde{W}_T^l \equiv p_l \widetilde{W}_T = \{p_l v:\ v \in \widetilde{W}_T\}.$$

It is clear that an arbitrary element of the space $\widetilde{W}_T^l$ has the form

$$v_l(x, t) = \sum_{k=1}^{l} \eta_k(t) e_k ,$$

where $\eta_k(t)$ is an absolutely continuous real function on $[0, T]$ such that

$$\eta_k(T) = 0 , \qquad \dot{\eta}_k(t) \in L^2(0, T).$$

If we multiply equation (3.8) by $\eta_j(t)$, summarize the result with respect to j from 1 to l, and integrate it with respect to t from 0 to T, then it is easy to find that

$$-\int_0^T (\dot{u}_m + \gamma u_m, \dot{v}_l)\, dt + \int_0^T \left(Au_m + M\left(\|A^{1/2}u_m\|^2\right)u_m, Av_l\right) dt + \int_0^T (Lu_m, v_l)\, dt =$$

$$= (u_1 + \gamma u_0, v_l(0)) + \int_0^T (p, v_l)\, dt$$

for $m \ge l$. The weak convergence of the sequence u_{m_k} to u in W_T as well as Lemma 3.1 and Exercise 3.3 enables us to pass to the limit in this equality and to show that the function $u(t)$ satisfies equation (3.4) for any function $v \in \widetilde{W}_T^l$, where $l = 1, 2, \ldots$. Further we use (cf. Exercise 2.1.11) the formula

$$\lim_{l\to\infty} \int_0^T \left(\|p_l \dot{v}(t) - \dot{v}(t)\|^2 + \|p_l A v(t) - A v(t)\|^2 \right) dt = 0$$

for any function $v(t) \in \widetilde{W}_T$ in order to turn from the elements v of $\widetilde{W}_T^l$ to the functions v from the space $\widetilde{W}_T$.

— Exercise *3.4* Prove that $u(t)|_{t=0} = u_0$.

Thus, every weak limit point $u(t)$ of the sequence of Galerkin approximations $\{u_m\}$ in the space W_T is a weak solution to problem (0.1) and (0.2).

If we compare equations (3.4) and (2.3), then we find that every weak solution $u(t)$ is simultaneously a weak solution to problem (2.1) and (2.2) with $b(t) = 0$ and

$$h(t) = -M\left(\|A^{1/2}u(t)\|^2\right) Au(t) - Lu(t) + p(t). \tag{3.17}$$

It is evident that $h(t) \in L^\infty(0, T; \mathcal{F}_0)$. Therefore, due to Theorem 2.1 equations (3.5) are valid for the function $u(t)$.

To prove energy equality (3.6) it is sufficient (due to (2.6)) to verify that for $h(t)$ of form (3.17) the equality

$$\int_0^t (h(\tau), \dot{u}(\tau))\, d\tau =$$

$$= \int_0^t (-Lu(\tau) + p(\tau), \dot{u}(\tau))\, d\tau - \frac{1}{2}\mathcal{M}\left(\|A^{1/2}u(\tau)\|^2\right)\Big|_{\tau=0}^{\tau=t} \tag{3.18}$$

holds. Here $u(t)$ is a vector-function possessing property (3.5). We can do that by first proving (3.18) for the function of the form $p_l u$ and then passing to the limit.

— Exercise *3.5* Let $u(t)$ be a weak solution to problem (0.1) and (0.2). Use equation (3.6) to prove that

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le (C_0 + C_1 E(u_0, u_1))\, e^{C_2 t}, \quad t > 0, \tag{3.19}$$

where $C_0, C_1, C_2 > 0$ are constants depending on the parameters of problem (0.1) and (0.2).

Let us prove the uniqueness of a weak solution to problem (0.1) and (0.2). We assume that $u_1(t)$ and $u_2(t)$ are weak solutions to problem (0.1) and (0.2) with the initial conditions $\{u_{01}, u_{11}\}$ and $\{u_{02}, u_{12}\}$, respectively. Then the function

$$u(t) = u_1(t) - u_2(t)$$

is a weak solution to problem (2.1) and (2.2) with the initial conditions $u_0 = u_{01} - u_{02}$, $u_1 = u_{11} - u_{12}$, the function $b(t) = M(\|A^{1/2}u_1(t)\|^2)$, and the right-hand side

$$h(t) = \left[M\left(\|A^{1/2}u_2(t)\|^2\right) - M\left(\|A^{1/2}u_1(t)\|^2\right)\right]Au_2(t) + L(u_2(t) - u_1(t)).$$

We use equation (3.19) to verify that

$$\|h(t)\| \le G_T(E(u_{01}, u_{11}) + E(u_{02}, u_{12}))\|A(u_1(t) - u_2(t))\|, \quad t \in [0, T],$$

where $G_T(\xi)$ is a positive monotonely increasing function of the parameter ξ. Therefore, equation (2.20) implies that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|^2 + \|u_1(t) - u_2(t)\|_1^2 \le$$

$$\le C_T\left(\|u_{11} - u_{12}\|^2 + \|u_{01} - u_{02}\|_1^2 + \int_0^t \|u_1(\tau) - u_2(\tau)\|_1^2 \, d\tau\right),$$

where $C_T > 0$ depends on T and the problem parameters and is a function of the variables $E(u_{0j}, u_{1j})$, $j = 1, 2$. We can assume that C_T is the same for all initial data such that $E(u_{0j}, u_{1j}) \le R$, $j = 1, 2$. Using Gronwall's lemma we obtain that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|^2 + \|u_1(t) - u_2(t)\|_1^2 \le C_1\left(\|u_{11} - u_{12}\|^2 + \|u_{01} - u_{02}\|_1^2\right)e^{aC_2}, \quad (3.20)$$

where $t \in [0, T]$ and $C > 0$ is a constant depending only on T, the problem parameters and the value $R > 0$ such that $\|u_{0j}\|_1^2 + \|u_{1j}\|^2 \le R$. In particular, this estimate implies the uniqueness of weak solutions to problem (0.1) and (0.2). The proof of Theorem 3.1 is complete.

Exercise *3.6* Show that a weak solution $u(t)$ satisfies equation (0.1) if we consider this equation as an equality of elements in $\mathcal{F}_{-1}$ for almost all t. Moreover, $\ddot{u}(t) \in C(0, T; \mathcal{F}_{-1})$ (*Hint:* see Exercise 2.5).

Exercise *3.7* Assume that the hypotheses of Theorem 3.1 hold. Let $u(t)$ be a weak solution to problem (0.1) and (0.2) on the segment $[0, T]$ and let $u_m(t)$ be the corresponding Galerkin approximation of the order m. Show that

$$u_m(t) \to u(t) \text{ weakly in } L^2(0, T; \mathcal{F}_1),$$

$$\dot{u}_m(t) \to \dot{u}(t) \text{ weakly in } L^2(0, T; \mathcal{F}_0),$$

$$u_m(t) \to u(t) \text{ strongly in } L^2(0, T; \mathcal{F}_s), \quad s < 1,$$

as $m \to \infty$.

In conclusion of the section we note that in case of stationary load $p(t) \equiv p \in H$ we can construct an evolutionary operator S_t of problem (0.1) and (0.2) in the space $\mathcal{H} \equiv \mathcal{H}_0 = \mathcal{F}_1 \times \mathcal{F}_0$ supposing that

$$S_t y = (u(t);\ \dot{u}(t))$$

for $y = (u_0,\ u_1)$, where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0;\ u_1)$. Due to the uniqueness of weak solutions we have

$$S_t \circ S_\tau = S_{t+\tau}, \qquad S_0 = I, \qquad t,\ \tau \geq 0\,.$$

By virtue of (3.20) the nonlinear mapping S_t is a continuous mapping of $\mathcal{H}$. Equation (3.5) implies that the vector-function $S_t y$ is strongly continuous with respect to t for any $y \in \mathcal{H}$. Moreover, for any $R > 0$ and $T > 0$ there exists a constant $C(R,\ T) > 0$ such that

$$\|S_t y_1 - S_t y_2\|_{\mathcal{H}} \ \leq\ C(R,\ T) \cdot \|y_1 - y_2\|_{\mathcal{H}} \tag{3.21}$$

for all $t \in [0,\ T]$ and for all $y_j \in \{y \in \mathcal{H}:\ \|y\|_{\mathcal{H}} \leq R\}$.

— Exercise *3.8* Use equation (3.21) to show that $(t,\ y) \to S_t y$ is a continuous mapping from $\mathbb{R}_+ \times \mathcal{H}$ into $\mathcal{H}$.

— Exercise *3.9* Prove the theorem on the existence and uniqueness of solutions to problem (0.1) and (0.2) for $\gamma \leq 0$. Use this fact to show that the collection of operators $\{S_t\}$ is defined for negative t and forms a group (*Hint:* cf. Exercises 2.9 and 2.10).

— Exercise *3.10* Prove that the mapping S_t is a homeomorphism in $\mathcal{H}$ for every $t > 0$.

— Exercise *3.11* Let $p(t) \in L^\infty(\mathbb{R}_+,\ H)$ be a periodic function: $p(t) = p(t + t_0)$, $t_0 > 0$. Define the family of operators S_m by the formula

$$S_m y\ = (u(m t_0);\ \dot{u}(m t_0)), \qquad m = 0,\ 1,\ 2,\ \ldots\,,$$

in the space $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$. Here $u(t)$ is a solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0,\ u_1)$. Prove that the pair $(\mathcal{H},\ S_m)$ is a discrete dynamical system. Moreover, $S_m = S_1^m$ and S_1 is a homeomorphism in $\mathcal{H}$.

§ 4 Smoothness of Solutions

In the study of smoothness properties of solutions constructed in Section 3 we use some ideas presented in paper [9]. The main result of this section is the following assertion.

Theorem 4.1

Let the hypotheses of Theorem 3.1 ***hold. We assume that*** $M(z) \in C^{l+1}(\mathbb{R}_+)$ ***and the load*** $p(t)$ ***lies in*** $C^l(0, T; \mathscr{F}_0)$ ***for some*** $l \geq 1$. ***Then for a weak solution*** $u(t)$ ***to problem*** (0.1) ***and*** (0.2) ***to possess the properties***

$$u^{(k)}(t) \in C(0, T; \mathscr{F}_2), \quad k = 0, 1, 2, \dots, l-1, \\ u^{(l)}(t) \in C(0, T; \mathscr{F}_1), \quad u^{(l+1)}(t) \in C(0, T; \mathscr{F}_0), \tag{4.1}$$

it is necessary and sufficient that the following compatibility conditions are fulfilled:

$$u^{(k)}(0) \in \mathscr{F}_2, \quad k = 0, 1, 2, \dots, l-1; \qquad u^{(l)}(0) \in \mathscr{F}_1. \tag{4.2}$$

Here $u^{(j)}(t)$ ***is a strong derivative of the function*** $u(t)$ ***with respect to*** t ***of the order*** j ***and the values*** $u^{(k)}(0)$ ***are recurrently defined by the initial conditions*** u_0 ***and*** u_1 ***with the help of equation*** (0.1)***:***

$$u^{(0)}(0) = u_0, \qquad u^{(1)}(0) = u_1,$$

$$u^{(k)}(0) = -\Big\{\gamma u^{(k-1)}(0) + A^2 u^{(k-2)}(0) + L u^{(k-2)}(0) + \\ + \frac{d^{k-2}}{dt^{k-2}}\Big(M\big(\|A^{1/2}u(t)\|^2\big) Au(t) - p(t)\Big)\Big|_{t=0}\Big\}, \tag{4.3}$$

where $k = 2, 3, \dots$.

Proof.

It is evident that if a solution $u(t)$ possesses properties (4.1) then compatibility conditions (4.2) are fulfilled. Let us prove that conditions (4.2) are sufficient for equations (4.1) to be satisfied. We start with the case $l = 1$. The compatibility conditions have the form: $u_0 \in \mathscr{F}_2$, $u_1 \in \mathscr{F}_1$. As in the proof of Theorem 3.1 we consider the Galerkin approximation

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k$$

of the order m for a solution to problem (0.1) and (0.2). It satisfies the equations

$$\ddot{u}_m(t) + \gamma \dot{u}_m(t) + A^2 u_m(t) +$$

$$+ M\Big(\|A^{1/2} u_m(t)\|^2\Big) A u_m(t) + p_m L u_m(t) \;=\; p_m p(t) \;, \tag{4.4}$$

$$u_m(0) \;=\; p_m u_0 \,, \qquad \dot{u}_m(0) \;=\; p_m u_1 \,,$$

where p_m is the orthoprojector onto the span of elements $e_1, \ldots e_m$. The structure of equation (3.10) implies that $u_m(t) \in C^2(0, T; \mathcal{F}_2)$. We differentiate equation (4.4) with respect to t to obtain that $v_m(t) = \dot{u}_m(t)$ satisfies the equation

$$\ddot{v}_m + \gamma \dot{v}_m + A^2 v_m + M\Big(\|A^{1/2} u_m\|^2\Big) A v_m + p_m L v_m =$$

$$= -2M'\Big(\|A^{1/2} u_m\|^2\Big)(A u_m,\ \dot{u}_m) A u_m + p_m \dot{p}(t) \tag{4.5}$$

and the initial conditions

$$v_m(0) \;=\; p_m u_1 \,,$$

$$\dot{v}_m(0) \;=\; -p_m \left\{\gamma u_1 + A^2 u_0 + M\Big(\|A^{1/2} p_m u_0\|^2\Big) A u_0 + L p_m u_0 - p(0)\right\}. \tag{4.6}$$

It is clear that

$$\dot{v}_m(0) = p_m u^{(2)}(0) + \Big[M\Big(\|A^{1/2} u_0\|^2\Big) - M\Big(\|A^{1/2} p_m u_0\|^2\Big)\Big] A p_m u_0 + L(u_0 - p_m u_0) \,,$$

where $u^{(2)}(0)$ is defined by (4.3). Therefore,

$$\|v_m(0) - p_m u^{(2)}(0)\| \;\le\; C(\|u_0\|_1) \|u_0 - p_m u_0\|_1 \,.$$

The compatibility conditions give us that $u_0 \in \mathcal{F}_2$ and hence $u^{(2)}(0) \in \mathcal{F}_0$. Thus, the initial condition $\dot{v}_m(0)$ possesses the property

$$\|\dot{v}_m(0) - u^{(2)}(0)\| \to 0 \,, \qquad m \to \infty \,. \tag{4.7}$$

We multiply (4.5) by $\dot{v}_m(t)$ scalarwise in H to find that

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot{v}_m(t)\|^2 + \|A v_m(t)\|^2\Big) + \gamma \|\dot{v}_m(t)\|^2 \;=\; (F_m(t),\ \dot{v}_m(t)) \,, \tag{4.8}$$

where

$$F_m(t) = -M\Big(\|A^{1/2} u_m\|^2\Big) A v_m - p_m L v_m -$$

$$-2M'\Big(\|A^{1/2} u_m\|^2\Big)(A u_m,\ \dot{u}_m) A u_m + p_m \dot{p}(t) \;. \tag{4.9}$$

Using a priori estimates (3.14) for $u_m(t)$ we obtain

$$\|F_m(t)\| \;\le\; C_T\Big(1 + \|A v_m(t)\|\Big), \qquad t \in [0,\ T] \,.$$

Thus, equation (4.8) implies that

$$\frac{d}{dt}\left(\|\dot{v}_m(t)\|^2 + \|Av_m(t)\|^2\right) \le C_T\left(1 + \|\dot{v}_m(t)\|^2 + \|Av_m(t)\|^2\right), \qquad t \in [0, T].$$

Equation (4.7) and the property $u_1 \in \mathcal{F}_1$ give us that the estimate

$$\|\dot{v}_m(0)\|^2 + \|Av_m(0)\|^2 < C$$

holds uniformly with respect to m. Therefore, we reason as in Section 3 and use Gronwall's lemma to find that

$$\|Av_m(t)\|^2 + \|\dot{v}_m(t)\|^2 \le C_T, \qquad t \in [0, T]. \tag{4.10}$$

Consequently,

$$\|A\dot{u}_m(t)\|^2 + \|\ddot{u}_m(t)\|^2 \le C_T, \qquad t \in [0, T]. \tag{4.11}$$

Equation (4.4) gives us that

$$\|A^2u_m(t)\| \le \|\ddot{u}_m(t)\| + \gamma\|\dot{u}_m(t)\| + \left|M(\|A^{1/2}u_m(t)\|^2)\right|\|Au_m(t)\| + \|Lu_m(t)\| + \|p(t)\|.$$

Therefore, (3.14) and (4.11) imply that

$$\|A^2u_m(t)\| \le C_T, \qquad t \in [0, T]. \tag{4.12}$$

Thus, the sequence $\{u_m(t)\}$ of approximate solutions to problem (0.1) and (0.2) possesses the properties (cf. Exercise 3.7):

$$\left.\begin{aligned} u_m(t) \to u(t) \quad &\text{weakly in} \quad L^2(0, T;\ D(A^2)); \\ \dot{u}_m(t) \to \dot{u}(t) \quad &\text{weakly in} \quad L^2(0, T;\ D(A)); \\ \ddot{u}_m(t) \to \ddot{u}(t) \quad &\text{weakly in} \quad L^2(0, T;\ H); \end{aligned}\right\} \tag{4.13}$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2). Moreover (see Exercise 1.13),

$$\lim_{m\to\infty} \int_0^T \|\dot{u}_m(t) - \dot{u}(t)\|_s^2 + \|u_m(t) - u(t)\|_{1+s}^2 \, dt = 0 \tag{4.14}$$

for every $s < 1$. If we use these equations and arguments similar to the ones given in Section 3, then it is easy to pass to the limit and to prove that the function $w(t) = \dot{u}(t)$ is a weak solution to the problem

$$\left\{\begin{aligned} &\ddot{w} + \gamma\dot{w} + A^2w + M\left(\|A^{1/2}u(t)\|^2\right)Aw + Lw = \\ &\quad = -2M'\left(\|A^{1/2}u(t)\|^2\right)(Au,\ \dot{u})Au + \dot{p}(t), \\ &w(0) = u_1, \quad \dot{w}(0) = u^{(2)}(0), \end{aligned}\right. \tag{4.15}$$

where $u^{(2)}(0)$ is defined by (4.3). Therefore, Theorem 2.1 gives us that

$$\dot{u}(t) = w(t) \in C(0, T; D(A)) \cap C^1(0, T; H).$$

This implies equation (4.1) for $l = 1$.

Further arguments are based on the following assertion.

Lemma 4.1

Let $u(t)$ be a weak solution to the linear problem

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) + A^2 u(t) + b(t) A u = h(t)\,, \\ u(0) = u_0\,, \end{cases} \tag{4.16}$$

where $b(t)$ is a scalar continuously differentiable function, $h(t) \in \in C^1(0, T; \mathcal{F}_0)$ and $u_0 \in \mathcal{F}_2$, $u_1 \in \mathcal{F}_1$. Then

$$u(t) \in C(0, T; \mathcal{F}_2) \cap C^1(0, T; \mathcal{F}_1) \cap C^2(0, T; \mathcal{F}_0) \tag{4.17}$$

and the function $v(t) = \dot{u}(t)$ is a weak solution to the problem obtained by the formal differentiation of (4.16) with respect to t and equiped with the initial conditions $v(0) = u_1$ and $\dot{v}(0) = \ddot{u}(0) \equiv -(\gamma u_1 + A^2 u_0 + + b(0) A u_0 - h(0))$.

Proof.

Let $u_m(t)$ be the Galerkin approximation of the order m of a solution to problem (4.16) (see (2.7)). It is clear that $u_m(t)$ is thrice differentiable with respect to t and $v_m(t) = \dot{u}_m(t)$ satisfies the equation

$$\ddot{v}_m + \gamma\dot{v}_m + A^2 v_m + b(t) A v_m = -\dot{b}(t) A u_m + p_m h(t)\,, \qquad t > 0\,,$$

and the initial conditions

$$v_m(0) = p_m u_1\,, \qquad \dot{v}_m(0) = -p_m(\gamma u_1 + A^2 u_0 + b(0) A u_0 - h(0))\,.$$

Therefore, as above, it is easy to prove the validity of equations (4.10)–(4.14) for the case under consideration and complete the proof of Lemma 4.1.

— Exercise *4.1* Assume that the hypotheses of Lemma 4.1 hold with $b(t) \in \in C^l(0, T)$ and $h(t) \in C^l(0, T; \mathcal{F}_0)$ for some $l \ge 1$. Let the compatibility conditions (4.2) be fulfilled with $u^{(0)}(0) = u_0$, $u^{(1)}(0) = = u_1$, and

$$u^{(k)}(0) = -\Bigg\{\gamma u^{(k-1)}(0) + A^2 u^{(k-2)}(0) + \\ + \sum_{j=0}^{k-2} C_{k-2}^{j} b^{(j)}(0) A u^{(k-2-j)}(0) + \\ + L u^{(k-2)}(0) - h^{(k-2)}(0)\Bigg\}$$

for $k = 2, 3, \ldots, l$. Show that the weak solution $u(t)$ to problem (4.16) possesses properties (4.1) and the function $v_k(t) = u^{(k)}(t)$ is a weak solution to the equation obtained by the formal differentiation of (4.16) k times with respect to t. Here $k = 0, 1, \ldots, l$.

In order to complete the proof of Theorem 4.1 we use induction with respect to l. Assume that the hypotheses of the theorem as well as equations (4.2) for $l = n+1$ hold. Assume that the assertion of the theorem is valid for $l = n \ge 1$. Since equations (4.1) hold for the solution $u(t)$ with $l = n$, we have

$$\frac{d^k}{dt^k}\left\{M\left(\|A^{1/2}u(t)\|^2\right)Au(t)\right\} = M\left(\|A^{1/2}u(t)\|^2\right)Au^{(k)}(t) + G_k(t),$$

where $G_k(t) \in C^1(0, T; \mathcal{F}_0)$, $k = 1, \ldots, n$. Therefore, we differentiate equation (0.1) $n-1$ times with respect to t to obtain that $v(t) = u^{(n-1)}(t)$ is a weak solution to problem (4.16) with

$$b(t) = M\left(\|A^{1/2}u(t)\|^2\right) \quad \text{and} \quad h(t) = -G_{n-1}(t) + p^{(n-1)}(t).$$

Consequently, Lemma 4.1 gives us that $w(t) = \dot{v}(t)$ is a weak solution to the problem which is obtained by the formal differentiation of equation (0.1) n times with respect to t:

$$\begin{cases} \ddot{w} + \gamma\dot{w} + A^2 w + M\left(\|A^{1/2}u(t)\|^2\right)Aw = p^{(n)}(t) - G_n(t), \\ w(0) = u^{(n)}(0), \quad \dot{w}(0) = u^{(n+1)}(0). \end{cases}$$

However, the hypotheses of Lemma 4.1 hold for this problem. Therefore (see (4.17)),

$$u^{(n)}(t) = w(t) \in C(0, T; \mathcal{F}_2) \cap C^1(0, T; \mathcal{F}_1) \cap C^2(0, T; \mathcal{F}_0),$$

i.e. equations (4.1) hold for $l = n+1$. **Theorem 4.1 is proved**.

Exercise *4.2* Show that if the hypotheses of Theorem 4.1 hold, then the function $v(t) = u^{(k)}(t)$ is a weak solution to the problem which is obtained by the formal differentiation of equation (0.1) k times with respect to t, $k = 1, 2, \ldots, l$.

Exercise *4.3* Assume that the hypotheses of Theorem 4.1 hold and $L \equiv 0$ in equation (0.1). Show that if the conditions

$$p(t) \in C^k([0, T]; \mathcal{F}_{l-k}), \quad k = 0, 1, \ldots, l, \tag{4.18}$$

are fulfilled, then a solution $u(t)$ to problem (0.1) and (0.2) possesses the properties

$$u(t) \in C^k([0, T]; \mathcal{F}_{l+1-k}), \quad k = 0, 1, \ldots, l+1.$$

Exercise *4.4* Assume that the hypotheses of Theorem 4.1 hold. We define the sets

$$\mathscr{V}_k = \left\{ (u_0,\ u_1) \in \mathscr{H}:\ \text{ equation (4.2) holds with } l = k \right\} \qquad (4.19)$$

in the space $\mathscr{H} = \mathscr{F}_1 \times \mathscr{F}_0$. Prove that

$$\mathscr{V}_1 = \mathscr{F}_2 \times \mathscr{F}_1 \quad \text{and} \quad \mathscr{V}_1 \supset \mathscr{V}_2 \supset \ldots \supset \mathscr{V}_l\ .$$

Exercise *4.5* Show that every set $\mathscr{V}_k$ given by equality (4.19) is invariant:

$$(u_0;\ u_1) \in \mathscr{V}_k \Rightarrow (u(t);\ \dot{u}(t)) \in \mathscr{V}_k\,, \qquad k = 1,\ \ldots,\ l\,.$$

Here $u(t)$ is a weak solution to problem (0.1) and (0.2).

Exercise *4.6* Assume that $L \equiv 0$ in equation (0.1) and the load $p(t)$ possesses property (4.18). Show that for $k = 1,\ 2,\ \ldots,\ l$ the set $\mathscr{V}_k$ of form (4.19) contains the subspace $\mathscr{F}_{k+1} \times \mathscr{F}_k$.

Exercise *4.7* Assume that the hypotheses of Theorem 3.1 hold and the operator L (in equation (0.1)) possesses the property

$$\|Lu\|_s \le C\|u\|_{1+s} \quad \text{for some} \quad 0 < s < 1\,. \qquad (4.20)$$

Let $u_0 \in \mathscr{F}_{1+s}$ and let $u_1 \in \mathscr{F}_s$. Show that the estimate

$$\|\dot{u}_m(t)\|_s^2 + \|u_m(t)\|_{1+s}^2 \le C_T\,, \qquad t \in [0,\ T]\,, \qquad (4.21)$$

is valid for the approximate Galerkin solution $u_m(t)$ to problem (0.1) and (0.2). Here the constant C_T does not depend on m (*Hint:* multiply equation (3.8) by $\lambda_j^{2s}\,\dot{g}_j(t)$ and summarize the result with respect to j; then use relation (3.14) to estimate the nonlinear term).

Exercise *4.8* Show that if the hypotheses of Exercise 4.7 hold, then problem (0.1) and (0.2) possesses a weak solution $u(t)$ such that

$$u(t) \in C(0,\ T;\ \mathscr{F}_{1+s}) \cap C^1(0,\ T;\ \mathscr{F}_s) \cap C^2(0,\ T;\ \mathscr{F}_{-1+s})\,,$$

where $s \in (0,\ 1)$ is the number from Exercise 4.7.

§ 5 Dissipativity and Asymptotic Compactness

In this section we prove the dissipativity and asymptotic compactness of the dynamical system $(\mathscr{H}, S_t)$ generated by weak solutions to problem (0.1) and (0.2) for $\gamma > 0$ in the case of a stationary load $p(t) \equiv p \in H = \mathscr{F}_0$. The phase space is $\mathscr{H} = \mathscr{F}_1 \times \mathscr{F}_0$. The evolutionary operator is defined by the formula

$$S_t y = (u(t),\ \dot{u}(t)), \tag{5.1}$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial condition $y = (u_0;\ u_1)$.

Theorem 5.1

Assume that in addition to (3.2) ***the following conditions are fulfilled:***

a) ***there exist numbers*** $a_j > 0$ ***such that***

$$zM(z) - a_1 \int_0^z M(\xi)\, d\xi \ \geq\ a_2 z^{1+\alpha} - a_3, \qquad z \geq 0 \tag{5.2}$$

with a constant $\alpha > 0$***;***

b) ***there exist*** $0 \leq \theta < 1$ ***and*** $C > 0$ ***such that***

$$\|Lu\| \leq C\|A^\theta u\|, \qquad u \in D(A^\theta). \tag{5.3}$$

Then the dynamical system $(\mathscr{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***for*** $\gamma > 0$ ***and for*** $p(t) \equiv p \in H$ ***is dissipative.***

To prove the theorem it is sufficient to verify (see Theorem 1.4.1 and Exercise 1.4.1) that there exists a functional $V(y)$ on $\mathscr{H}$ which is bounded on the bounded sets of the space $\mathscr{H}$, differentiable along the trajectories of system (0.1) and (0.2), and such that

$$V(y) \ \geq\ \alpha\|y\|^2_{\mathscr{H}} - \mathscr{D}_1, \tag{5.4}$$

$$\frac{d}{dt}(V(S_t y)) + \beta V(S_t y) \ \leq\ \mathscr{D}_2, \tag{5.5}$$

where $\alpha, \beta > 0$ and $\mathscr{D}_1,\ \mathscr{D}_2 \geq 0$ are constants. To construct the functional $V(y)$ we use the method which is widely-applied for finite-dimensional systems (we used it in the proof of estimate (2.13)).

Let

$$V(y) = E(y) + \nu\Phi(y),$$

where $y = (u_0;\ u_1) \in \mathscr{H}$. Here $E(y) = E(u_0;\ u_1)$ is the energy of system (0.1) and (0.2) defined by the formula (3.7),

$$\Phi(y) = (u_0;\ u_1) + \frac{\gamma}{2}\|u_0\|^2,$$

and the parameter $\nu > 0$ will be chosen below. It is evident that

$$-\frac{1}{2\gamma}\|u_1\|^2 \le \Phi(y) \le \frac{1}{2\gamma}\|u_1\|^2 + \gamma\|u_0\|^2.$$

For $0 < \nu < \gamma$ this implies estimate (5.4) and the inequality

$$V(y) \le b_1 E(y) + b_2 \tag{5.6}$$

with the constants $b_1,\ b_2 > 0$ independent of ν. This inequality guarantees the boundedness of $V(y)$ on the bounded sets of the space $\mathcal{H}$.

Energy equality (3.6) implies that the function $E(y(t))$, where $y(t) = S_t y$, is continuously differentiable and

$$\frac{d}{dt}E(y(t)) = -\gamma\|\dot{u}(t)\|^2 + (-Lu(t) + p,\ \dot{u}(t)).$$

Therefore, due to (5.3) we have that

$$\frac{d}{dt}E(y(t)) \le -\frac{\gamma}{2}\|\dot{u}\|^2 + C_1\|A^\theta u\|^2 + C_2.$$

We use interpolation inequalities (see Exercises 2.1.12 and 2.1.13) to obtain that

$$\|A^\theta u\|^2 \le \delta\|Au\|^2 + C_\delta\|A^{1/2}u\|^2, \qquad \delta > 0.$$

Thus, the estimate

$$\frac{d}{dt}E(y(t)) \le -\frac{\gamma}{2}\|\dot{u}\|^2 + \frac{\varepsilon}{2}\|Au\|^2 + C_\varepsilon\|A^{1/2}u\|^2 + C_2 \tag{5.7}$$

holds for any $\varepsilon > 0$.

Lemma 5.1

Let $u(t)$ *be a weak solution to problem* (0.1) *and* (0.2) *and let* $y(t) = (u(t),\ \dot{u}(t))$. *Then the function* $\Phi(y(t))$ *is continuously differentiable and*

$$\frac{d}{dt}\Phi(y(t)) = \|\dot{u}(t)\|^2 + (\ddot{u}(t) + \gamma\dot{u}(t),\ u(t)). \tag{5.8}$$

We note that since $\ddot{u}(t) \in C(\mathbb{R}_+,\ \mathcal{F}_{-1})$ (see Exercise 3.6), equation (5.8) is correctly defined.

Proof.

It is sufficient to verify that

$$(u(t),\ \dot{u}(t)) = (u_0,\ u_1) + \int_0^t \left\{(\ddot{u}(\tau),\ u(\tau)) + \|\dot{u}(\tau)\|^2\right\}d\tau. \tag{5.9}$$

Let p_l be the orthoprojector onto the span of elements $\{e_k,\ k = 1, 2, \ldots, l\}$ in $\mathcal{F}_0$. Then it is evident that the vector-function $u_l(t) = p_l u(t)$ is twice continuously differentiable with respect to t. Therefore,

$$(u_l(t),\ \dot{u}_l(t)) = (u_l(0),\ \dot{u}_l(0)) + \int_0^t \left\{ (\ddot{u}_l(\tau),\ u_l(\tau)) + \|\dot{u}_l(\tau)\|^2 \right\} d\tau .$$

The properties of the projector p_l (see Exercise 2.1.11) enable us to pass to the limit $l \to \infty$ and to obtain (5.9). Lemma 5.1 is proved.

Since $u(t)$ is a solution to equation (0.1), relation (5.8) implies that

$$\frac{d}{dt}\Phi(y(t)) = \|\dot{u}\|^2 - \left\{ \|Au\|^2 + M\left(\|A^{1/2}u\|^2\right)\cdot \|A^{1/2}u\|^2 + (Lu - p,\ u) \right\}.$$

Therefore, equation (5.2) and the evident inequality

$$|(Lu - p,\ u)| \le \frac{1}{2}\|Au\|^2 + C_1\|u\|^2 + \|p\|^2$$

give us that

$$\frac{d}{dt}\Phi(y(t)) \le \|\dot{u}\|^2 - \frac{1}{2}\|Au\|^2 - a_1\mathcal{M}\left(\|A^{1/2}u\|^2\right) - a_2\|A^{1/2}u\|^{2+2\alpha} + C_1\|u\|^2 + C_2 .$$

Hence, (5.6) and (5.7) enable us to obtain the estimate

$$\frac{d}{dt}V(y(t)) + \delta V(y(t)) \le -\frac{1}{2}(\gamma - \delta b_1 - 2\nu)\|\dot{u}\|^2 -$$
$$-\frac{1}{2}(\nu - \delta b_1 - \varepsilon)\|Au\|^2 - \left(\nu a_1 - \frac{\delta b_1}{2}\right)\mathcal{M}\left(\|A^{1/2}u\|^2\right) + R(u;\ \nu, \varepsilon, \delta) ,$$

where $\delta > 0$ and

$$R(u;\ \nu, \varepsilon, \delta) = -\nu a_2\|A^{1/2}u\|^{2+2\alpha} + C_\varepsilon\|A^{1/2}u\|^2 + \nu C_1\|u\|^2 + C_\delta .$$

Therefore, for any $0 < \nu < \gamma/2$ estimate (5.5) holds, provided δ and ε are chosen appropriately. Thus, **Theorem 5.1 is proved**.

Exercise *5.1* Prove that if the hypotheses of Theorem 5.1 hold, then the assertion on the dissipativity of solutions to problem (0.1) and (0.2) remains true in the case of a nonstationary load $p(t) \in L^\infty(\mathbb{R}_+,\ H)$.

Theorem 5.2

Let the hypotheses of Theorem 5.1 *hold and assume that for some* $\sigma > 0$

$$p \in \mathcal{F}_\sigma, \qquad LD(A) \subset D(A^\sigma), \qquad \|A^\sigma L u\| \le C\|Au\|. \tag{5.10}$$

Then there exists a positively invariant bounded set K_σ ***in the space*** $\mathcal{H}_\sigma = \mathcal{F}_{1+\sigma} \times \mathcal{F}_\sigma$ ***which is closed in*** $\mathcal{H}$ ***and such that***

$$\sup\left\{\operatorname{dist}_{\mathcal{H}}(S_t y,\ K_\sigma):\ y \in B\right\} \le C e^{-\frac{\gamma}{4}(t - t_B)} \tag{5.11}$$

for any bounded set B ***in the space*** $\mathcal{H}$***,*** $t > t_B$.

Due to the compactness of the embedding of $\mathcal{H}_\sigma$ in $\mathcal{H}$ for $\sigma > 0$, this theorem and Lemma 1.4.1 imply that the dynamical system $(\mathcal{H},\ S_t)$ is asymptotically compact.

Proof.

Since the system $(\mathcal{H},\ S_t)$ is dissipative, there exists $R > 0$ such that for all $y \in B$ and $t \ge t_0 = t_0(B)$

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le R^2\ , \tag{5.12}$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0;\ u_1) \in B$. We consider $u(t)$ as a solution to linear problem (2.1) and (2.2) with $b(t) = M(\|A^{1/2}u(t)\|^2)$ and $h(t) = -Lu(t) + p$. It is easy to verify that $b(t)$ is a continuously differentiable function and

$$|b(t)| + |\dot{b}(t)| \le C_R\ , \qquad t \ge t_0\,.$$

Moreover, equation (2.27) implies that

$$S_t y = U(t,\ t_0)\,y(t_0) + G(t,\ t_0;\ y)\,, \tag{5.13}$$

where

$$G(t,\ t_0;\ y) = -\int_{t_0}^{t} U(t,\ \tau)(0;\ Lu(\tau) - p)\,d\tau\,.$$

Here $U(t,\ \tau)$ is the evolutionary operator of the homogeneous problem (2.1) and (2.2) with $h(t) = 0$ and $b(t) = M(\|A^{1/2}u(t)\|^2)$. By virtue of Theorem 2.2 there exists $N_0 \ge 0$ such that

$$\left\|(1 - P_N)\,U(t,\ \tau)h\right\|_{\mathcal{H}_\sigma} \le \sqrt{3}\,\|h\|_{\mathcal{H}_\sigma} \exp\left\{-\frac{\gamma}{4}(t - \tau)\right\}, \tag{5.14}$$

where $N \ge N_0$, $t \ge \tau \ge t_0$, and P_N is the orthoprojector $\mathcal{H}$ onto

$$\mathcal{L}_N = \operatorname{Lin}\{(e_k;\ 0),\ (0;\ e_k)\,,\quad k = 1,\ 2,\ \ldots,\ N\}\,.$$

This implies that

$$\left\|(1 - P_{N_0})\,G(t,\ t_0;\ y)\right\|_{\mathcal{H}_\sigma} \le \sqrt{3}\int_{t_0}^{t} \exp\left\{-\frac{\gamma}{4}(t - \tau)\right\}\|Lu(\tau) - p\|_\sigma\,d\tau\,.$$

Therefore, we use (5.10) to obtain that

$$\left\|(1 - P_{N_0})\,G(t,\ t_0;\ y)\right\|_{\mathcal{H}_\sigma} \le C(R)\,. \tag{5.15}$$

It is also easy to find that

$$\left\|P_{N_0} S_t y\right\|_{\mathcal{H}_\sigma} \le N_0^\sigma \left\|S_t y\right\|_{\mathcal{H}} \le R N_0^\sigma , \qquad t \ge t_0 .$$

Consequently, there exists a number R_σ depending on the radius of dissipativity R and the parameters of the problem such that the value

$$S_t y - (1-P_{N_0})\, U(t,\ t_0) S_{t_0} y_0 = P_{N_0} S_t y + (1-P_{N_0}) G(t,\ t_0;\ y) \qquad (5.16)$$

lies in the ball

$$B_\sigma = \{y:\ \|y\|_{\mathcal{H}_\sigma} \le R_\sigma\}$$

for $t \ge t_0$. Therefore, with the help of (5.12) and (2.23) we have that

$$\mathrm{dist}_{\mathcal{H}}(S_t y,\ B_\sigma) \le \left\|(1-P_{N_0})\, U(t,\ t_0) S_{t_0} y\right\| \le R\sqrt{3}\, e^{-\frac{\gamma}{4}(t-t_0)} . \qquad (5.17)$$

Let $K_\sigma = \gamma^+(B_\sigma) \equiv \bigcup_{t \ge 0} S_t(B_\sigma)$. Evidently equation (5.11) is valid. Moreover, K_σ is positively invariant. The continuity of $S_t y$ with recpect to the both variables $(t;\ y)$ in the space $\mathcal{H}$ (see Exercise 3.8) and attraction property (5.17) imply that K_σ is a closed set in $\mathcal{H}$. Let us prove that K_σ is bounded in $\mathcal{H}_\sigma$. First we note that K_σ is bounded in $\mathcal{H}$. Indeed, by virtue of the dissipativity we have that $\left\|S_t y\right\| \le R$ for all $y \in B_\sigma$ and $t \ge t_\sigma \equiv t(B_\sigma)$. Since $S_t y$ is continuous with respect to the variables $(t;\ y)$, its maximum is attained on the compact $[0,\ t_\sigma] \times B_\sigma$. Thus, there exists $\overline{R} > 0$ such that $\|y\| \le \overline{R}$ for all $y \in K_\sigma$. Let us return to equality (5.16) for $t_0 = 0$ and $y_0 \in B_\sigma$. It is evident that the norm of the right-hand side in the space $\mathcal{H}_\sigma$ is bounded by the constant $C = C(\sigma,\ \overline{R})$. However, equation (5.14) implies that

$$\left\|(1-P_{N_0})\, U(t,\ 0) y_0\right\|_{\mathcal{H}_\sigma} \le \sqrt{3} R_\sigma , \qquad t \ge 0 , \qquad y_0 \in B_\sigma .$$

Therefore, equation (5.16) leads to the uniform estimate

$$\left\|S_t y\right\|_{\mathcal{H}_\sigma} \le \overline{R}_\sigma , \qquad t \ge 0 , \qquad y \in B_\sigma .$$

Thus, the set K_σ is bounded in $\mathcal{H}_\sigma$. **Theorem 5.2 is proved**.

— Exercise *5.2* Show that for any $0 \le s \le \sigma$ a bounded set of $\mathcal{H}_s$ is attracted by K_σ at an exponential rate with respect to the metric of the space $\mathcal{H}_s$. Thus, we can replace $\mathrm{dist}_{\mathcal{H}}$ by $\mathrm{dist}_{\mathcal{H}_s}$ in (5.11).

— Exercise *5.3* Prove that if the hypotheses of Theorem 5.2 hold, then the assertion on the asymptotic compactness of solutions to problem (0.1) and (0.2) remains true in the case of nonstationary load $p(t) \in L^\infty(\mathbb{R}_+,\ \mathcal{F}_\sigma)$ (see also Exercise 5.1).

— Exercise *5.4* Prove that the hypotheses of Theorem 5.1 and 5.2 hold for problem (0.3) and (0.4) for any $0 < \sigma < 1/4$, provided that $\gamma > 0$ and $p(x,\ t) \equiv p(x)$ lies in the Sobolev space $H_0^1(\Omega)$.

Let us consider the dissipativity properties of smooth solutions (see Section 4) to problem (0.1) and (0.2).

Theorem 5.3

Let the hypotheses of Theorem 5.1 ***hold. Assume that*** $M(z) \in C^{l+1}(\mathbb{R}_+)$ ***and the initial conditions*** $y_0 = (u_0;\ u_1)$ ***are such that equations*** (4.1) ***(and hence*** (4.2)***) are valid for the solution*** $(u(t);\ \dot{u}(t)) = S_t y_0$. ***Then there exists*** $R_l > 0$ ***such that for any initial data*** $y_0 = (u_0;\ u_1)$ ***possessing the property***

$$\|u^{(k+1)}(0)\|^2 + \|Au^{(k)}(0)\|^2 + \|A^2 u^{(k-1)}(0)\|^2 < \rho^2, \quad k = 1, 2, \dots, l, \tag{5.18}$$

the solution $(u(t);\ \dot{u}(t)) = S_t y_0$ ***admits the estimate***

$$\|u^{(k+1)}(t)\|^2 + \|Au^{(k)}(t)\|^2 + \|A^2 u^{(k-1)}(t)\|^2 < R_l^2 \tag{5.19}$$

for all $k = 1, 2, \dots, l$ ***as soon as*** $t \ge t_0(\rho)$.

We use induction to prove the theorem. The proof is based on the following assertion.

Lemma 5.2

Assume that the hypotheses of Theorem 5.3 *hold for* $l = 1$. *Then the dynamical system* $(\mathcal{H}_1,\ S_t)$ *generated by problem* (0.1) *and* (0.2) *in the space* $\mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$ *is dissipative.*

Proof.

Let $(u(t);\ \dot{u}(t)) = S_t y_0$ be a semitrajectory of the dynamical system $(\mathcal{H}_1,\ S_t)$ and let $y_0 = (u_0;\ u_1) \in \mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$. If the hypotheses of the lemma hold, then the function $w(t) = \dot{u}(t)$ is a weak solution to problem (4.15) obtained by formal differentiation of (0.1) with respect to t (as we have shown in Section 4). By virtue of Theorem 2.1 the energy equality of the form (2.6) holds for the function $w(t)$. We rewrite it in the differential form:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t;\ w(t),\ \dot{w}(t)) + \gamma \|\dot{w}(t)\|^2 = \Psi(u(t),\ w(t)), \tag{5.20}$$

where

$$F(t;\ w,\ \dot{w}) = \frac{1}{2}\left(\|\dot{w}\|^2 + \|Aw\|^2 + M\left(\|A^{1/2}u(t)\|^2\right) \cdot \|A^{1/2}w\|^2\right) \tag{5.21}$$

and

$$\Psi(u(t),\ w(t)) = -(Lw(t),\ \dot{w}(t)) +$$

$$+ M'\left(\|A^{1/2}u(t)\|^2\right)(Au(t),\ \dot{u}(t))\left[\|A^{1/2}w(t)\|^2 - 2(Au(t),\ \dot{w}(t))\right].$$

The dissipativity property of $(\mathcal{H},\ S_t)$ given by Theorem 5.1 leads to the estimates

$$\left| M\left(\|A^{1/2}u(t)\|^2\right)\right| \|A^{1/2}w\|^2 \;\le\; C_R\|Aw(t)\|\,,$$

$$|\Psi(u(t),\ w(t))| \;\le\; \|Lw\|\,\|\dot{w}\| + C_R(\|Aw(t)\| + \|\dot{w}(t)\|)$$

for all $(u_0;\ u_1)$ with the property

$$\|Au_0\|^2 + \|u_1\|^2 \;\le\; \rho^2$$

and for all $t \ge t_0(\rho)$ large enough. Hereinafter R is the radius of dissipativity of the system $(\mathcal{H},\ S_t)$. These estimates imply that for $t \ge t_0(\rho)$ we have

$$\frac{1}{4}\left(\|\dot{w}\|^2 + \|Aw\|^2\right) - \alpha_1 \;\le\; F(t;\ w,\ \dot{w}) \;\le\; \alpha_2\left(\|w(t)\|^2 + \|\dot{w}(t)\|^2\right) + \alpha_3 \quad (5.22)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t;\ w,\ \dot{w}) + \frac{\gamma}{2}\|\dot{w}\|^2 \;\le\; \beta\|A^{\theta}w\|^2 + \alpha_4\|Aw\| + \alpha_5\,, \quad (5.23)$$

where the constants $\alpha_j > 0$ depend on R. Here $\theta < 1$. Similarly, we use Lemma 5.1 to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(w(t),\ \dot{w}(t)) + \frac{\gamma}{2}\|w(t)\|^2\right\} \;\le\; \|\dot{w}(t)\|^2 - \frac{1}{2}\|Aw(t)\|^2 + C_R$$

for $t \ge t_0(\rho)$. Consequently, the function

$$V(t) = F(t;\ w,\ \dot{w}) + \nu\left\{(w,\ \dot{w}) + \frac{\gamma}{2}\|w\|^2\right\}$$

possesses the properties

$$\frac{\mathrm{d}V}{\mathrm{d}t} + \omega V \;\le\; C_R\,, \qquad \omega > 0\,,$$

and

$$\frac{1}{4}\left(\|\dot{w}\|^2 + \|Aw\|^2\right) - a_1 \;\le\; V \;\le\; a_2\left(\|w\|^2 + \|\dot{w}\|^2\right) + a_3$$

for $t \ge t_0(\rho)$ and for $\nu > 0$ small enough. This implies that

$$\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \;\le\; C_1\left(\|\dot{w}(t_0)\|^2 + \|Aw(t_0)\|^2\right)e^{-\omega t} + C_2 \quad (5.24)$$

for $t \ge t_0 = t_0(\rho)$, provided that

$$\|Au_0\|^2 + \|u_1\|^2 \;\le\; \rho^2\,. \quad (5.25)$$

If we use (5.4)–(5.6), then it is easy to find that

$$\|Au(t)\|^2 + \|\dot{u}(t)\|^2 \;\le\; C_{\rho,\ R}\,, \qquad t > 0\,,$$

under condition (5.25). Using the energy equality for the weak solutions to problem (4.15) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|Aw(t)\|^2 + \|\dot{w}(t)\|^2\Big) \le C_1(\rho,\, R)\cdot\|Aw\|\cdot\|\dot{w}\| + C_2(\rho,\, R)\,.$$

Therefore, standard reasoning in which Gronwall's lemma is used leads to

$$\|Aw(t)\|^2 + \|\dot{w}(t)\|^2 \le \Big(\|Aw(0)\|^2 + \|\dot{w}(0)\|^2 + a\Big)e^{bt}\,,$$

provided that equation (5.25) is valid. Here a and b are some positive constants depending on ρ and R. This and equation (5.24) imply that

$$\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \le C_1(\rho,\, R)\Big((1+\|\dot{w}(0)\|^2) + \|Aw(0)\|^2\Big)e^{-\omega t} + C_2\,, \quad (5.26)$$

where C_2 depends on R only. Since

$$\|\dot{w}(0)\|^2 + \|Aw(0)\|^2 \le C_\rho\,, \qquad \|u_1\|^2 + \|Au_0\|^2 \le \left(\frac{\rho}{\lambda_1}\right)^2$$

provided that

$$\|Au_1\|^2 + \|A^2u_0\|^2 \le \rho^2\,,$$

equation (5.26) gives us the estimate

$$\|\ddot{u}(t)\|^2 + \|A\dot{u}(t)\|^2 \le b_R^2\,, \qquad t \ge \bar{t}_0(\rho)\,.$$

This easily implies the dissipativity of the dynamical system $(\mathscr{H}_1,\, S_t)$. Thus, Lemma 5.2 is proved.

Exercise 5.5 Prove that the dynamical system $(\mathscr{H}_1,\, S_t)$ generated by problem (0.1) and (0.2) with the initial data $y_0 = (u_0;\, u_1) \in \mathscr{H}_1 = \mathscr{F}_2 \times \mathscr{F}_1$ is asymptotically compact provided that equations (5.10) hold.

In order to complete the proof of Theorem 5.3, we should note first that Lemma 5.2 coincides with the assertion of the theorem for $l = 1$ and second we should use the fact that the derivatives $u^{(k)}(t)$ are weak solutions to the problem obtained by differentiation of the original equation. The main steps of the reasoning are given in the following exercises.

Exercise 5.6 Assume that the hypotheses of Theorem 5.3 hold for $l = n+1$ and its assertion is valid for $l = n$. Show that $w(t) = u^{(n+1)}(t)$ is a weak solution to the problem of the form

$$\begin{cases} \ddot{w}(t) + \gamma\dot{w}(t) + A^2w(t) + M\|A^{1/2}u\|^2Aw + Lw = G_{n+1}(t)\,, \\ w(0) = u^{(n+1)}(0), \quad \dot{w}(0) = u^{(n+2)}(0)\,, \end{cases}$$

where

$$\|G_{n+1}(t)\| \le C(R_n) \quad \text{for all} \quad t > t_0(\rho)\,.$$

— Exercise *5.7* Use the result of Exercise 5.6 and the method given in the proof of Lemma 5.2 to prove that $w(t) = u^{(n+1)}(t)$ can be estimated as follows:

$$\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \le$$

$$\le C_1\Big(\|\dot{w}(0)\|^2 + \|Aw(0)\|^2 + C_2\Big)e^{-\omega t} + C_3 \ , \qquad (5.27)$$

where $\omega > 0$, the numbers C_j depend on ρ and R_n, $j = 1, 2$, and the constant C_3 depends on R_n only.

— Exercise *5.8* Use the induction assumption and equation (5.27) to prove the assertion of Theorem 5.3 for $l = n+1$.

§ 6 *Global Attractor and Inertial Sets*

The above given properties of the evolutionary operator S_t generated by problem (0.1) and (0.2) in the case of stationary load $p(t) \equiv p$ enable us to apply the general assertions proved in Chapter 1 (see also [10]).

Theorem 6.1

Assume that conditions (3.2), (5.2), (5.3), ***and*** (5.10) ***are fulfilled. Then the dynamical system*** $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***possesses a global attractor*** $\mathcal{A}$ ***of a finite fractal dimension. This attractor is a connected compact set in*** $\mathcal{H}$ ***and is bounded in the space*** $\mathcal{H}_\sigma = \mathcal{F}_{1+\sigma} \times \mathcal{F}_\sigma$, ***where*** $\sigma > 0$ ***is defined by condition*** (5.10).

Proof.

By virtue of Theorems 5.1, 5.2, and 1.5.1 we should prove only the finite dimensionality of the attractor. The corresponding reasoning is based on Theorem 1.8.1 and the following assertions.

Lemma 6.1.

Assume that conditions (3.2), (5.2), *and* (5.3) *are fulfilled. Let* $p \in H$. *Then for any pair of semitrajectories* $\{S_t y_j : t \ge 0\}$, $j = 1, 2$, *possessing the property* $\|S_t y_j\| \le R$ *for all* $t \ge 0$ *the estimate*

$$\|S_t y_1 - S_t y_2\|_{\mathcal{H}} \le \exp(a_0 t)\|y_1 - y_2\|_{\mathcal{H}} \ , \qquad t \ge 0 \ , \qquad (6.1)$$

holds with the constant a_0 *depending on* R.

Proof.

If $u_1(t)$ and $u_2(t)$ are solutions to problem (0.1) and (0.2) with the initial conditions $y_1 = (u_{01};\ u_{11})$ and $y_2 = (u_{02};\ u_{12})$, then the function $v(t) = u_1(t) - u_2(t)$ satisfies the equation

$$\ddot{v} + \gamma \dot{v} + A^2 v = \mathcal{F}(u_1,\ u_2,\ t)\,, \tag{6.2}$$

where

$$\mathcal{F}(u_1,\ u_2,\ t) = M\left(\left\|A^{1/2}u_2(t)\right\|^2\right)Au_2(t) - M\left(\left\|A^{1/2}u_1(t)\right\|^2\right)Au_1(t) - Lv(t)\,.$$

It is evident that the estimate

$$\left\|\mathcal{F}(u_1,\ u_2,\ t)\right\| \ \le\ C_R \cdot \left\|A(u_1(t) - u_2(t))\right\|$$

holds, provided that $\|y_i(t)\|^2_{\mathcal{H}} \ =\ \|\dot{u}_i(t)\|^2 + \|u_i(t)\|_1^2 \ \le\ R^2$. Therefore, (2.20) implies that

$$\left\|S_t y_1 - S_t y_2\right\|^2_{\mathcal{H}} \ \le\ \left\|y_1 - y_2\right\|^2_{\mathcal{H}} + a_1 \int_0^t \left\|S_t y_1 - S_t y_2\right\|^2_{\mathcal{H}}\,d\tau\ .$$

Gronwall's lemma gives us equation (6.1).

Lemma 6.2

Assume that the hypotheses of Theorem 6.1 *hold. Let* K_σ *be the compact positively invariant set constructed in Theorem* 5.2. *Then for any* $y_1,\ y_2 \in K_\sigma$ *the inequality*

$$\left\|Q_N(S_t y_1 - S_t y_2)\right\|_{\mathcal{H}} \ \le\ a_2\, e^{-\frac{\gamma}{4}t}\left(1 + \frac{L_\sigma}{\lambda^\sigma_{N+1}}\, e^{a_3 t}\right)\left\|y_1 - y_2\right\|_{\mathcal{H}} \tag{6.3}$$

is valid, where $Q_N = 1 - P_N$, $N \ge N_0$, *the orthoprojector* P_N *and the number* N_0 *are defined as in* (5.14), L_σ *and* a_i *are positive constants which depend on the parameters of the problem.*

Proof.

It is evident that

$$Q_N S_t y_i \ = (q_N u_i(t);\ q_N \dot{u}_i(t))\,,$$

where q_N is the orthoprojector onto the closure of the span of elements $\{e_k : k = N+1,\ N+2,\ \ldots\}$ in $\mathcal{F}_0 = H$. Moreover, the function $w_N(t) = q_N(u_1(t) - u_2(t))$ is a solution to the equation

$$\ddot{w}_N + \gamma \dot{w}_N + A^2 w_N + M\left(\left\|A^{1/2}u_1\right\|^2\right)Aw_N = \Phi_N(u_1,\ u_2,\ t)\,,$$

where

$$\Phi_N(u_1,\ u_2,\ t) = -\left[M\left(\left\|A^{1/2}u_1\right\|^2\right) - M\left(\left\|A^{1/2}u_2\right\|^2\right)\right]q_N A u_2 - q_N L(u_1 - u_2)\,.$$

Let us estimate the value Φ_N. Since $\|y\|_{\mathcal{H}_\sigma} \le \overline{R}_\sigma$ for $y \in K_\sigma$ (see the proof of Theorem 5.2), we have

$$\|q_N A u_2\| = \|q_N u_2\|_1 \le \lambda_{N+1}^{-\sigma} \|q_N u_2\|_{1+\sigma} \le \lambda_{N+1}^{-\sigma} \overline{R}_\sigma .$$

Using equation (5.10) we similarly obtain that

$$\|q_N L v\| \le \lambda_{N+1}^{-\sigma} \|A^\sigma L v\| \le C \lambda_{N+1}^{-\sigma} \|A v\| .$$

Therefore,

$$\|q_N L (u_1(t) - u_2(t))\| \le C \lambda_{N+1}^{-\sigma} \|S_t y_1 - S_t y_2\|_{\mathcal{H}} .$$

Consequently,

$$\|\Phi_N(u_1, u_2, t)\| \le \frac{C(\overline{R}_\sigma)}{\lambda_{N+1}^{\sigma}} \|S_t y_1 - S_t y_2\|_{\mathcal{H}} . \tag{6.4}$$

Using equation (2.27) we obtain that

$$Q_N(S_t y_1 - S_t y_2) = U(t, 0)\, Q_N(y_1 - y_2) + \int_0^t U(t, \tau)(0;\ \Phi_N(\tau))\, d\tau ,$$

where $U(t, \tau)$ is the evolutionary operator of homogeneous problem (2.1) with $b(t) = M(\|A^{1/2} u_1(t)\|^2)$ and $h(t) = 0$. Therefore, (2.23) and (6.4) imply that

$$\|Q_N(S_t y_1 - S_t y_2)\|_{\mathcal{H}} \le$$

$$\le \sqrt{3}\, e^{-\frac{\gamma}{4} t} \left\{ \|y_1 - y_2\|_{\mathcal{H}} + \frac{C(\overline{R}_\sigma)}{\lambda_{N+1}^{\sigma}} \int_0^t e^{\frac{\gamma}{4}\tau} \|S_\tau y_1 - S_\tau y_2\|\, d\tau \right\} . \tag{6.5}$$

We substitute (6.1) in this equation to obtain (6.3). Lemma 6.2 is proved.

Let us choose t_0 and $N \ge N_0$ such that

$$a_2 \exp\left\{-\frac{\gamma}{4} t_0\right\} = \frac{\delta}{2}, \qquad L_\sigma \lambda_{N+1}^{-\sigma} \exp\{a_3 t_0\} \le 1 , \qquad \delta < 1 .$$

Then Lemmata 6.1 and 6.2 enable us to state that

$$\|S_{t_0} y_1 - S_{t_0} y_2\|_{\mathcal{H}} \le l \|y_1 - y_2\|_{\mathcal{H}}$$

and

$$\|Q_N(S_{t_0} y_1 - S_{t_0} y_2)\|_{\mathcal{H}} \le \delta \|y_1 - y_2\|_{\mathcal{H}} ,$$

where $l = e^{a_0 t_0}$ and the elements y_1 and y_2 lie in the global attractor $\mathcal{A}$. Hence, we can use Theorem 1.8.1 with $M = \mathcal{A}$, $V = S_{t_0}$, and $P = P_N$. Therefore, the fractal dimension of the attractor $\mathcal{A}$ is finite. Thus, **Theorem 6.1 is proved**.

Theorem 5.2 and Lemmata 6.1 and 6.2 enable us to use Theorem 1.9.2 to obtain an assertion on the existence of the inertial set (fractal exponential attractor) for the dynamical system $(\mathcal{H}, S_t)$ generated by problem (0.1) and (0.2).

Theorem 6.2

Assume that the hypotheses of Theorem 6.1 ***hold. Then there exists a compact positively invariant set*** $\mathcal{A}_{\exp} \subset \mathcal{H}$ ***of the finite fractal dimension such that***

$$\sup\left\{\operatorname{dist}_{\mathcal{H}}(S_t y,\ \mathcal{A}_{\exp}):\ y \in B\right\} \le C e^{-\nu(t-t_B)}$$

for any bounded set B ***in*** $\mathcal{H}$ ***and*** $t \ge t_B$. ***Here*** C ***and*** ν ***are positive numbers. The inertial set*** $\mathcal{A}_{\exp}$ ***is bounded in the space*** $\mathcal{H}_\sigma$.

To prove the theorem we should only note that relations (5.11), (6.1), and (6.3) coincide with conditions (9.12)–(9.14) of Theorem 1.9.2.

Using (1.8.3) and (1.9.18) we can obtain estimates (involving the parameters of the problem) for the dimensions of the attractor and the inertial set by an accurate observing of the constants in the proof of Theorems 5.1 and 5.2 and Lemmata 6.1 and 6.2. However, as far as problem (0.3) and (0.4) is concerned, it is rather difficult to evaluate these estimates for the values of parameters that are very interesting from the point of view of applications. Moreover, these estimates appear to be quite overstated. Therefore, the assertions on the finite dimensionality of an attractor and inertial set should be considered as qualitative results in this case. In particular, this assertions mean that the nonlinear flutter of a plate is an essentially finite-dimensional phenomenon. The study of oscillations caused by the flutter can be reduced to the study of the structure of the global attractor of the system and the properties of inertial sets.

Exercise *6.1* Prove that the global attractor of the dynamical system generated by problem (0.1) and (0.2) is a uniformly asymptotically stable set (*Hint:* see Theorem 1.7.1).

We note that theorems analogous to Theorems 6.1 and 6.2 also hold for a class of retarded perturbations of problem (0.1) and (0.2). For example, instead of (0.1) and (0.2) we can consider (cf. [11–13]) the following problem

$$\ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu + q(u_t) = p\ ,$$

$$u\big|_{t=0} = u_0\,,\quad \dot{u}\big|_{t=0} = u_1\,,\quad u\big|_{t\in(-r,\,0)} = \varphi(t)\,.$$

Here A, M, and L are the same as in Theorems 6.1 and 6.2, the symbol u_t denotes the function on $[-r, 0]$ which is given by the equality $u_t(\sigma) = u(t+\sigma)$ for $\sigma \in [-r, 0]$, the parameter r is a delay value, and $q(\cdot)$ is a linear mapping from $L^2(-r, 0; \mathcal{F}_{1+\alpha})$ into H possessing the property

$$\|A^{\alpha} q(v)\|^2 \le c_0 \int_{-r}^{0} \|A^{1+\alpha} v(\sigma)\|^2 \, d\sigma$$

for c_0 small enough and for all $\alpha \in [0, \alpha_0]$, where α_0 is a positive number. Such a formulation of the problem corresponds to the case when we use the model of the linearized potential gas flow (see [11–14]) to take into account the aerodynamic pressure in problem (0.3) and (0.4).

The following assertion gives the time smoothness of trajectories lying in the attractor of problem (0.1) and (0.2).

Theorem 6.3

Assume that conditions (3.2) ***and*** (5.2) ***are fulfilled and the linear operator*** L ***possesses the property***

$$\|A^{\sigma} L u\| \le C \|A^{\sigma + 1/2} u\|, \qquad u \in D(A) \tag{6.6}$$

for all $\sigma \in [-1/2, 1/2)$. ***Let*** $p \in \mathcal{F}_{1/2}$. ***Then the assertions of Theorem*** 6.1 ***are valid for any*** $\sigma \in (0, 1/2)$. ***Moreover, if*** $M(z) \in C^{l+1}(\mathbb{R}_+)$ ***for some*** $l \ge 1$, ***then the trajectories*** $y = (u(t), \dot{u}(t))$ ***lying in the global attractor*** $\mathcal{A}$ ***of the system*** $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***for*** $\gamma > 0$ ***possess the property***

$$\|u^{(k+1)}(t)\|^2 + \|A u^{(k)}(t)\|^2 + \|A^2 u^{(k-1)}(t)\|^2 \le R_l^2 \tag{6.7}$$

for all $-\infty < t < \infty$, $k = 1, 2, \ldots, l$, ***where*** R_l ***is a constant depending on the problem parameters only.***

Proof.

It is evident that conditions (5.3) and (5.10) follow from (6.6). Therefore, we can apply Theorem 6.1 which guarantees the existence of a global attractor $\mathcal{A}$. Let us assume that $M(z) \in C^{l+1}(\mathbb{R}_+)$, $l \ge 1$. Let $y(t) = (u(t), \dot{u}(t))$ be a trajectory in $\mathcal{A}$, $-\infty < t < \infty$. We consider a function $u_m(t) = p_m u(t)$, where p_m is the orthoprojector onto the span of the basis vectors $\{e_1, \ldots, e_m\}$ in $\mathcal{F}_0$ for m large enough. It is clear that $u_m(t) \in C^2(\mathbb{R}; \mathcal{F}_2)$ and satisfies the equation

$$\ddot{u}_m + \gamma \dot{u}_m + A^2 u_m + M\left(\|A^{1/2} u(t)\|^2\right) A u_m + p_m L u = p_m p. \tag{6.8}$$

Equation (6.6) for $\sigma = -1/2$ implies that $p_m L u$ is a continuously differentiable function. It is also evident that $M(\|A^{1/2} u(t)\|^2) \in C^1(\mathbb{R})$. Therefore, we differentiate equation (6.8) with respect to t to obtain the equation

$$\ddot{w}_m + \gamma \dot{w}_m + A^2 w_m + M\left(\|A^{1/2}u(t)\|^2\right) A w_m = -p_m L\dot{u}(t) + F_m(t)$$

for the function $w_m(t) = \dot{u}_m(t) = p_m \dot{u}(t)$. Here

$$F_m(t) = -2M'\left(\|A^{1/2}u(t)\|^2\right)(Au(t),\ \dot{u}(t))Au_m(t).$$

Since any trajectory $y = (u(t),\ \dot{u}(t))$ lying in the attractor possesses the property

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le R_0^2, \qquad -\infty < t < \infty, \tag{6.9}$$

it is clear that

$$\|F_m(t)\| \le C_{R_0}, \qquad -\infty < t < \infty. \tag{6.10}$$

Relation (6.9) also implies that the function

$$b(t) = M\left(\|A^{1/2}u(t)\|^2\right) \tag{6.11}$$

possesses the property

$$|b(t)| + |\dot{b}(t)| \le C_{R_0}, \qquad -\infty < t < \infty.$$

Therefore, as in the proof of Theorem 2.2, we find that there exists N_0 such that

$$\|(1-P_N)\,U(t,\ \tau)y\|_{\mathscr{H}_S} \le C\|(1-P_N)y\|_{\mathscr{H}_S} e^{-\frac{\gamma}{4}(t-\tau)} \tag{6.12}$$

for all real s, where $\mathscr{H}_s = \mathscr{F}_{1+s} \times \mathscr{F}_s$, P_N is the orthoprojector onto

$$L_N = \mathrm{Lin}\{(e_k,\ 0);\ (0,\ e_k):\ k = 1,\ 2,\ \ldots,\ N\},$$

$N \ge N_0$, and $U(t,\ \tau)$ is the evolutionary operator of the problem

$$\begin{cases} \ddot{u} + \gamma\dot{u} + A^2 u + b(t)Au = 0, \\ u|_{t=0} = u_0, \quad \dot{u}|_{t=0} = u_1, \end{cases}$$

with $b(t)$ of the form (6.10). Moreover, $z_m(t) = (w_m(t),\ \dot{w}_m(t))$ can be presented in the form

$$z_m(t) = U(t,\ t_0)\,z_m(t_0) + \int_{t_0}^{t} U(t,\ \tau)(0;\ \ -p_m L\dot{u}(\tau) + F_m(\tau))\,d\tau. \tag{6.13}$$

Then for $m > N > N_0$ we have

$$\|(1-P_N)\,z_m(t)\|_{\mathscr{H}_{-1/2}} \le Ce^{-\frac{\gamma}{4}(t-t_0)}\|z_m(t_0)\|_{\mathscr{H}_{-1/2}} +$$

$$+\ C\int_{t_0}^{t} e^{-\frac{\gamma}{4}(t-\tau)}\|p_m L\dot{u}(\tau) + F_m(\tau)\|_{-1/2}\,d\tau.$$

Therefore, we use (6.9), (6.10), and (6.6) for $\sigma = -1/2$ to obtain that

$$\|(1-P_N)z_m(t)\|_{\mathcal{H}_{-1/2}} \le C_{R_{01}m}\, e^{-\frac{\gamma}{4}(t-t_0)} + C_{R_0}\int_{t_0}^{t} e^{-\frac{\gamma}{4}(t-\tau)}\, d\tau\,.$$

We tend $t_0 \to -\infty$ in this inequality to find that

$$\sup_{t\,\in\,R}\|(1-P_N)z_m(t)\|_{\mathcal{H}_{-1/2}} \le C\,,$$

where $C > 0$ does not depend on m. It further follows from (6.9) and equation (0.1) that

$$\sup_{t\,\in\,R}\|P_N\, z_m(t)\|_{\mathcal{H}_{-1/2}} \le C\,,$$

where the constant C can depend on N. Hence,

$$\sup_{t\,\in\,R}\left(\left\|p_m A^{1/2}\dot{u}(t)\right\|^2 + \left\|A^{-1/2}p_m\ddot{u}(t)\right\|^2\right) \le C\,.$$

We tend $m \to \infty$ to find that any trajectory $y(t) = (u(t),\ \dot{u}(t))$ lying in the attractor possesses the property

$$\|\dot{u}(t)\|_{1/2} = \|A^{1/2}\dot{u}(t)\| \le C_{R_0}\,, \qquad -\infty < t < \infty\,.$$

By virtue of (6.6) we have

$$\|L\dot{u}(t)\| \le C_{R_0}\,, \qquad -\infty < t < \infty\,.$$

Therefore, we reason as above to find that equation (6.13) implies

$$\|(1-P_N)z_m(t)\|_{\mathcal{H}} \le C_{R_0 m}\, e^{-\frac{\gamma}{4}(t-t_0)} + C_{R_0}\,.$$

Similarly we get

$$\|\ddot{u}(t)\|^2 + \|A\dot{u}(t)\|^2 \le R_1^2$$

for all $t \in (-\infty,\infty)$. Consequently, using equation (0.1) we obtain estimate (6.7) for $k = 1$. In order to prove (6.7) for the other values of k we should use induction with respect to k and similar arguments. We offer the reader to make an independent detailed study as an exercise.

Exercise *6.2* In addition to the hypotheses of Theorem 6.3 we assume that $L \equiv 0$ and $\rho \in \mathcal{F}_l \equiv D(A^l)$. Prove that the global attractor $\mathcal{A}$ of the system $(\mathcal{H},\ S_t)$ lies in $\mathcal{F}_{l+1} \times \mathcal{F}_l$.

§ 7 Conditions of Regularity of Attractor

Unfortunately, the structure of the global attractor of problem (0.1) and (0.2) can be described only under additional conditions that guarantee the existence of the Lyapunov function (see Section 1.6). These conditions require that $L \equiv 0$ and assume the stationarity of the transverse load $p(t)$. For the Berger system (0.3) and (0.4) these hypotheses correspond to $\rho = 0$ and $p(x,\ t) \equiv p(x)$, i.e. to the case of plate oscillations in a motionless stationary medium.

Thus, let us assume that the operator L is identically equal to zero and $p(t) \equiv p$ in (0.1). Assume that the hypotheses of Theorem 3.1 hold. Then energy equality (3.6) implies that

$$E(y(t_2)) - E(y(t_1)) = -\gamma \int_{t_1}^{t_2} \|\dot{u}(\tau)\|^2 \, d\tau + \int_{t_1}^{t_2} (p,\ \dot{u}(\tau))^2 \, d\tau \ , \tag{7.1}$$

where $y(t) = S_t y = (u(t),\ \dot{u}(t))$, the function $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0;\ u_1)$, and $E(y)$ is the energy of the system defined by formula (3.7).

Let us prove that the functional $\Psi(y) = E(y) - (p,\ u_0)$ with $y = (u_0;\ u_1)$ is a Lyapunov function (for definition see Section 1.6) of the dynamical system $(\mathcal{H},\ S_t)$. Indeed, it is evident that the functional $\Psi(y)$ is continuous on $\mathcal{H}$. By virtue of (7.1) it is monotonely increasing. If $E(y(t_0)) = E(y)$ for some $t_0 > 0$, then

$$\int_0^{t_0} \|\dot{u}(\tau)\|^2 \, d\tau = 0 \, .$$

Therefore, $\dot{u}(\tau) = 0$ for $\tau \in [0,\ t_0]$, i.e. $u(t) = \overline{u}$ is a stationary solution to problem (0.1) and (0.2). Hence, $y = (\overline{u};\ 0)$ is a fixed point of the semigroup S_t.

Therefore, Theorems 1.6.1 and 6.1 give us the following assertion.

Theorem 7.1

Assume that $\gamma > 0$, $L \equiv 0$, ***and*** $p \in \mathcal{F}_\sigma$ ***for some*** $\sigma > 0$. ***We also assume that the function*** $M(z)$ ***satisfies conditions*** (3.2) ***and*** (5.2). ***Then the global attractor*** $\mathcal{A}$ ***of the dynamical system*** $(\mathcal{H},\ S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***has the form***

$$\mathcal{A} = \boldsymbol{M}_+(\mathcal{N}) \ , \tag{7.2}$$

where $\mathcal{N}$ ***is the set of fixed points of the semigroup*** S_t, ***i.e.***

$$\mathcal{N} = \left\{ (u;\ 0) \colon \ u \in \mathcal{F}_1, \ \ A^2 u + M\left(\|A^{1/2} u\|^2\right) A u = p \right\} , \tag{7.3}$$

and $\boldsymbol{M}_+(\mathcal{N})$ ***is the unstable set emanating from*** $\mathcal{N}$ *(for definition see Section 1.6).*

— Exercise *7.1* Let $p \equiv 0$. Prove that if the hypotheses of Theorem 7.1 hold, then any fixed point z of problem (0.1) and (0.2) either equals to zero, $z = (0;\ 0)$, or has the form $z = (c \cdot e_k;\ 0)$, where the constant c is the solution to the equation $M(c^2 \lambda_k) + \lambda_k = 0$.

— Exercise *7.2* Assume that $p \equiv 0$ and $M(z) = -\Gamma + z$. Then problem (0.1) and (0.2) has a unique fixed point $z_0 = (0;\ 0)$ for $\Gamma \le \lambda_1$. If $\lambda_n < \Gamma \le \lambda_{n+1}$, then the number of fixed points is equal to $2n+1$ and all of them have the form $z_k = (w_k;\ 0)$, $k = 0,\ \pm 1,\ \ldots,\ \pm n$, where

$$w_0 = 0\,, \qquad w_{\pm k} = \pm\sqrt{\frac{\Gamma - \lambda_k}{\lambda_k}} \cdot e_k\,, \qquad k = 1,\ 2,\ \ldots,\ n\,.$$

— Exercise *7.3* Show that if the hypotheses of Exercise 7.2 hold, then the energy $E(z_k)$ of each fixed point z_k has the form

$$E(z_0) = 0\,, \qquad E(z_{\pm k}) = -\tfrac{1}{4}(\Gamma - \lambda_k)^2\,, \qquad k = 1,\ 2,\ \ldots,\ n\,,$$

for $\lambda_n < \Gamma \le \lambda_{n+1}$.

— Exercise *7.4* Assume that the hypotheses of Theorem 7.1 hold. Show that if the set

$$\mathcal{D}_c = \left\{ y = (u_0;\ u_1) \in \mathcal{H}\colon\ \Psi(y) \equiv E(y) - (p,\ u_0) \le c \right\} \tag{7.4}$$

is not empty, then it is a closed positively invariant set of the dynamical system $(\mathcal{H},\ S_t)$ generated by weak solutions to problem (0.1) and (0.2).

— Exercise *7.5* Assume that the hypotheses of Theorem 7.1 hold and the set $\mathcal{D}_c$ defined by equality (7.4) is not empty. Show that the dynamical system $(\mathcal{D}_c,\ S_t)$ possesses a compact global attractor $\mathcal{A}_c = \boldsymbol{M}_+(\mathcal{N}_c)$, where $\mathcal{N}_c$ is the set of fixed points of S_t satisfying the condition $\Psi(z) \le c$.

— Exercise *7.6* Show that if the hypotheses of Theorem 7.1 hold, then the global minimal attractor $\mathcal{A}_{\min}$ (for definition see Section 1.3) of problem (0.1) and (0.2) coincides with the set $\mathcal{N}$ of the fixed points (see (7.3)).

Further we prove that if the hypotheses of Theorem 7.1 hold, then the attractor $\mathcal{A}$ of problem (0.1) and (0.2) is regular in generic case. As in Section 2.5, the corresponding arguments are based on the results obtained by A. V. Babin and M. I. Vishik (see also Section 1.6). These results prove that in generic case the number of fixed points is finite and all of them are hyperbolic.

Lemma 7.1.

Assume that conditions (3.2) *and* (5.2) *are fulfilled. Then the problem*

$$\mathcal{L}[u] \equiv A^2 u + M\left(\|A^{1/2}u\|^2\right)Au = p, \qquad u \in D(A^{2+\sigma}), \tag{7.5}$$

possesses a solution for any $p \in \mathcal{F}_\sigma$, *where* $\sigma \ge 0$. *If* $\mathcal{B}$ *is a bounded set in* $\mathcal{F}_\sigma$, *then its preimage* $\mathcal{L}^{-1}(\mathcal{B})$ *is bounded in* $\mathcal{F}_{2+\sigma} = D(A^{2+\sigma})$. *If* $\mathcal{B}$ *is a compact in* $\mathcal{F}_\sigma$, *then* $\mathcal{L}^{-1}(\mathcal{B})$ *is a compact in* $D(A^{2+\sigma})$, *i.e. the mapping* $\mathcal{L}$ *is proper.*

Proof.

We follow the line of arguments given in the proof of Lemma 2.5.3. Let us consider the continuous functional

$$W(u) = \frac{1}{2}\left\{(Au,\ Au) + \mathcal{M}\left(\|A^{1/2}u\|^2\right)\right\} - (p,\ u) \tag{7.6}$$

on $\mathcal{F}_1 = D(A)$, where $\mathcal{M}(z) = \int_0^z M(\xi)\,d\xi$ is a primitive of the function $M(z)$. Equation (3.2) implies that

$$\begin{aligned} W(u) \ &\ge\ \frac{1}{2}\left(\|Au\|^2 - a\|A^{1/2}u\|^2 - b\right) - \|A^{-1}p\| \cdot \|Au\| \ \ge \\ &\ge\ \frac{1}{4}\left(1 - \frac{a}{\lambda_1}\right)\|Au\|^2 - \frac{b}{2} - \left(1 - \frac{a}{\lambda_1}\right)^{-1}\|A^{-1}p\|^2 . \end{aligned} \tag{7.7}$$

Thus, the functional $W(u)$ is bounded below. Let us consider it on the subspace $p_m\mathcal{F}_1$, where p_m is the orthoprojector onto $\mathrm{Lin}\{e_1, \ldots, e_m\}$ as before. Since $W(u) \to +\infty$ as $\|Au\| \to \infty$, there exists a minimum point u_m on the subspace $p_m\mathcal{F}_1$. This minimum point evidently satisfies the equation

$$A^2 u_m + M\left(\|A^{1/2}u_m\|^2\right)Au_m \ =\ p_m p . \tag{7.8}$$

Equation (7.7) gives us that

$$\|Au_m\|^2 \ \le\ c_1 + c_2 \inf\left\{W(u):\ \ u \in p_m\mathcal{F}_1\right\} + c_3\|A^{-1}p\|^2$$

with the constants being independent of m. Therefore, it follows from (7.8) that $\|A^2 u_m\| \le C_R$, provided $\|p\| \le R$. This estimate enables us to pass to the limit in (7.8) and to prove that if $\sigma = 0$, then equation (7.5) is solvable for any $p \in \mathcal{F}_\sigma$. Equation (7.5) implies that

$$\|A^{2+\sigma}u_m\| \le C_R \quad \text{for} \quad \|p\|_\sigma \le R ,$$

i.e. $\mathcal{L}^{-1}(\mathcal{B})$ is bounded in $D(A^{2+\sigma})$ if $\mathcal{B}$ is bounded. In order to prove that the mapping $\mathcal{L}$ is proper we should reason as in the proof of Lemma 2.5.3. We give the reader an opportunity to follow these reasonings individually, as an exercise. Lemma 7.1 is proved.

Lemma 7.2

Let $u \in \mathcal{F}_1$. *Then the operator* $\mathcal{L}'[u]$ *defined by the formula*

$$\mathcal{L}'[u]w = A^2 w + 2M'\left(\|A^{1/2}u\|^2\right)(Au,\ w)Au + M\left(\|A^{1/2}u\|^2\right)Aw \qquad (7.9)$$

with the domain $D(\mathcal{L}'[u]) = D(A^2)$ *is selfadjoint and* $\dim \mathcal{K}er\, \mathcal{L}'[u] < \infty$.

Proof.

It is clear that $\mathcal{L}'[u]$ is a symmetric operator on $D(A^2)$. Moreover, it is easy to verify that

$$\|\mathcal{L}'[u]w - A^2 w\| \le C(u)\cdot\|Aw\|, \qquad w \in D(A^2), \qquad (7.10)$$

i.e. $\mathcal{L}'[W]$ is a relatively compact perturbation of the operator A^2. Therefore, $\mathcal{L}'[u]$ is selfadjoint. It is further evident that

$$\mathcal{K}er\, \mathcal{L}'[u] = \mathcal{K}er\left[I + A^{-2}\left(\mathcal{L}'[u] - A^2\right)\right].$$

However, due to (7.10) the operator $A^{-2}(\mathcal{L}'[u] - A^2)$ is compact. Therefore, $\dim \mathcal{K}er\, \mathcal{L}'[u] < \infty$. Lemma 7.2 is proved.

— Exercise *7.7* Prove that for any $u \in \mathcal{F}_1$ the operator $\mathcal{L}'[u]$ is bounded below and has a discrete spectrum, i.e. there exists an orthonormal basis $\{f_k\}$ in $\mathcal{H}$ such that

$$\mathcal{L}'[u]f_k = \mu_k f_k, \quad k = 1,\ 2,\ \dots, \quad \mu_1 \le \mu_2 \le \dots \quad \lim_{n\to\infty} \mu_n = \infty.$$

— Exercise *7.8* Assume that $u = c_0 e_{k_0}$, where c_0 is a constant and e_{k_0} is an element of the basis $\{e_k\}$ of eigenfunctions of the operator A. Show that $\mathcal{L}'[u]e_k = \bar{\mu}_k e_k$ for all $k = 1,\ 2,\ \dots$, where

$$\bar{\mu}_k = \lambda_k^2\left[1 + 2\delta_{k k_0} c_0^2 M'(c_0^2 \lambda_{k_0})\right] + M(c_0^2 \lambda_{k_0})\lambda_k.$$

Here $\delta_{k k_0} = 1$ for $k = k_0$ and $\delta_{k k_0} = 0$ for $k \ne k_0$.

As in Section 2.5, Lemmata 7.1 and 7.2 enable us to use the Sard-Smale theorem (see, e.g., the book by A. V. Babin and M. I. Vishik [10]) and to state that the set

$$\mathcal{R}_\sigma = \left\{h \in \mathcal{F}_\sigma :\ \exists[\mathcal{L}'[u]]^{-1} \quad \text{for all} \quad u \in \mathcal{L}^{-1}[h]\right\}$$

of regular values of the operator $\mathcal{L}$ is an open everywhere dense set in $\mathcal{F}_\sigma$ for $\sigma \ge 0$.

— Exercise *7.9* Show that the set of solutions to equation (7.5) is finite for $p \in \mathcal{R}_\sigma$ (*Hint:* see the proof of Lemma 2.5.5).

Let us consider the linearization of problem (0.1) and (0.2) on a solution $u \in D(A^2)$ to problem (7.5):

$$\begin{aligned} &\ddot{w} + \gamma\dot{w} + \mathscr{L}'[u]w = 0 \ , \\ &w\big|_{t=0} = w_0, \quad \dot{w}\big|_{t=0} = w_1 \ . \end{aligned} \tag{7.11}$$

Here $\mathscr{L}'[u]$ is given by formula (7.9).

— Exercise *7.10* Prove that problem (7.11) has a unique weak solution on any segment $[0, T]$ if $w_0 \in \mathscr{F}_1$, $w_1 \in \mathscr{F}_0$, and the function $M(z) \in C^2(\mathbb{R}_+)$ possesses property (3.2).

Thus, problem (7.11) defines a strongly continuous linear evolutionary semigroup $T_t[u]$ in the space $\mathscr{H} = \mathscr{F}_1 \times \mathscr{F}_0$ by the formula

$$T_t[u](w_0;\ w_1) = (w(t);\ \dot{w}(t)), \tag{7.12}$$

where $w(t)$ is a weak solution to problem (7.11).

— Exercise *7.11* Let $\{f_k\}$ be the orthonormal basis of eigenelements of the operator $\mathscr{L}'[u]$ and let μ_k be the corresponding eigenvalues. Then each subspace

$$\mathscr{H}_k = \mathrm{Lin}\{(f_k;\ 0),\ (0;\ f_k)\} \subset \mathscr{H}$$

is invariant with respect to $T_t[u]$. The eigenvalues of the restriction of the operator $T_t[u]$ onto the subspace $\mathscr{H}_k$ have the form

$$\exp\left\{-\left(\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \mu_k}\right)t\right\}.$$

Lemma 7.3

Let $L \equiv 0$. *Assume that* $M(z) \in C^2(\mathbb{R}_+)$ *possesses property* (3.2). *Then the evolutionary operator* S_t *of problem* (0.1) *and* (0.2) *is Frechét differentiable at each fixed point* $\overline{y} = (\overline{u};\ 0)$. *Moreover,* $S_t'[u] = T_t[\overline{u}]$, *where* $T_t[u]$ *is defined by equality* (7.12).

Proof.

Let

$$z(t) = S_t[\overline{y} + h] - \overline{y} - T_t[\overline{u}]h \ ,$$

where $h = (h_0;\ h_1) \in \mathscr{H}$, $\overline{y} = (\overline{u};\ 0)$, and $\overline{u}$ is a solution to equation (7.5). It is clear that $z(t) = (v(t);\ \dot{v}(t))$, where $v(t) \equiv u(t) - \overline{u} - w(t)$ is a weak solution to problem

$$\begin{cases} \ddot{v} + \gamma v + A^2 v = F(u(t),\ \overline{u},\ w(t)) \ , \\ v\big|_{t=0} = 0, \quad \dot{v}\big|_{t=0} = 0 \ . \end{cases} \tag{7.13}$$

Here

$$F(u(t),\ \overline{u},\ w(t)) = M\left(\|A^{1/2}\overline{u}\|^2\right)A\overline{u} - M\left(\|A^{1/2}u(t)\|^2\right)Au(t) +$$

$$+M\left(\|A^{1/2}\overline{u}\|^2\right)Aw(t) + 2M'\left(\|A^{1/2}\overline{u}\|\right)(A\overline{u},\ w(t))A\overline{u}\ ,$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y_0 = \overline{y} + h \equiv (\overline{u} + h_0;\ h_1)$ and $w(t)$ is a solution to problem (7.11) with $w_0 = h_0$ and $w_1 = h_1$. It is evident that

$$F(u(t),\ \overline{u},\ w(t)) = -M\left(\|A^{1/2}\overline{u}\|^2\right)Av + F_1(t) - M'\left(\|A^{1/2}\overline{u}\|^2\right)F_2(t)\ , \quad (7.14)$$

where

$$F_1(t) = -\Big\{M\left(\|A^{1/2}u(t)\|^2\right) - M\left(\|A^{1/2}\overline{u}\|^2\right) -$$

$$-M'\left(\|A^{1/2}\overline{u}\|^2\right)\left(\|A^{1/2}u(t)\|^2 - \|A^{1/2}\overline{u}\|^2\right)\Big\}Au(t)\ ,$$

$$F_2(t) = \left(\|A^{1/2}u(t)\|^2 - \|A^{1/2}\overline{u}\|^2\right)Au(t) - 2(A\overline{u},\ w(t))A\overline{u}\,.$$

It is also evident that the value $F_1(t)$ can be estimated in the following way

$$\|F_1(t)\| \le c_1 \cdot \max\left\{|M''(z)| : z \in [0,\ c_2]\right\} \left|\|A^{1/2}u(t)\|^2 - \|A^{1/2}\overline{u}\|^2\right|^2$$

for $t \in [0,\ T]$ and for $\|h\|_{\mathcal{H}} \le R$, where the constants c_1 and c_2 depend on T, R, and $\overline{u}$. This implies that

$$\|F_1(t)\| \le C(T,\ R,\ \overline{u})\|A(u(t) - \overline{u})\|^2\ . \quad (7.15)$$

Let us rewrite the value $F_2(t)$ in the form

$$F_2(t) = (u(t) + \overline{u},\ A(u(t) - \overline{u}))\cdot A(u(t) - \overline{u}) +$$

$$+\|A^{1/2}(u(t) - \overline{u})\|^2 A\overline{u} + 2(\overline{u},\ Av(t))A\overline{u}\ .$$

Consequently, the estimate

$$\|F_2(t)\| \le C_1(T,\ R,\ \overline{u})\|A(u(t) - \overline{u})\|^2 + C_2(\overline{u})\|Av(t)\| \quad (7.16)$$

holds for $t \in [0,\ T]$ and for $\|h\|_{\mathcal{H}} \le R$. Therefore, equations (7.14)–(7.16) give us that

$$\|F(u(t),\ \overline{u},\ w(t))\| \le C_1\|Av(t)\| + C_2\|A(u(t) - \overline{u})\|^2$$

on any segment $[0,\ T]$. Here $C_1 = C_1(\overline{u})$ and $C_2 = C_2(T,\ R,\ \overline{u})$. We use continuity property (3.20) of a solution to problem (0.1) and (0.2) with respect to the initial conditions to obtain that

$$\|A(u(t)-\overline{u})\| \;\le\; C_{T,\,R}\|h\|_{\mathcal{H}}, \qquad t \in [0,\,T], \qquad \|h\|_{\mathcal{H}} \le R\,.$$

Therefore,

$$\|F(u(t),\ \overline{u},\ w(t))\| \;\le\; C_1\|Av(t)\| + C_2\|h\|^2_{\mathcal{H}}$$

for $t \in [0,\,T]$ and for $\|h\|_{\mathcal{H}} \le R$. Hence, the energy equality for the solutions to problem (7.13) gives us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|Av(t)\|^2 + \|\dot{v}(t)\|^2\Big) \;\le\; a\Big(\|Av(t)\|^2 + \|\dot{v}(t)\|^2\Big) + C\|h\|^4\,.$$

Therefore, Gronwall's lemma implies that

$$\|Av(t)\|^2 + \|\dot{v}(t)\|^2 \;\le\; C\|h\|^4\,, \qquad t \in [0,\,T]\,.$$

This equation can be rewritten in the form

$$\big\|S_t[\overline{y}+h] - \overline{y} - T_t[\overline{u}]h\big\|_{\mathcal{H}} \;\le\; C\|h\|^2\,.$$

Thus, Lemma 7.3 is proved.

— Exercise *7.12* Use the arguments given in the proof of Lemma 7.3 to verify that under condition (3.2) for $M(z) \in C^2(\mathbb{R}_+)$ the evolutionary operator S_t of problem (0.1) and (0.2) in $\mathcal{H}$ belongs to the class C^1 and

$$\big\|S'_t[y_1] - S'_t[y_2]\big\|_{\mathcal{L}(\mathcal{H},\ \mathcal{H})} \;\le\; C\big\|y_1 - y_2\big\|_{\mathcal{H}}$$

for any $t > 0$ and $y_j \in \mathcal{H}$.

— Exercise *7.13* Use the results of Exercises 7.7 and 7.11 to prove that for a regular value p of the mapping $\mathcal{L}[u]$ the spectrum of the operator $T_t[u]$ does not intersect the unit circumference while the eigensubspace E_+ which corresponds to the spectrum outside the unit disk does not depend on t and is finite-dimensional.

The results presented above enable us to prove the following assertion (see Chapter V of the book by A. V. Babin and M. I. Vishik [10]).

Theorem 7.2

Assume that the hypotheses of Theorem 7.1 ***hold. Then there exists an open dense set*** $\mathcal{R}_\sigma$ ***in*** $\mathcal{F}_\sigma$ ***such that the dynamical system*** $(\mathcal{H},\ S_t)$ ***possesses a regular global attractor*** $\mathcal{A}$ ***for every*** $p \in \mathcal{R}_\sigma$***, i. e.***

$$\mathcal{A} = \bigcup_{j=1}^{N} \boldsymbol{M}_+(z_j),$$

where $\boldsymbol{M}_+(z_j)$ ***is the unstable manifold of the evolutionary operator*** S_t ***emanating from the fixed point*** z_j***. Moreover, each set*** $\boldsymbol{M}_+(z_j)$ ***is a finite-dimensional surface of the class*** C^1.

In the case of a zero transverse load ($p \equiv 0$) Theorem 7.2 is not applicable in general. However, this case can be studied by using the structure of the problem. For example, we can guarantee finiteness of the set of fixed points if we assume (see Exercise 7.1) that the equation $M(c^2\lambda_k)+\lambda_k = 0$, first, is solvable with respect to c only for a finite number of the eigenvalues λ_k and, second, possesses not more than a finite number of solutions for every k. The solutions to equation (7.5) are either $\bar{u} \equiv 0$, or $\bar{u} = c_0 e_{k_0}$, where c_0 and k_0 satisfy $M(c_0^2\lambda_{k_0})+\lambda_{k_0} = 0$. The eigenvalues of the operator $\mathcal{L}'[u]$ have the form

$$\bar{\mu}_k = \lambda_k^2 + M(0)\lambda_k \quad \text{if} \quad \bar{u} \equiv 0$$

and

$$\bar{\mu}_k = \lambda_k\left[\lambda_k - \lambda_{k_0} + 2\delta_{kk_0}\lambda_{k_0}c_0^2 M'(c_0^2\lambda_{k_0})\right] \quad \text{if} \quad \bar{u} = c_0 e_{k_0}.$$

Therefore, the result of Exercise 7.11 implies that the fixed points are hyperbolic if all the numbers $\bar{\mu}_k$ are nonzero, i.e. if

$$M(0) \neq -\lambda_k, \quad k = 1, 2, \dots; \qquad \lambda_k \neq \lambda_{k_0}, \quad k \neq k_0; \qquad M(c_0^2\lambda_{k_0}) \neq 0$$

for all c_0 and k_0 such that $M(c_0^2\lambda_{k_0})+\lambda_{k_0} = 0$. In particular, if $M(z) = -\Gamma + z$, then for any real Γ there exists a finite number of fixed points (see Exercise 7.2) and all of them are hyperbolic, provided that $\Gamma \neq \lambda_k$ for all k and the eigenvalues λ_j satisfying the condition $\lambda_j < \Gamma$ are simple. Moreover, we can prove that for $\lambda_n < \Gamma < \lambda_{n+1}$ the unstable manifold $\boldsymbol{M}_+(z_k)$, $k = 0, \pm 1, \dots, \pm n$, emanating from the fixed point z_k (see Exercise 7.2) possesses the property

$$\dim \boldsymbol{M}_+(z_0) = n, \qquad \dim \boldsymbol{M}_+(z_k) = |k| - 1.$$

§ 8 On Singular Limit in the Problem of Oscillations of a Plate

In this section we consider problem (0.1) and (0.2) in the following form:

$$\mu\ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu = p, \quad t > 0, \tag{8.1}$$

$$u\big|_{t=0} = u_0, \quad \dot{u}\big|_{t=0} = u_1. \tag{8.2}$$

Equation (8.1) differs from equation (0.1) in that the parameter $\mu > 0$ is introduced. It stands for the mass density of the plate material. The introduction of a new time $t' = t/\sqrt{\mu}$ transforms equation (8.1) into (0.1) with the medium resistance parameter $\gamma' = \gamma/\sqrt{\mu}$ instead of γ. Therefore, all the above results mentioned above remain true for problem (8.1) and (8.2) as well.

The main question discussed in this section is the asymptotic behaviour of the solution to problem (8.1) and (8.2) for the case when the inertial forces are small with respect to the medium resistance forces $\mu \ll \gamma$. Formally, this assumption leads to a quasistatic statement of problem (8.1) and (8.2):

$$\gamma \dot{u} + A^2 u + M\left(\|A^{1/2} u\|^2\right) Au + Lu = p, \quad t > 0, \tag{8.3}$$

$$u\big|_{t=0} = u_0 . \tag{8.4}$$

Here we prove that the global attractor of problem (8.1) and (8.2) is close to the global attractor of the dynamical system generated by equations (8.3) and (8.4) in some sense.

Without loss of generality we further assume that $\gamma = 1$. We also note that problem (8.3) and (8.4) belongs to the class of equations considered in Chapter 2.

— Exercise *8.1* Assume that conditions (3.2) and (3.3) are fulfilled and $p \in \mathcal{F}_0 = H$. Show that problem (8.3) and (8.4) has a unique mild (in $\mathcal{F}_1 = D(A)$) solution on any segment $[0, T]$, i.e. there exists a unique function $u(t) \in C(0, T; \mathcal{F}_1)$ such that

$$u(t) = e^{-A^2 t} u_0 -$$

$$- \int_0^t e^{-A^2(t-\tau)} \left\{ M\left(\|A^{1/2} u(\tau)\|^2\right) Au(\tau) + Lu(\tau) - p \right\} d\tau .$$

(*Hint:* see Theorem 2.2.4 and Exercise 2.2.10).

Let us consider the Galerkin approximations of problem (8.3) and (8.4):

$$\dot{u}_m(t) + A^2 u_m(t) + M\left(\|A^{1/2} u_m(t)\|^2\right) Au_m + p_m L u_m(t) = p_m p , \tag{8.5}$$

$$u_m(0) = p_m u_0 , \tag{8.6}$$

where p_m is the orthoprojector onto the first m eigenvectors of the operator A and $u_m(t) \in \mathrm{Lin}\{e_1, \ldots, e_m\}$.

— Exercise *8.2* Assume that conditions (3.2) and (3.3) are fulfilled and $p \in \mathcal{F}_0 = H$. Then problem (8.5) and (8.6) is solvable on any segment $[0, T]$ and

$$\max_{[0, T]} \|A(u(t) - u_m(t))\| \to 0 , \qquad m \to \infty . \tag{8.7}$$

Theorem 8.1

Let $p \in H$ ***and assume that conditions*** (3.2), (5.2), ***and*** (5.3) ***are fulfilled. Then the dynamical system*** $(\mathcal{F}_1, S_t)$ ***generated by weak solutions to problem*** (8.3) ***and*** (8.4) ***possesses a compact connected global attractor*** $\mathcal{A}$. ***This attractor is a bounded set in*** $\mathcal{F}_{1+\beta}$ ***for*** $0 \le \beta < 1$ ***and has a finite fractal dimension.***

Proof.

First we prove that the system $(\mathscr{F}_1, S_t)$ is dissipative. To do that we consider the Galerkin approximations (8.5) and (8.6). We multiply (8.5) by $u_m(t)$ scalarwise and find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|^2 + \|Au_m(t)\|^2 + M\Big(\|A^{1/2}u_m(t)\|^2\Big)\|A^{1/2}u_m(t)\|^2 + $$
$$+ (Lu_m(t),\ u_m(t)) = (p,\ u_m(t))\ .$$

Using equation (5.2) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|^2 + \|Au_m(t)\|^2 + a_1\,\mathscr{M}\Big(\|A^{1/2}u_m(t)\|^2\Big) \le$$
$$\le a_3 - a_2\|A^{1/2}u_m(t)\|^{2+2\alpha} - (Lu_m(t),\ u_m(t)) + (p,\ u_m(t))\ .$$

We use equation (5.3) and reason in the same way as in the proof of Theorem 5.1 to find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + b_0\Big(\|Au_m\|^2 + \mathscr{M}\Big(\|A^{1/2}u_m\|^2\Big)\Big) + b_1\|A^{1/2}u_m\|^{2+2\alpha} \le b_2 \qquad (8.8)$$

with some positive constants b_j, $j = 0,\ 1,\ 2$. Multiplying equation (8.5) by $\dot{u}_m(t)$ we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Pi(u_m(t)) + \|\dot{u}_m(t)\|^2 + (Lu_m(t),\ \dot{u}_m(t)) = (p,\ \dot{u}_m(t)),$$

where

$$\Pi(u) = \|Au\|^2 + \mathscr{M}\Big(\|A^{1/2}u\|^2\Big).$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(u_m(t)) + \|\dot{u}_m(t)\|^2 \le 2\|p\|^2 + C\|A^\theta u_m(t)\|^2\ . \qquad (8.9)$$

If we summarize (8.8) and (8.9), then it is easy to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|u_m\|^2 + \Pi(u_m)\Big\} + b_0\Big\{\|u_m\|^2 + \Pi(u_m)\Big\} \le C\ .$$

This implies that

$$\|u_m(t)\|^2 + \Pi(u_m(t)) \le \Big\{\|p_m u_0\|^2 + \Pi(p_m u_0)\Big\}e^{-b_0 t} + C\,.$$

We use (8.7) to pass to the limit as $m \to \infty$ and to obtain that

$$\|u(t)\|^2 + \Pi(u(t)) \le \Big\{\|u_0\|^2 + \Pi(u_0)\Big\}e^{-b_0 t} + C\ .$$

This implies the dissipativity of the dynamical system $(\mathscr{F}_1, S_t)$ generated by problem (8.3) and (8.4). In order to complete the proof of the theorem we use Theorem 2.4.1.

We note that the dissipativity also implies that the dynamical system $(\mathscr{F}_1, S_t)$ possesses a fractal exponential attractor (see Theorem 2.4.2).

Exercise *8.3* Assume that the hypotheses of Theorem 8.1 hold and $L \equiv 0$. Show that for generic $p \in H$ the attractor of the dynamical system $(\mathscr{F}_1, S_t)$ generated by equations (8.3) and (8.4) is regular (see the definition in the statement of Theorem 7.2). *Hint:* see Section 2.5.

We assume that $M(z) \in C^2(\mathbb{R}_+)$ and conditions (3.2), (5.2), (5.3), and (5.10) are fulfilled. Let us consider the dynamical system $(\mathscr{H}_1, S_t^\mu)$ generated by problem (8.1) and (8.2) in the space $\mathscr{H}_1 = \mathscr{F}_2 \times \mathscr{F}_1 \equiv D(A^2) \times D(A)$. Lemma 5.2 and Exercise 5.5 imply that $(\mathscr{H}_1, S_t^\mu)$ possesses a compact global attractor $\mathscr{A}_\mu$ for any $\mu > 0$.

The main result of this section is the following assertion on the closeness of attractors of problem (8.1) and (8.2) and problem (8.3) and (8.4) for small $\mu > 0$.

Theorem 8.2

Assume that $M(z) \in C^2(\mathbb{R}_+)$ ***and conditions*** (3.2), (5.2), (5.3), ***and*** (5.10) ***concerning*** $M(z)$, L, ***and*** p ***are fulfilled. Then the equation***

$$\lim_{\mu \to 0} \sup\{\operatorname{dist}_{\mathscr{H}}(y, \mathscr{A}^*) : y \in \mathscr{A}_\mu\} = 0 \tag{8.10}$$

is valid, where $\mathscr{A}_\mu$ ***is a global attractor of the dynamical system*** $(\mathscr{H}_1, S_t^\mu)$ ***generated by problem*** (8.1) ***and*** (8.2),

$$\mathscr{A}^* = \left\{(z_0;\ z_1) :\ z_0 \in \mathscr{A},\ z_1 = -A^2 z_0 - M\left(\|A^{1/2} z_0\|^2\right) A z_0 - L z_0 + p\right\}.$$

Here $\mathscr{A}$ ***is a global attractor of problem*** (8.3) ***and*** (8.4) ***in*** $\mathscr{F}_1$ ***and*** $\operatorname{dist}_{\mathscr{H}}(y, A)$ ***is the distance between the element*** y ***and the set*** A ***in the space*** $\mathscr{H} = \mathscr{F}_1 \times \mathscr{F}_0$. ***We remind that*** $\gamma = 1$ ***in equations*** (8.1) ***and*** (8.3).

The proof of the theorem is based on the following lemmata.

Lemma 8.1

The dynamical system $(\mathscr{H}_1, S_t^\mu)$ *is uniformly dissipative in* $\mathscr{H}$ *with respect to* $\mu \in (0, \mu_0]$ *for some* $\mu_0 > 0$, *i.e. there exists* $\mu_0 > 0$ *and* $R > 0$ *such that for any set* $B \subset \mathscr{H}_1$ *which is bounded in* $\mathscr{H}$ *we have*

$$S_t^\mu B \subset \left\{y = (u_0;\ u_1) :\ \mu\|u_1\|^2 + \|A u_0\|^2 \le R^2\right\} \tag{8.11}$$

for all $t \ge t(B, \mu)$, $\mu \in (0, \mu_0]$.

Proof.

We use the arguments from the proof of Theorem 5.1 slightly modifying them. Let

$$V(y) = E(y) + \nu \Phi(y), \qquad y = (u_0;\ u_1),$$

where

$$E(y) = \frac{1}{2}\Big(\mu \|u_1\|^2 + \|Au_0\|^2 + \mathcal{M}\Big(\|A^{1/2}u_0\|^2\Big)\Big)$$

and

$$\Phi(y) = \mu(u_0,\ u_1) + \frac{1}{2}\|u_0\|^2 .$$

As in the proof of Theorem 5.1 it is easy to find that the inequalities

$$\frac{d}{dt}E(y(t)) \le -\frac{1}{2}\|u\|^2 + \frac{\varepsilon}{2}\|Au\|^2 + C_\varepsilon\|A^{1/2}u\|^2 + C_1 \tag{8.12}$$

and

$$\frac{d}{dt}\Phi(y(t)) \le \mu\|\dot{u}\|^2 - \frac{1}{2}\|Au\|^2 - a_1\mathcal{M}\Big(\|A^{1/2}u\|^2\Big) - \\ - a_2\|A^{1/2}u\|^{2+2\alpha} + C_2\|u\|^2 + C_3 \tag{8.13}$$

are valid for $y(t) = (u(t);\ \dot{u}(t)) = S_t^{\mu} y_0$. Here $\varepsilon > 0$ is an arbitrary number, the constants a_j and c_j do not depend on μ. Moreover, it is also evident that

$$V(y) \le \frac{1}{2}\beta_0\Big(\mu\|u_1\|^2 + \|Au_0\|^2 + \mathcal{M}\Big(\|A^{1/2}u_0\|^2\Big)\Big) + \beta_1 \tag{8.14}$$

for $\mu \in (0,\ \mu_0]$ and for any μ_0. Here β_0 and β_1 do not depend on $\mu \in \in (0,\ \mu_0]$ and $\nu \le 1$. Equations (8.12)–(8.14) lead us to the inequality

$$\frac{d}{dt}V(y(t)) + \delta V(y(t)) \le -\frac{1}{2}(1-(\delta\beta_0+\nu)\mu)\|\dot{u}\|^2 - \\ -\frac{1}{2}(\nu - \delta\beta_0 - \varepsilon)\|Au\|^2 - \Big(\nu a_1 - \frac{\delta\beta_0}{2}\Big)\mathcal{M}\Big(\|A^{1/2}u\|^2\Big) + \mathcal{D},$$

where the constant $\mathcal{D}$ does not depend on $\mu \in (0,\ \mu_0]$. If we choose μ_0 small enough, then we can take $\delta > 0$ and $\nu > 0$ independent of $\mu \in (0,\ \mu_0]$ and such that

$$\frac{d}{dt}V(y(t)) + \delta V(y(t)) \le \mathcal{D}_1, \tag{8.15}$$

where $\mathcal{D}_1 > 0$ does not depend on $\mu \in (0,\ \mu_0]$. Moreover, we can assume (due to the choice of μ_0) that

$$V(y(t)) \ge \beta_3\Big(\mu\|\dot{u}(t)\|^2 + \|Au(t)\|^2\Big) - C, \tag{8.16}$$

where β_3 and C do not depend on $\mu \in (0,\ \mu_0]$. Using equations (8.14)–(8.16) we obtain the assertion of Lemma 8.1.

Lemma 8.2

Let $u(t)$ be a solution to problem (8.1) *and* (8.2) *such that* $\|Au(t)\| \le R$ *for all* $t \ge 0$. *Then the estimate*

$$\frac{1}{2}\int_0^t \|\dot{u}(\tau)\|^2 e^{-\beta(t-\tau)}\,\mathrm{d}\tau \;\le\; \left(\mu\|\dot{u}(0)\|^2 + \|Au(0)\|^2\right)e^{-\beta t} + C(R,\,\beta)$$

is valid for $t \ge 0$, *where* β *is a positive constant such that* $\beta\mu \le 1/2$.

Proof.

It is evident that the estimate

$$\left\| M\left(\|A^{1/2}u(t)\|^2\right)Au(t) + Lu(t) - p \right\| \;\le\; C_R$$

holds, provided $\|Au(t)\| \le R$. Therefore, equaiton (8.1) easily implies the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right) + \frac{1}{2}\|\dot{u}(t)\|^2 \;\le\; C_R$$

for the solution $u(t)$. We multiply this inequality by $2\exp(\beta t)$. Then by virtue of the fact that $\|Au(t)\| \le R$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\beta t}\left(\mu\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right)\right] + \frac{1}{2}\|\dot{u}(t)\|^2 e^{\beta t} \;\le\; C_{R,\,\beta}\,e^{\beta t}$$

for $\mu\beta \le 1/2$. We integrate this equation from 0 to t to obtain the assertion of the lemma.

Lemma 8.3

Let $u(t)$ be a solution to problem (8.1) *and* (8.2) *with the initial conditions* $(u_0;\,u_1) \in \mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$ *and such that* $\|Au(t)\| \le R$ *for* $t \ge 0$. *Then the estimate*

$$\mu\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \;\le\; C_1\left(1 + \|\dot{w}(0)\|^2 + \|Aw(0)\|^2\right)e^{-\beta_0 t} + C_2 \tag{8.17}$$

is valid for the function $w(t) = \dot{u}(t)$. *Here* $\mu \in (0,\,\mu_0]$, μ_0 *is small enough,* $\beta_0 > 0$, *and the numbers* C_1 *and* C_2 *do not depend on* $\mu \in (0,\,\mu_0]$.

Proof.

Let us consider the function

$$W(t) = \frac{1}{2}\left(\mu\|\dot{w}(t)\|^2 + \|Aw(t)\|^2\right) + \nu\left(\mu(\dot{w},\,w) + \frac{1}{2}\|w\|^2\right)$$

for $\nu > 0$. It is clear that

$$\frac{1}{2}\mu(1-\nu\mu)\|\dot{w}\|^2+\frac{1}{2}\|Aw\|^2 \le W(t) \le$$
$$\le \frac{1}{2}\mu(1+\nu\mu)\|\dot{w}\|^2+\left(\frac{1}{2}+\nu\lambda_1^{-2}\right)\|Aw\|^2 . \tag{8.18}$$

Since the function $w(t)$ is a weak solution to the equation obtained by the differentiation of (8.1) with respect to t (cf. (4.15)):

$$\mu\ddot{w}(t)+\dot{w}(t)+A^2w(t)+M\left(\|A^{1/2}u(t)\|^2\right)Aw(t)+Lw(t) = F(t),$$

where

$$F(t)=-2M'\left(\|A^{1/2}u(t)\|^2\right)(Au(t),\ w(t))\,Au(t),$$

then we have that

$$\frac{d}{dt}W(t) = -(1-\nu\mu)\|\dot{w}\|^2-M(\|A^{1/2}u(t)\|^2)(Aw,\ \dot{w})-(Lw,\ \dot{w})+(F,\ \dot{w})-$$
$$-\ \nu\left\{\|A^2w\|^2+M(\|A^{1/2}u(t)\|^2)\|A^{1/2}w\|^2+(Lw,\ w)-(F,\ w)\right\}.$$

It follows that

$$\frac{d}{dt}W(t) \le -\left(\frac{1}{2}-\nu\mu\right)\|\dot{w}\|^2-\frac{1}{2}(\nu-C_R^{(1)})\|Aw\|^2+\nu C_R^{(2)}\|w\|^2 .$$

We take $\nu = C_R^{(1)}+2$ and choose μ_0 small enough to obtain with the help of (8.18) that

$$\frac{d}{dt}W(t)+\beta_0W(t) \le C\|w(t)\|^2, \qquad t\ge 0, \qquad \mu\in(0,\ \mu_0],$$

where the constants $\beta_0>0$ and C do not depend on μ. Consequently,

$$W(t) \le W(0)\,e^{-\beta_0 t}+C\int_0^t \|\dot{u}(\tau)\|^2e^{-\beta_0(t-\tau)}\,d\tau .$$

Therefore, estimate (8.17) follows from equation (8.18) and Lemma 8.2. Thus, Lemma 8.3 is proved.

Lemma 8.3 and equations (8.11) imply the existence of a constant R_1 such that for any bounded set B in $\mathcal{H}_1$ there exists $t_0 = t_0(B,\ \mu)$ such that

$$\mu\|\ddot{u}(t)\|^2+\|A\dot{u}(t)\|^2 \le R_1^2, \qquad \mu\in(0,\ \mu_0), \tag{8.19}$$

where $u(t)$ is a solution to problem (8.1) and (8.2) with the initial conditions from B. However, due to (8.1) equations (8.11) and (8.19) imply that $\|A^2u(t)\|\le C$ for $t\ge t_0(B,\ \mu)$. Thus, there exists $R_2>0$ such that

$$\mu\|\ddot{u}(t)\|^2 + \|A\dot{u}(t)\|^2 + \|A^2u(t)\|^2 \le R_2^2\,, \quad t \ge t_0(B,\ \mu)\,, \tag{8.20}$$

where $u(t)$ is a solution to system (8.1) and (8.2) with the initial conditions from the bounded set B in $\mathcal{H}_1$, R_2 does not depend on $\mu \in (0,\ \mu_0)$, and μ_0 is small enough. Equation (8.20) and the invariance property of the attractor $\mathcal{A}_\mu$ imply the estimate

$$\mu\left\|\ddot{u}_\mu(t)\right\|^2 + \left\|A\dot{u}_\mu(t)\right\|^2 + \left\|A^2u_\mu(t)\right\|^2 \le R_2^2 \tag{8.21}$$

for any trajectory $S_t^\mu y_0 = (u_\mu(t);\ \dot{u}_\mu(t))$ lying in $\mathcal{A}_\mu$ for all $t \in (-\infty, \infty)$.

Let us prove (8.10). It is evident that there exists an element $y_\mu = (u_{0\,\mu};\ u_{1\,\mu})$ from $\mathcal{A}_\mu$ such that

$$d(y_\mu) \equiv \text{dist}_{\mathcal{H}}(y_\mu,\ \mathcal{A}^*) = \sup\{\text{dist}_{\mathcal{H}}(y,\ \mathcal{A}^*):\ y \in \mathcal{A}_\mu\}\,.$$

Let $y_\mu(t) = (u_\mu(t);\ \dot{u}_\mu(t))$ be a trajectory of system (8.1) and (8.2) lying in the attractor $\mathcal{A}_\mu$ and such that $y_\mu(0) = y_\mu$. Equation (8.21) implies that there exist a subsequence $\{y_{\mu_n}(t)\}$ and an element $y(t) = (u(t);\ \dot{u}(t)) \in L^\infty(-\infty, \infty;\ \mathcal{H}_1)$ such that for any segment $[a,\ b]$ the sequence $y_{\mu_n}(t)$ converges to $y(t)$ in the $*$-weak topology of the space $L^\infty(a,\ b;\ \mathcal{H}_1)$ as $\mu_n \to 0$. Equation (8.21) gives us that the subsequence $\{Au_{\mu_n}(t)\}$ is uniformly continuous and uniformly bounded in H. Therefore (cf. Exercise 1.14),

$$\lim_{\mu_n \to 0}\ \max_{t\,\in[a,\ b]}\left\|A(u_{\mu_n}(t) - u(t))\right\| = 0 \tag{8.22}$$

for any $a < b$. However, it follows from (8.21) that $\mu\left\|\ddot{u}_\mu(t)\right\| \to 0$ as $\mu \to 0$. Therefore, we pass to the limit $\mu \to 0$ in equation (8.1) and obtain that the function $u(t)$ is a bounded (on the whole axis) solution to problem (8.3) and (8.4). Hence, it lies in the attractor $\mathcal{A}$ of the system $(\mathcal{F}_1,\ S_t)$. With the help of (8.21) and (8.22) it is easy to find that

$$d(y_{\mu_n}) \le \left\|y_{\mu_n} - y_0\right\|_{\mathcal{H}} \to 0\,, \qquad \mu_n \to 0\ ,$$

where

$$y_0 = \Big(u(0);\ \ -Au(0) - M\Big(\|A^{1/2}u(0)\|^2\Big)Au_0 - Lu_0 + p\Big) \in \mathcal{A}^*\,.$$

Thus, **Theorem 8.2 is proved**.

§ 9 On Inertial and Approximate Inertial Manifolds

The considerations of this section are based on the results presented in Sections 3.7, 3.8, and 3.9. For the sake of simplicity we further assume that $p(t) \equiv g \in H$.

Theorem 9.1

Assume that conditions (3.2), (5.2), ***and*** (5.3) ***are fulfilled. We also assume that eigenvalues of the operator*** A ***possess the properties***

$$\inf_N \frac{\lambda_N}{\lambda_{N+1}} > 0 \quad \textit{and} \quad \lambda_{N(k)+1} = c_0 k^\rho (1 + o(1)), \qquad \rho > 0, \quad k \to \infty, \tag{9.1}$$

for some sequence $\{N(k)\} \to \infty$. ***Then there exist numbers*** $\gamma_0 > 0$ ***and*** $k_0 > 0$ ***such that the conditions***

$$\gamma > \gamma_0 \quad \textit{and} \quad \lambda^2_{N(k)+1} - \lambda^2_{N(k)} \ge k_0 \lambda_{N(k)+1} \tag{9.2}$$

imply that the dynamical system $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***possesses a local inertial manifold, i.e. there exists a finite-dimensional manifold*** $\mathcal{M}$ ***in*** $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$ ***of the form***

$$\mathcal{M} = \{ y = w + \Phi(w) : \quad w \in P\mathcal{H}, \ \Phi(w) \in (1-P)\mathcal{H} \}, \tag{9.3}$$

where $\Phi(\cdot)$ ***is a Lipschitzian mapping from*** $P\mathcal{H}$ ***into*** $(1-P)\mathcal{H}$ ***and*** P ***is a finite-dimensional projector in*** $\mathcal{H}$. ***This manifold possesses the properties:***

1) ***for any bounded set*** B ***in*** $\mathcal{H}$ ***and for*** $t \ge t_0(B)$

$$\sup \{ \operatorname{dist}(S_t y, \mathcal{M}) : y \in \mathcal{B} \} \le C \exp\{ -\beta (t - t_0(\boldsymbol{B})) \} ; \tag{9.4}$$

2) ***there exists*** $R > 0$ ***such that the conditions*** $y \in \mathcal{M}$ ***and*** $\|S_t y\|_{\mathcal{H}} \le R$ ***for*** $t \in [0, t_0]$ ***imply that*** $S_t y \in \mathcal{M}$ ***for*** $t \in [0, t_0]$;
3) ***if the global attractor of the system*** $(\mathcal{H}, S_t)$ ***exists, then the set*** $\mathcal{M}$ ***contains it*** *(see* Theorem 6.1*)*.

Proof.

Conditions (3.2), (5.2), and (5.3) imply (see Theorem 5.1) that the dynamical system $(\mathcal{H}, S_t)$ is dissipative, i.e. there exists $R > 0$ such that

$$\|S_t y\|_{\mathcal{H}} \le R, \qquad y \in B, \qquad t \ge t_0(B) \tag{9.5}$$

for any bounded set $B \in \mathcal{H}$. This enables us to use the dynamical system $(\mathcal{H}, \tilde{S}_t)$ generated by an equation of the type

$$\begin{cases} \ddot{u} + \gamma \dot{u} + A^2 u = B_R(u), \\ u|_{t=0} = u_0, \quad \dot{u}|_{t=0} = u_1, \end{cases} \tag{9.6}$$

to describe the asymptotic behaviour of solutions to problem (0.1) and (0.2). Here

$$B_R(u) = \chi\left((2R)^{-1} \|Au\| \right) \cdot \left\{ g - M\left(\|A^{1/2} u\|^2 \right) Au - Lu \right\}$$

and $\chi(s)$ is an infinitely differentiable function on $\mathbb{R}_+$ possessing the properties

$$0 \le \chi(s) \le 1 ; \quad |\chi'(s)| \le 2 ;$$
$$\chi(s) = 1 , \quad 0 \le s \le 1 ; \qquad \chi(s) = 0 , \quad s \ge 2 .$$

It is easy to find that there exists a constant C_R such that

$$\|B_R(u)\| \le C_R$$

and

$$\|B_R(u_1) - B_R(u_2)\| \le C_R \|A(u_1 - u_2)\| .$$

Therefore, we can apply Theorem 3.7.2 to the dynamical system $(\mathcal{H}, \tilde{S}_t)$ generated by equation (9.6). This theorem guarantees the existence of an inertial manifold of the system $(\mathcal{H}, \tilde{S}_t)$ if the hypotheses of Theorem 9.1 hold. However, inside the dissipativity ball $\{y : \|y\|_{\mathcal{H}} \le R\}$ problem (9.6) coincides with problem (0.1) and (0.2). This easily implies the assertion of Theorem 9.1.

— Exercise *9.1* Show that the hypotheses of Theorem 9.1 hold for the problem on oscillations of an infinite panel in a supersonic flow of gas:

$$\begin{cases} \partial_t^2 u + \gamma \partial_t u + \partial_x^4 u + \left(\Gamma - \int\limits_0^\pi |\partial_x u(x, t)|^2 \, dx \right) \partial_x^2 u + \rho \partial_x u = g(x) , \quad x \in (0, \pi), \ t > 0 , \\ u\big|_{x=0, \ x=\pi} = \partial_x^2 u\big|_{x=0, \ x=\pi} = 0 , \quad u\big|_{t=0} = u_0(x) , \quad \partial_t u\big|_{t=0} = u_1(x) . \end{cases}$$

Here Γ and ρ are real parameters and $g(x) \in L^2(0, \pi)$.

It is evident that the most essential assumption of Theorem 9.1 that restricts its application is condition (9.2). In this connection the following assertion concerning the case when problem (0.1) and (0.2) possesses a regular attractor is of some interest.

Theorem 9.2

Assume that in equation (0.1) ***we have*** $L \equiv 0$ ***and*** $p(t) \equiv g \in \mathrm{Lin}\{e_1, \ldots, e_{N_0}\}$ ***for some*** N_0. ***We also assume that conditions*** (3.2) ***and*** (5.2) ***are fulfilled. Then there exists*** $N_1 \ge N_0$ ***such that for all*** $N \ge N_1$ ***the subspace***

$$\mathcal{H}_N = \mathrm{Lin}\{(e_k; \ 0), \ (0; \ e_k) : \ k = 1, 2, \ldots, N\} \tag{9.7}$$

is an invariant and exponentially attracting set of the dynamical system $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2)***:***

$$\mathrm{dist}(S_t y, \mathcal{H}_N) \le C_B \|(1 - P_N) y\|_{\mathcal{H}} \, e^{-\frac{\gamma}{4}(t - t_0(B))} , \qquad y \in B \tag{9.8}$$

for $t \ge t_0(B)$ ***and for any bounded set*** B ***in*** $\mathcal{H}$. ***Here*** P_N ***is the orthoprojector onto*** $\mathcal{H}_N$.

Proof.

Since $p_N g = g$ for $N \ge N_0$, where p_N is the orthoprojector onto the span of $\{e_1, \ldots, e_{N_0}\}$, the uniqueness theorem implies the invariance of $\mathcal{H}_N$. Let us prove attraction property (9.8). It is sufficient to consider a trajectory $(u(t), \dot{u}(t))$ lying in the ball of dissipativity $\{y: \|y\|_{\mathcal{H}} \le R\}$. Evidently the function $v(t) = (1 - p_N) u(t)$ satisfies the equation

$$\begin{cases} \ddot{v} + \gamma \dot{v} + A^2 v + M\left(\|A^{1/2} u(t)\|^2\right) A v = 0 \,, \\ v\big|_{t=0} = (1 - p_N) u_0 \,, \quad \dot{v}\big|_{t=0} = (1 - p_N) u_1 \,. \end{cases} \tag{9.9}$$

It is also clear that the conditions

$$\left| M\left(\|A^{1/2} u(t)\|^2\right) \right| \le b_0(R) \quad \text{and} \quad \left| \frac{\mathrm{d}}{\mathrm{d}t} M\left(\|A^{1/2} u(t)\|^2\right) \right| \le b_1(R)$$

hold in the ball of dissipativity. This fact enables us to use Theorem 2.2 with $b(t) = M(\|A^{1/2} u(t)\|^2)$. In particular, equation (2.23) guarantees the existence of a number $N_1 \ge N_0$ which depends on γ, $b_0(R)$, and $b_1(R)$ and such that

$$\|y(t)\|_{\mathcal{H}} = \|(1 - P_N) y(t)\|_{\mathcal{H}} \le \sqrt{3} \|(1 - P_N) y(0)\| \, e^{-\frac{\gamma}{4} t} \,, \qquad t > 0 \,,$$

for all $N \ge N_1$, where P_N is the orthoprojector onto $\mathcal{H}_N$ and $y(t) = (v(t), \dot{v}(t))$. This implies estimate (9.8). **Theorem 9.2 is proved**.

— Exercise *9.2* Assume that the hypotheses of Theorem 9.2 hold. Show that for any semitrajectory $S_t y$ there exists an induced trajectory in $\mathcal{H}_N$, i.e. there exists $\bar{y} \in \mathcal{H}_N$ such that

$$\|S_t y - S_t \bar{y}\|_{\mathcal{H}} \le C_R \, e^{-\frac{\gamma}{4}(t - t_0)}$$

for $t \ge t_0$ and for some $t_0 = t_0(\|y\|_{\mathcal{H}})$.

— Exercise *9.3* Write down an inertial form of problem (0.1) and (0.2) in the subspace $\mathcal{H}_N$, provided the hypotheses of Theorem 9.2 hold. Prove that the inertial form coincides with the Galerkin approximation of the order N of problem (0.1) and (0.2).

— Exercise *9.4* Show that if the hypotheses of Theorem 9.2 hold, then the global attractor of problem (0.1) and (0.2) coincides with the global attractor of its Galerkin approximation of a sufficiently large order.

Let us now turn to the question on the construction of approximate inertial manifolds for problem (0.1) and (0.2). In this case we can use the results of Section 3.8 and the theorems on the regularity proved in Sections 4 and 5.

— Exercise *9.5* Assume that $M(z) \in C^{m+1}(\mathbb{R}_+)$, $m \ge 1$ and

$$B(u) = p - M\Big(\|A^{1/2}u\|^2\Big)Au - Lu\,,$$

where $p \in H$ and $u \in \mathscr{D}(A) = \mathscr{F}_1$. Show that the mapping $B(\cdot)$ has the Frechét derivatives $B^{(k)}$ up to the order l inclusive. Moreover, the estimates

$$\Big\|\langle B^{(k)}[u];\ w_1,\ \ldots,\ w_k\rangle\Big\| \ \le\ C_R \prod_{j=1}^{k} \|Aw_j\| \tag{9.10}$$

and

$$\Big\|\langle B^{(k)}[u] - B^{(k)}[u^*];\ w_1,\ \ldots,\ w_k\rangle\Big\| \ \le$$

$$\le\ C_R\|A(u-u^*)\| \prod_{j=1}^{k} \|Aw_j\| \tag{9.11}$$

are valid, where $k = 0,\ 1,\ \ldots,\ m$, $\|Au\| \le R$, $\|Au^*\| \le R$, and $w_j \in \mathscr{D}(A)$. Here $\langle B^{(k)}[u];\ w_1,\ \ldots,\ w_k\rangle$ is the value of the Frechét derivative on the elements $w_1,\ \ldots,\ w_k$.

We consider equations (9.10) and (9.11) as well as Theorem 5.3 which guarantees nonemptiness of the classes $L_{m,\,R}$ corresponding to the problem considered when $R > 0$ is large enough. They enable us to apply the results of Section 3.8.

Let P be the orthoprojector onto the span of elements $\{e_1,\ \ldots,\ e_N\}$ in H and let $Q = 1 - P$. We define the sequences $\{h_n(p,\ \dot p)\}_{n=0}^{\infty}$ and $\{l_n(p,\ \dot p)\}_{n=0}^{\infty}$ of mappings from $PH \times PH$ into QH by the formulae

$$h_0(p,\ \dot p)\ =\ l_0(p,\ \dot p)\ =\ 0\,, \tag{9.12}$$

$$A^2 h_k(p,\ \dot p)\ =\ a_0 - M_{k-1}(p,\ \dot p)Ah_{k-1} - QL(p+h_{k-1}) - \gamma\, l_{\nu(k)} -$$

$$-\langle \delta_p l_{k-1};\ \dot p\rangle + \langle \delta_{\dot p} l_{k-1};\ \gamma\dot p + A^2 p - b_0 + M_{k-1}(p,\ \dot p)Ap + PL(p+h_{k-1})\rangle\,, \tag{9.13}$$

$$l_k(p,\ \dot p)\ =\ \langle \delta_p h_{k-1};\ \dot p\rangle\ -$$

$$-\ \langle \delta_{\dot p} h_{k-1};\ \gamma\dot p + A^2 p - b_0 + M_{k-1}(p,\ \dot p)Ap + PL(p+h_{k-1})\rangle\,. \tag{9.14}$$

Here $M_k(p,\ \dot p) = M\Big(\|A^{1/2}p\|^2 + \big\|A^{1/2}h_k(p,\ \dot p)\big\|^2\Big)$, δ_p and $\delta_{\dot p}$ are the Frechét derivatives with respect to the corresponding variables, $a_0 = Qg$, $b_0 = Pg$, where $g \equiv p(t)$ is a stationary transverse load in (0.1), $k = 1,\ 2,\ \ldots,\ m$, the numbers $\nu(k)$ are chosen to fulfil the inequality $k - 1\ \le\ \nu(k) \le\ k$.

— Exercise *9.6* Evaluate the functions $h_1(p,\ \dot p)$ and $l_1(p,\ \dot p)$.

Theorem 3.8.2 implies the following assertion.

Theorem 9.3

Assume that $p(t) \equiv g \in H$, $M(z) \in C^{m+1}(\mathbb{R}_+)$, $m \geq 2$, *and conditions* (3.2), (5.2), *and* (5.3) *are fulfilled. Then for all* $k = 1, \ldots, m$ *the collection of mappings* (h_n, l_n) *given by equalities* (9.12)–(9.14) *possesses the properties*

1) *there exist constants* $M_j = M_j(n, \rho)$ *and* $L_j(n, \rho)$, $j = 1, 2$ *such that*

$$\left\|A^2 h_n(p_0, \dot{p}_0)\right\| \leq M_1, \qquad \left\|A l_n(p_0, \dot{p}_0)\right\| \leq M_2,$$

$$\left\|A^2(h_n(p_1, \dot{p}_1) - h_n(p_2, \dot{p}_2))\right\| \leq L_1(\left\|A^2(p_1 - p_2)\right\| + \left\|A(\dot{p}_1 - \dot{p}_2)\right\|),$$

$$\left\|A(l_n(p_1, \dot{p}_1) - l_n(p_2, \dot{p}_2))\right\| \leq L_2(\left\|A^2(p_1 - p_2)\right\| + \left\|A(\dot{p}_1 - \dot{p}_2)\right\|),$$

for all p_j *and* $\dot{p}_j$ *from* PH *and such that*

$$\left\|A^2 p_j\right\|^2 + \left\|A\dot{p}_j\right\|^2 \leq \rho, \quad j = 0, 1, 2, \quad \rho > 0\ ;$$

2) *for any solution* $u(t)$ *to problem* (0.1) *and* (0.2) *which satisfies compatibility conditions* (4.3) *with* $l = m$ *the estimate*

$$\left\{\left\|A^2(u(t) - u_n(t))\right\|^2 + \left\|A(u(t) - \bar{u}_n(t))\right\|^2\right\}^{1/2} \leq C_n \lambda_{N+1}^{-n}$$

is valid for $n \leq m - 1$ *and for* t *large enough. Here*

$$u_n(t) = p(t) + h_n(p(t), \dot{p}(t)),$$

$$\bar{u}_n(t) = \dot{p}(t) + l_n(p(t), \dot{p}(t)),$$

λ_N *is the* N*-th eigenvalue of the operator* A *and the constant* C_n *depends on the radius of dissipativity.*

In particular, Theorem 9.3 means that the manifold

$$\mathcal{M}_n = \{(p + h_n(p, \dot{p});\ \dot{p} + l_n(p, \dot{p})):\quad p, \dot{p} \in PH\}$$

attracts sufficiently smooth trajectories of the dynamical system $(\mathcal{H}, S_t)$ generated by problem (0.1) and (0.2) into a small vicinity (of the order $C_n \lambda_{N+1}^{-n}$) of $\mathcal{M}_n$.

— Exercise *9.7* Assume that the hypotheses of Theorem 6.3 hold (this theorem guarantees the existence of the global attractor $\mathcal{A}$ consisting of smooth trajectories of problem (0.1) and (0.2)). Prove that

$$\sup\{\operatorname{dist}(y, \mathcal{M}_n):\ y \in \mathcal{A}\} \leq C_n \lambda_{N+1}^{-n}$$

for all $n \leq m - 1$ (the number m is defined by the condition $M(z) \in C^{m+1}(\mathbb{R}_+)$).

— Exercise *9.8* Prove the analogue of Theorem 3.9.1 on properties of the nonlinear Galerkin method for problem (0.1) and (0.2).

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