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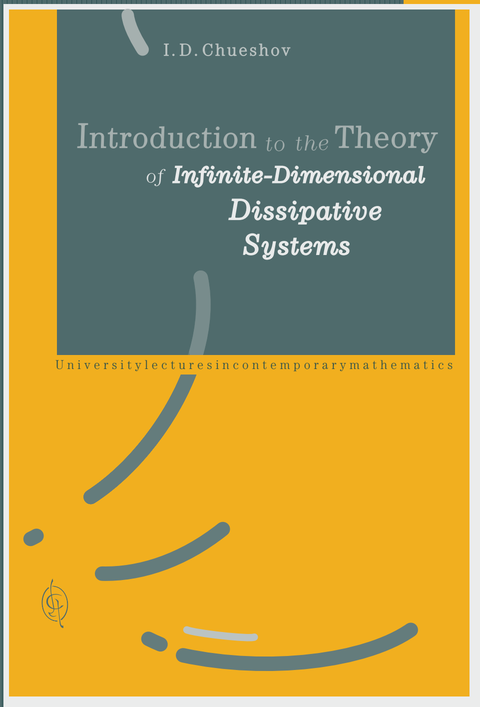

This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

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Chapter 3

Inertial Manifolds

Contents

If an infinite-dimensional dynamical system possesses a global attractor of finite dimension (see the definitions in Chapter 1), then there is, at least theoretically, a possibility to reduce the study of its asymptotic regimes to the investigation of properties of a finite-dimensional system. However, as the structure of attractor cannot be described in details for the most interesting cases, the constructive investigation of this finite-dimensional system cannot be carried out. In this respect some ideas related to the method of integral manifolds and to the reduction principle are very useful. They have led to appearance and intensive use of the concept of inertial manifold of an infinite-dimensional dynamical system (see [1]–[8] and the references therein). This manifold is a finite-dimensional invariant surface, it contains a global attractor and attracts trajectories exponentially fast. Moreover, there is a possibility to reduce the study of limit regimes of the original infinite-dimensional system to solving of a similar problem for a class of ordinary differential equations.

In this chapter we present one of the approaches to the construction of inertial manifolds (IM) for an evolutionary equation of the type:

$$\frac{du}{dt} + Au = B(u,\ t), \qquad u\big|_{t=0} = u_0 . \tag{0.1}$$

Here $u(t)$ is a function of the real variable t with the values in a separable Hilbert space H. We pay the main attention to the case when A is a positive linear operator with discrete spectrum and $B(u,\ t)$ is a nonlinear mapping of H subordinated to A in some sense. The approach used here for the construction of inertial manifolds is based on a variant of the Lyapunov-Perron method presented in the paper [2]. Other approaches can be found in [1], [3]–[7], [9], and [10]. However, it should be noted that all the methods for the construction of IM known at present time require a quite strong condition on the spectrum of the operator A: the difference $\lambda_{N+1} - \lambda_N$ of two neighbouring eigenvalues of the operator A should grow sufficiently fast as $N \to \infty$.

§ 1 Basic Equation and Concept of Inertial Manifold

In a separable Hilbert space H we consider a Cauchy problem of the type

$$\frac{du}{dt} + Au = B(u,\ t), \qquad t > s, \quad u\big|_{t=s} = u_0, \quad s \in \mathbb{R}, \tag{1.1}$$

where A is a positive operator with discrete spectrum (for the definition see Section 1 of Chapter 2) and $B(\cdot,\ \cdot)$ is a nonlinear continuous mapping from $D(A^{\theta}) \times \mathbb{R}$

into H, $0 \le \theta < 1$, possessing the properties

$$\|B(u,\ t)\| \le M(1 + \|A^\theta u\|) \tag{1.2}$$

and

$$\|B(u_1,\ t) - B(u_2,\ t)\| \le M\|A^\theta(u_1 - u_2)\| \tag{1.3}$$

for all u, u_1, and u_2 from the domain $\mathcal{F}_\theta = D(A^\theta)$ of the operator A^θ. Here M is a positive constant independent of t and $\|\cdot\|$ is the norm in the space H. Further it is assumed that $\{e_k\}$ is the orthonormal basis in H consisting of the eigenfunctions of the operator A:

$$A e_k = \lambda_k e_k, \qquad 0 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{k \to \infty} \lambda_k = \infty.$$

Theorem 2.3 of Chapter 2 implies that for any initial condition $u_0 \in \mathcal{F}_\theta$ problem (1.1) has a unique mild (in $\mathcal{F}_\theta$) solution $u(t)$ on every half-interval $[s,\ s+T)$, i.e. there exists a unique function $u(t) \in C(s,\ s+T;\ \mathcal{F}_\theta)$ which satisfies the integral equation

$$u(t) = e^{-(t-s)A} u_0 + \int_s^t e^{-(t-\tau)A} B(u(\tau),\ \tau)\, d\tau \tag{1.4}$$

for all $t \in [s,\ s+T)$. This solution possesses the property (see (2.6) in Chapter 2)

$$\|A^\beta(u(t+\sigma) - u(t))\| \le C\sigma^{\theta-\beta}, \qquad 0 \le \beta \le \theta$$

for $0 < \sigma < 1$ and $t > s$. Moreover, for any pair of mild solutions $u_1(t)$ and $u_2(t)$ to problem (1.1) the following inequalities hold (see (2.2.15)):

$$\|A^\theta u(t)\| \le a_1 e^{a_2(t-s)} \|A^\theta u(s)\|, \qquad t \ge s \tag{1.5}$$

and (cf. (2.2.18))

$$\|Q_N A^\theta u(t)\| \le \left\{ e^{-\lambda_{N+1}(t-s)} + M(1+k) a_1\, \lambda_{N+1}^{-1+\theta}\, e^{a_2(t-s)} \right\} \|A^\theta u(s)\|, \tag{1.6}$$

where $u(t) = u_1(t) - u_2(t)$, a_1 and a_2 are positive numbers depending on θ, λ_1, and M only. Hereinafter $Q_N = I - P_N$, where P_N is the orthoprojector onto the first N eigenvectors of the operator A. Moreover, we use the notation

$$k = \theta^\theta \int_0^\infty \xi^{-\theta} e^{-\xi}\, d\xi \quad \text{for } \theta > 0 \quad \text{and} \quad k = 0 \ \text{ for } \theta = 0. \tag{1.7}$$

Further we will also use the following so-called dichotomy estimates proved in Lemma 1.1 of Chapter 2:

$$\|A^\theta e^{-tA} P_N\| \le \lambda_N^\theta\, e^{\lambda_N |t|}, \qquad t \in \mathbb{R};$$

$$\|e^{-tA} Q_N\| \le e^{-\lambda_{N+1} t}, \qquad t \ge 0; \tag{1.8}$$

$$\left\| A^{\theta} e^{-tA} Q_N \right\| \le [(\theta/t)^{\theta} + \lambda_{N+1}^{\theta}]\, e^{-\lambda_{N+1} t}, \qquad t > 0, \quad \theta > 0.$$

The ***inertial manifold*** (IM) of problem (1.1) is a collection of surfaces $\{\mathbf{M}_t,\ t \in \mathbb{R}\}$ in H of the form

$$\mathbf{M}_t = \{p + \Phi(p,\ t):\ p \in P_N H,\ \ \Phi(p,\ t) \in (1 - P_N)\mathcal{F}_{\theta}\},$$

where $\Phi(p,\ t)$ is a mapping from $P_N H \times \mathbb{R}$ into $(1 - P_N)\mathcal{F}_{\theta}$ satisfying the Lipschitz condition

$$\left\| A^{\theta}(\Phi(p_1,\ t) - \Phi(p_2,\ t)) \right\| \le C \left\| A^{\theta}(p_1 - p_2) \right\| \tag{1.9}$$

with the constant C independent of p_j and t. We also require the fulfillment of the invariance condition (if $u_0 \in \mathbf{M}_s$, then the solution $u(t)$ to problem (1.1) possesses the property $u(t) \in \mathbf{M}_t$, $t \ge s$) and the condition of the uniform exponential attraction of bounded sets: there exists $\gamma > 0$ such that for any bounded set $B \subset H$ there exist numbers C_B and $t_B > s$ such that

$$\sup \left\{ \mathrm{dist}_{\mathcal{F}_{\theta}}(u(t,\ u_0),\ \mathbf{M}_t):\ u_0 \in B \right\} \le C_B\, e^{-\gamma(t - t_B)}$$

for all $t \ge t_B$. Here $u(t,\ u_0)$ is a mild solution to problem (1.1).

From the point of view of applications the existence of an inertial manifold (IM) means that a regular separation of fast (in the subspace $(I - P_N)H$) and slow (in the subspace $P_N H$) motions is possible. Moreover, the subspace of slow motions turns out to be finite-dimensional. It should be noted in advance that such separation is not unique. However, if the global attractor exists, then every IM contains it.

When constructing IM we usually use the methods developed in the theory of integral manifolds for central and central-unstable cases (see [11], [12]).

If the inertial manifold exists, then it continuously depends on t, i.e.

$$\lim_{t \to s} \left\| A^{\theta}(\Phi(p,\ s) - \Phi(p,\ t)) \right\| = 0$$

for any $p \in P_N H$ and $s \in \mathbb{R}$. Indeed, let $u(t)$ be the solution to problem (1.1) with $u_0 = p + \Phi(p,\ s)$, $p \in P_N H$. Then $u(t) \in \mathbf{M}_t$ for $t \ge s$ and hence

$$u(t) = P_N u(t) + \Phi(P_N u(t),\ t).$$

Therefore,

$$\Phi(p,\ t) - \Phi(p,\ s) = [\Phi(p,\ t) - \Phi(P_N u(t),\ t)] +$$
$$+ [u(t) - u_0] + [p - P_N u(t)].$$

Consequently, Lipschitz condition (1.9) leads to the estimate

$$\left\| A^{\theta}(\Phi(p,\ s) - \Phi(p,\ t)) \right\| \le C \left\| A^{\theta}(u(t) - u_0) \right\|.$$

Since $u(t) \in C(s,\ +\infty,\ D(A^{\theta}))$, this estimate gives us the required continuity property of $\Phi(p,\ t)$.

— Exercise *1.1* Prove that the estimate

$$\|A^{\beta}(\Phi(p,\ t+\sigma)-\Phi(p,\ t))\| \le C_{\beta}(p,\ N)\sigma^{\theta-\beta}$$

holds for $\Phi(p,\ t)$ when $0 \le \sigma \le 1$, $0 \le \beta \le \theta$, $t \in \mathbb{R}$.

The notion of the inertial manifold is closely related to the notion of the ***inertial form***. If we rewrite the solution $u(t)$ in the form $u(t) = p(t) + q(t)$, where $p(t) = P_N u(t)$, $q(t) = Q_N u(t)$, and $Q_N = I - P_N$, then equation (1.1) can be rewritten as a system of two equations

$$\begin{cases} \dfrac{d}{dt}p(t) + Ap(t) = P_N B(p(t)+q(t)) , \\ \dfrac{d}{dt}q(t) + Aq(t) = Q_N B(p(t)+q(t)) , \\ p\big|_{t=s} = p_0 \equiv P_N u_0 , \quad q\big|_{t=s} = q_0 \equiv Q_N u_0 . \end{cases}$$

By virtue of the invariance property of IM the condition $(p_0,\ q_0) \in \mathbf{M}_s$ implies that $(p(t),\ q(t)) \in \mathbf{M}_t$, i.e. the equality $q_0 = \Phi(p_0,\ s)$ implies that $q(t) = \Phi(p(t),\ t)$. Therefore, if we know the function $\Phi(p,\ t)$ that gives IM, then the solution $u(t)$ lying in $\mathbf{M}_t$ can be found in two stages: at first we solve the problem

$$\frac{d}{dt}p(t) + Ap(t) = P_N B(p(t) + \Phi(p(t),\ t)) , \qquad p\big|_{t=s} = p_0 , \tag{1.10}$$

and then we take $u(t) = p(t) + \Phi(p(t),\ t)$. Thus, the qualitative behaviour of solutions $u(t)$ lying in IM is completely determined by the properties of differential equation (1.10) in the finite-dimensional space $P_N H$. Equation (1.10) is said to be the inertial form (IF) of problem (1.1). In the autonomous case ($B(u,\ t) \equiv B(u)$) one can use the attraction property for IM and the reduction principle (see Theorem 7.4 of Chapter 1) in order to state that the finite-dimensional IF completely determines the asymptotic behaviour of the dynamical system generated by problem (1.1).

— Exercise *1.2* Let $\Phi(p,\ t)$ give the inertial manifold for problem (1.1). Show that IF (1.10) is uniquely solvable on the whole real axis, i.e. there exists a unique function $p(t) \in C(-\infty,\ \infty;\ P_N H)$ such that equation (1.10) holds.

— Exercise *1.3* Let $p(t)$ be a solution to IF (1.10) defined for all $t \in \mathbb{R}$. Prove that $u(t) = p(t) + \Phi(p(t),\ t)$ is a mild solution to problem (1.1) defined on the whole time axis and such that $u\big|_{t=s} = p_0 + \Phi(p_0,\ t)$.

— Exercise *1.4* Use the results of Exercises 1.2 and 1.3 to show that if IM $\{\mathbf{M}_t\}$ exists, then it is strictly invariant, i.e. for any $u \in \mathbf{M}_t$ and $s < t$ there exists $u_0 \in \mathbf{M}_t$ such that $u = u(t)$ is a solution to problem (1.1).

In the sections to follow the construction of IM is based on a version of the Lyapunov-Perron method presented in the paper by Chow-Lu [2]. This method is based on the following simple fact.

Lemma 1.1.

Let $f(t)$ be a continuous function on $\mathbb{R}$ with the values in H such that

$$\|Q_N f(t)\| \le C, \qquad t \in \mathbb{R}.$$

Then for the mild solution $u(t)$ (on the whole axis) to equation

$$\frac{d}{dt} u + A u = f(t) \tag{1.11}$$

to be bounded in the subspace $Q_N \mathcal{F}_\theta$ it is necessary and sufficient that

$$u(t) = e^{-(t-s)A} p + \int_s^t e^{-(t-\tau)A} P_N f(\tau)\, d\tau + \int_{-\infty}^t e^{-(t-\tau)A} Q_N f(\tau)\, d\tau \tag{1.12}$$

for $t \in \mathbb{R}$, where p is an element from $P_N H$ and s is an arbitrary real number.

We note that the solution to problem (1.11) on the whole axis is a function $u(t) \in$ $\in C(\mathbb{R}, H)$ satisfying the equation

$$u(t) = e^{-(t-s)A} u(s) + \int_s^t e^{-(t-\tau)A} f(\tau)\, d\tau$$

for any $s \in \mathbb{R}$.

Proof.

It is easy to prove (do it yourself) that equation (1.12) gives a mild solution to (1.11) with the required property of boundedness. Vice versa, let $u(t)$ be a solution to equation (1.11) such that $\|Q_N u(t)\|_\theta$ is bounded. Then the function $q(t) = Q_N u(t)$ is a bounded solution to equation

$$\frac{d}{dt} q(t) + A q(t) = Q_N f(t)\,.$$

Consequently, Lemma 2.1.2 implies that

$$q(t) = \int_{-\infty}^t e^{-(t-\tau)A} Q_N f(\tau)\, d\tau\,.$$

Therefore, in order to prove (1.12) it is sufficient to use the constant variation formula for a solution to the finite-dimensional equation

$$\frac{dp}{dt} + Ap = P_N f(t)\,, \qquad p(t) = P_N u(t)\,.$$

Thus, Lemma 1.1 is proved.

Lemma 1.1 enables us to obtain an equation to determine the function $\Phi(p, t)$. Indeed, let us assume that $B(u, t)$ is bounded and there exists $\mathbf{M}_t$ with the function $\Phi(p, t)$ possessing the property $\|A^\theta \Phi(p, t)\| \le C$ for all $p \in P_N H$ and $t \in \mathbb{R}$. Then the solution to problem (1.1) lying in $\mathbf{M}_t$ has the form

$$u(t) = p(t) + \Phi(p(t), t).$$

It is bounded in the subspace $Q_N H$ and therefore it satisfies the equation of the form

$$u(t) = e^{-(t-s)A} p + \\ + \int_s^t e^{-(t-\tau)A} P_N B(u(\tau), \tau)\, d\tau + \int_{-\infty}^t e^{-(t-\tau)A} Q_N B(u(\tau), \tau)\, d\tau\,, \quad (t \in \mathbb{R}). \quad (1.13)$$

Moreover,

$$\Phi(p, s) = Q_N u(s) = \int_{-\infty}^s e^{-(s-\tau)A} Q_N B(u(\tau), \tau)\, d\tau\,. \quad (1.14)$$

Actually it is this fact that forms the core of the Lyapunov-Perron method. It is proved below that under some conditions (i) integral equation (1.13) is uniquely solvable for any $p \in P_N H$ and (ii) the function $\Phi(p, s)$ defined by equality (1.14) gives IM.

In the construction of IM with the help of the Lyapunov-Perron method an important role is also played by the results given in the following exercises.

Exercise *1.5* Assume that $\sup\{e^{-\gamma(s-t)}\|f(t)\|:\ t < s\} < \infty$, where γ is any number from the interval $(\lambda_N, \lambda_{N+1})$ and $s \in \mathbb{R}$. Let $u(t)$ be a mild solution (on the whole axis) to equation (1.11). Show that $u(t)$ possesses the property

$$\sup_{t<s}\{e^{-\gamma(s-t)}\|A^\theta u(t)\|\} < \infty$$

if and only if equation (1.12) holds for $t < s$.
Hint: consider the new unknown function

$$w(t) = e^{\gamma(t-s)} u(t)$$

instead of $u(t)$.

Exercise *1.6* Assume that $f(t)$ is a continuous function on the semiaxis $[s, +\infty)$ with the values in H such that for some γ from the interval $(\lambda_N, \lambda_{N+1})$ the equation

$$\sup\{e^{-\gamma(s-t)}\|f(t)\|:\ t \in [s, +\infty)\} < \infty$$

holds. Prove that for a mild solution $u(t)$ to equation (1.11) on the semiaxis $[s, +\infty)$ to possess the property

$$\sup\{e^{-\gamma(s-t)}\|A^{\theta}u(t)\|:\ t\in[s,\ +\infty)\}<\infty$$

it is necessary and sufficient that

$$u(t) = e^{-(t-s)A}q + \int_s^t e^{-(t-\tau)A}Q_N f(\tau)\,\mathrm{d}\tau - \\ - \int_t^{+\infty} e^{-(t-\tau)A}P_N f(\tau)\,\mathrm{d}\tau \quad , \tag{1.15}$$

where $t \geq s$ and q is an element of $Q_N D(A^{\theta})$. *Hint*: see the hint to Exercise 1.5.

§ 2 *Integral Equation for Determination of Inertial Manifold*

In this section we study the solvability and the properties of solutions to a class of integral equations which contains equation (1.13) as a limit case. Broader treatment of the equation of the type (1.13) is useful in connection with some problems of the approximation theory for IM.

For $s \in \mathbb{R}$ and $0 < L \leq \infty$ we define the space $C_s \equiv C_{\gamma,\,\theta}(s-L,\ s)$ as the set of continuous functions $v(t)$ on the segment $[s-L,\ s]$ with the values in $D(A^{\theta})$ and such that

$$|v|_s \equiv \sup_{t\in[s-L,\ s]}\{e^{-\gamma(s-t)}\|A^{\theta}u(t)\|\} < \infty .$$

Here γ is a positive number. In this space we consider the integral equation

$$v(t) = \mathbf{B}_p^{s,\ L}[v](t), \qquad s-L \leq t \leq s, \tag{2.1}$$

where

$$\mathbf{B}_p^{s,\,L}[v](t) = e^{-(t-s)A}p - \int_t^s e^{-(t-\tau)A}PB(v(\tau),\ \tau)\,\mathrm{d}\tau\ + \\ + \int_{s-L}^t e^{-(t-\tau)A}QB(v(\tau),\ \tau)\,\mathrm{d}\tau\ .$$

Hereinafter the index N of the projectors P_N and Q_N is omitted, i.e. P is the orthoprojector onto $\mathrm{Lin}\{e_1,\ \ldots,\ e_N\}$ and $Q = 1-P$. It should be noted that the most significant case for the construction of IM is when $L = \infty$.

Lemma 2.1.

Let at least one of two conditions be fulfilled:

$$0 < L < \infty \quad \textit{and} \quad M\Big(\frac{\theta^\theta}{1-\theta} L^{1-\theta} + L\lambda_{N+1}^\theta\Big) \le q < 1 \tag{2.2}$$

or $0 < L \le \infty$ *and*

$$\lambda_{N+1} - \lambda_N \ge \frac{2M}{q}((1+k)\lambda_{N+1}^\theta + \lambda_N^\theta), \quad 0 < q < 1, \tag{2.3}$$

where k *is defined by equation* (1.7). *Then for any fixed* $s \in \mathbb{R}$ *there exists a unique function* $v_s(t;\, p) \in C_s$ *satisfying equation* (2.1) *for all* $t \in [s-L,\, s]$, *where* γ *is an arbitrary number from the segment* $[\lambda_N, \lambda_{N+1}]$ *in the case of* (2.2) *and* $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$ *in the case of* (2.3). *Moreover,*

$$\big|v(\,\cdot\,;\; p_1) - v(\,\cdot\,;\; p_1)\big|_s \le (1-q)^{-1}\big\|A^\theta(p_1 - p_2)\big\| \tag{2.4}$$

and

$$|v_s|_s \le (1-q)^{-1}\{D_1 + \|A^\theta p\|\}, \tag{2.5}$$

where

$$D_1 = M(1+k)\lambda_{N+1}^{-1+\theta} + M\lambda_N^{-1+\theta}. \tag{2.6}$$

Proof.

Let us apply the fixed point method to equation (2.1). Using (1.8) it is easy to check (similar estimates are given in Chapter 2) that

$$\Big\|A^\theta(\mathrm{B}_{p_1}^{s,\,L}(v_1)(t) - \mathrm{B}_{p_2}^{s,\,L}(v_2)(t))\Big\| \le$$

$$\le e^{\lambda_N(s-t)}\big\|A^\theta(p_1 - p_2)\big\| + \int_t^s \lambda_N^\theta e^{\lambda_N(\tau - t)} M\|v_1(\tau) - v_2(\tau)\|_\theta \,\mathrm{d}\tau +$$

$$+ \int_{s-L}^t \Big[\Big(\frac{\theta}{t-\tau}\Big)^\theta + \lambda_{N+1}^\theta\Big] e^{-\lambda_{N+1}(t-\tau)} M\|v_1(\tau) - v_2(\tau)\|_\theta \,\mathrm{d}\tau \le$$

$$\le e^{\lambda_N(s-t)}\big\|A^\theta(p_1 - p_2)\big\| + (q_1(s,\,t) + q_2(s,\,t))\, e^{\gamma(s-t)}|v_1 - v_2|_s\,,$$

where

$$q_1(s,\,t) = M\int_{s-L}^t \Big[\Big(\frac{\theta}{t-\tau}\Big)^\theta + \lambda_{N+1}^\theta\Big] e^{-(\lambda_{N+1}-\gamma)(t-\tau)}\,\mathrm{d}\tau \tag{2.7}$$

and

$$q_2(s,\ t) = M\int_t^s \lambda_N^\theta e^{(\lambda_N-\gamma)(\tau-t)}\mathrm{d}\tau\,. \tag{2.8}$$

Therefore, if the estimate

$$q_1(s,\ t)+q_2(s,\ t)\ \le\ q\ , \qquad s-L\le t\le s \tag{2.9}$$

holds, then

$$\left|\mathbf{B}_{p_1}^{s,\,L}[v_1]-\mathbf{B}_{p_2}^{s,\,L}[v_2]\right|_s\ \le\ \left\|A^\theta(p_1-p_2)\right\|+q\left|v_1-v_2\right|_s. \tag{2.10}$$

Let us estimate the values $q_1(s,\ t)$ and $q_2(s,\ t)$. Assume that (2.2) is fulfilled. Then it is evident that

$$q_1(s,\ t)\ \le\ M\theta^\theta\int_{s-L}^{t}(t-\tau)^{-\theta}\mathrm{d}\tau+M\lambda_{N+1}^\theta(t-s+L)\ =$$

$$=M\frac{\theta^\theta}{1-\theta}(t-s+L)^{1-\theta}+M\lambda_{N+1}^\theta(t-s+L)$$

and

$$q_2(s,\ t)\ \le\ M\lambda_N^\theta(s-t)\ \le\ M\lambda_{N+1}^\theta(s-t)$$

for $\lambda_N\le\gamma\le\lambda_{N+1}$. Therefore,

$$q_1(s,\ t)+q_2(s,\ t)\le M\Big(\frac{\theta^\theta}{1-\theta}(t-s+L)^{1-\theta}+\lambda_{N+1}^\theta L\Big).$$

Consequently, equation (2.2) implies (2.9). Now let the spectral condition (2.3) be fulfilled. Then

$$q_1(s,\ t)\le\int_{-\infty}^{t}\frac{M\theta^\theta}{(t-\tau)^\theta}e^{-(\lambda_{N+1}-\gamma)(t-\tau)}\mathrm{d}\tau+\frac{M\lambda_{N+1}^\theta}{\lambda_{N+1}-\gamma}$$

for all $\gamma<\lambda_{N+1}$. We change the variable in integration $\xi=(\lambda_{N+1}-\gamma)(t-\tau)$ and find that

$$q_1(s,\ t)\ \le\ \frac{Mk}{(\lambda_{N+1}-\gamma)^{1-\theta}}\ +\ \frac{M\lambda_{N+1}^\theta}{\lambda_{N+1}-\gamma}\ ,$$

where the constant k is defined by (1.7). It is also evident that

$$q_2(s,\ t)\ \le\ \frac{M\lambda_N^\theta}{\gamma-\lambda_N}$$

provided that $\gamma>\lambda_N$. Equation (2.3) implies that $\gamma=\lambda_N+(2M/q)\lambda_N^\theta$ lies in the interval $(\lambda_N,\ \lambda_{N+1})$. If we choose the parameter γ in such way, then we get

$$q_1(s,\ t)+q_2(s,\ t)\ \le\ \frac{M(1+k)\lambda_{N+1}^{\theta}}{\lambda_{N+1}-\lambda_N-\frac{2M}{q}\lambda_N^{\theta}}+\frac{q}{2}.$$

Hence, equation (2.3) implies (2.9). Therefore, estimate (2.10) is valid, provided that the hypotheses of the lemma hold. Moreover, similar reasoning enables us to show that

$$\left|\mathrm{B}_p^{s,\,L}[v]\right|_s\ \le\ D_1+\|A^{\theta}p\|+q\,|v|_s\,, \tag{2.11}$$

where D_1 is defined by formula (2.6). In particular, estimates (2.10) and (2.11) mean that when s, L, and p are fixed, the operator $\mathrm{B}_p^{s,\,L}$ maps C_s into itself and is contractive. Therefore, there exists a unique fixed point $v_s(t,\ p)$. Evidently it possesses properties (2.4) and (2.5). Lemma 2.1 is proved.

Lemma 2.1 enables us to define a collection of manifolds $\{\mathrm{M}_s^L\}$ by the formula

$$\mathrm{M}_s^L=\{p+\Phi^L(p,\ s):\ p\in PH\},$$

where

$$\Phi^L(p,\ s)=\int\limits_{s-L}^{s}e^{-(s-\tau)A}QB(v(\tau),\ \tau)\,\mathrm{d}\tau\ \equiv\ Qv(s;\ p). \tag{2.12}$$

Here $v(t)=v(t;\ p)$ is the solution to integral equation (2.1). Some properties of the manifolds $\{\mathrm{M}_s^L\}$ and the function $\Phi^L(p,\ s)$ are given in the following assertion.

Theorem 2.1.

Assume that at least one of two conditions (2.2) ***and*** (2.3) ***is satisfied. Then the mapping*** $\Phi^L(\cdot,\ s)$ ***from*** PH ***into*** QH ***possesses the properties***

a)
$$\|A^{\theta}\Phi^L(p,\ s)\|\ \le\ D_2+q(1-q)^{-1}\{D_1+\|A^{\theta}p\|\} \tag{2.13}$$

for any $p\in PH$, ***hereinafter*** D_1 ***is defined by formula*** (2.6) ***and***

$$D_2=M(1+k)\,\lambda_{N+1}^{-1+\theta}\ ; \tag{2.14}$$

b) ***the manifold*** M_s^L ***is a Lipschitzian surface and***

$$\left\|A^{\theta}\Phi^L(p_1,\ s)-\Phi^L(p_2,\ s)\right\|\ \le\ \frac{q}{1-q}\left\|A^{\theta}(p_1-p_2)\right\| \tag{2.15}$$

for all $p_1,\ p_2\in PH$ ***and*** $s\in\mathbb{R}$***;***

c) ***if*** $u(t)\equiv u(t,\ s;\ p+\Phi_s^L(p))$ ***is the solution to problem*** (1.1) ***with the initial data*** $u_0=p+\Phi^L(p,\ s)$, $p\in PH$, ***then*** $Qu(t)=\Phi^L(Pu(t),\ t)$ ***for*** $L=\infty$. ***In case of*** $L<\infty$ ***the inequality***

$$\|A^{\theta}(Qu(t)-\Phi^L(Pu(t),\ t))\|\ \le$$
$$\le\ D_2(1-q)^{-1}e^{-\gamma L}+q(1-q)^{-2}e^{-\gamma(t-s)}\{D_1+\|A^{\theta}p\|\} \tag{2.16}$$

holds for all $s \le t \le s+L$, ***where*** γ ***is an arbitrary number from the segment*** $[\lambda_N, \lambda_{N+1}]$ ***if*** (2.2) ***is fulfilled and*** $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$ ***when*** (2.3) ***is fulfilled;***

d) ***if*** $B(u, t) \equiv B(u)$ ***does not depend on*** t, ***then*** $\Phi^L(p, s) \equiv \Phi^L(p)$, ***i.e.*** $\Phi^L(p, t)$ ***is independent of*** t.

Proof.

Equations (2.12) and (1.8) imply that

$$\|A^\theta \Phi^L(p, s)\| \le M \int_{s-L}^{s} \left[\left(\frac{\theta}{s-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(s-\tau)} (1 + \|A^\theta v(\tau)\|)\,d\tau \le$$

$$\le M \int_{s-L}^{s} \left[\left(\frac{\theta}{s-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(s-\tau)}\,d\tau + q_1(s, s)|v|_s .$$

By virtue of (2.9) we have that $q_1(s, s) < q$. Therefore, when we change the variable in integration $\xi = \lambda_{N+1}(s-\tau)$ with the help of equation (2.5) we obtain (2.13). Similarly, using (2.4) and (1.8) one can prove property (2.15).

Let us prove assertion (c). We fix $t_0 \in [s, s+L]$ and assume that $w(t)$ is a function on the segment $[s, s+L]$ such that $w(t) = u(t)$ for $t \in [s, t_0]$ and $w(t) = v_s(t)$ for $t \in [s-L, s]$. Here $v_s(t)$ is the solution to integral equation (2.1). Using equations (1.4) and (2.1) we obtain that

$$w(t) = e^{-(t-s)A}(p + \Phi^L(p, s)) + \int_s^t e^{-(t-\tau)A} B(w(\tau), \tau)\,d\tau =$$

$$= e^{-(t-s)A} p + \int_s^t e^{-(t-\tau)A} PB(w(\tau), \tau)\,d\tau + \int_{s-L}^t e^{-(t-\tau)A} QB(w(\tau), \tau)\,d\tau \qquad (2.17)$$

for $s \le t \le t_0$. Evidently, equation (2.17) also remains true for $t \in [s-L, s]$. Equation (1.4) gives us that

$$p = e^{-(s-t_0)A} p(t_0) + \int_{t_0}^{s} e^{-(s-\tau)A} PB(w(\tau), \tau)\,d\tau .$$

Therefore, the substitution in (2.17) gives us that

$$w(t) = \mathbf{B}_{p(t_0)}^{t_0, L}[w](t) + b_L(t_0, s; t) \qquad (2.18)$$

for all $t \in [t_0-L, t_0]$, where $p(t) = Pu(t)$ and

$$b_L(t_0, s; t) = \int_{s-L}^{t_0-L} e^{-(t-\tau)A} QB(v_s(\tau), \tau)\,d\tau . \qquad (2.19)$$

In particular, if $L = \infty$ equation (2.18) turns into equation (2.1) with $s = t_0$ and $p = p(t_0)$. Therefore, equation (2.12) implies the invariance property $Qu(t_0) = \Phi^\infty(Pu(t_0), t_0)$. Let us estimate the value (2.19). If we reason in the same way as in the proof of Lemma 2.1, then we obtain that

$$\|A^\theta b_L(t_0, s;\ t)\| \le e^{-(t-t_0+L)\lambda_{N+1}}\left\{q_1^*(s, t_0-L) + q_1(s, t_0-L)\, e^{\gamma(s-t_0+L)}|v_s|_s\right\},$$

where $q_1(s, t)$ is defined by formula (2.7) and

$$q_1^*(s, t) = M\int_{s-L}^{t}\left(\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right)e^{-\lambda_{N+1}(t-\tau)}d\tau\,. \tag{2.20}$$

Therefore, simple calculations give us that

$$\|A^\theta b_L(t_0, s;\ t)\| \le e^{-(t-t_0+L)\lambda_{N+1}}\left\{D_2 + e^{\gamma(s-t_0+L)}q|v_s|_s\right\}, \tag{2.21}$$

where D_2 is defined by formula (2.14). Let $v_{t_0}(t)$ be the solution to integral equation (2.1) for $s = t_0$ and $p = Pu(t_0)$. Then using (2.12), (2.18), and (2.1) we find that

$$Qu(t_0) - \Phi^L(Pu(t_0), t_0) = Q(w(t_0) - v_{t_0}(t_0))\,. \tag{2.22}$$

However, for all $t \in [t_0-L, t_0]$ we have that

$$w(t) - v_{t_0}(t) = \mathrm{B}_{p(t_0)}^{t_0, L}[w](t) - \mathrm{B}_{p(t_0)}^{t_0, L}[v_{t_0}](t) + b_L(t_0, s;\ t)\,.$$

Therefore, the contractibility property of the operator $\mathrm{B}_p^{t_0, L}$ gives us that

$$(1-q)|w - v_{t_0}|_{t_0} \le \sup_{t\in[t_0-L,\ t_0]}\left\{e^{-\gamma(t_0-t)}\|A^\theta b_L(t_0, s;\ t)\|\right\}.$$

Hence, it follows from (2.21) and (2.22) that

$$\|A^\theta(Qu(t_0)) - \Phi^L(Pu(t_0), t_0)\| \le \|A^\theta(w(t_0) - v_{t_0}(t_0))\| \le$$

$$\le |w - v_{t_0}|_{t_0} \le (1-q)^{-1}\left\{e^{-\gamma L}D_2 + q\,e^{-\gamma(t_0-s)}|v_s|_s\right\}.$$

This and equation (2.5) imply (2.16). Therefore, assertion (c) is proved.

In order to prove assertion (d) it should be kept in mind that if $\mathcal{B}(u, t) \equiv \mathcal{B}(u)$, then the structure of the operator $\mathrm{B}_p^{s, L}$ enables us to state that

$$\mathrm{B}_p^{s, L}[v](t-h) = \mathrm{B}_p^{s+h, L}[v_h](t)$$

for $s+h-L \le t \le s+h$, where $v_h(t) = v(t-h)$. Therefore, if $v(t) \in C_{\gamma, \theta}(s-L, s)$ is a solution to integral equation (2.1), then the function

$$v_h(t) \equiv v(t-h) \in C_{\gamma, \theta}(s+h-L, s+h)$$

is its solution when $s+h$ is written instead of s. Consequently, equation (2.12) gives us that

$$\Phi^L(p,\ s+h) = Qv_h(s+h) = Qv(s) = \Phi^L(p,\ s)\ .$$

Thus, **Theorem 2.1 is proved**.

— Exercise *2.1* Show that if $\|B(u,\ t)\| \le M$, then inequalities (2.13) and (2.16) can be replaced by the relations

$$\|A^\theta \Phi^L(p,\ s)\| \le D_2\,, \tag{2.23}$$

$$\|A^\theta(Qu(t) - \Phi^L(Pu((t),\ t)))\| \le D_2(1-q)^{-1}e^{-\gamma L}\,, \tag{2.24}$$

where D_2 is defined by formula (2.14).

§ 3 *Existence and Properties of Inertial Manifolds*

In particular, assertion (c) of Theorem 2.1 shows that if the spectral gap condition

$$\lambda_{N+1} - \lambda_N \ge \frac{2M}{q}((1+k)\lambda_{N+1}^\theta + \lambda_N^\theta)\,, \qquad 0 < q < 1\,, \tag{3.1}$$

is fulfilled, then the collection of surfaces

$$\mathbf{M}_s = \{p + \Phi(p,\ s):\ p \in PH\}\,, \qquad s \in \mathbb{R}\,, \tag{3.2}$$

is invariant, i.e.

$$U(t,\ s)\mathbf{M}_s \subset \mathbf{M}_t\,, \qquad -\infty < s \le t < \infty\,. \tag{3.3}$$

Here $\Phi(p,\ s) = \Phi^\infty(p,\ s)$ is defined by formula (2.12) for $L = \infty$ and $U(t,\ s)$ is the evolutionary operator corresponding to problem (1.1). It is defined by the formula $U(t,\ s)u_0 = u(t)$, where $u(t)$ is a mild solution to problem (1.1).

In this section we show that collection (3.2) possesses the property of exponential uniform attraction. Hence, $\{\mathbf{M}_t\}$ is an inertial manifold for problem (1.1). Moreover, Theorem 3.1 below states that $\{\mathbf{M}_t\}$ is an ***exponentially asymptotically complete*** IM, i.e. for any solution $u(t) = U(t,\ s)u_0$ there exists a solution $\tilde{u}(t) = U(t,\ s)\tilde{u}_0$ lying in the manifold ($\tilde{u}(t) \in \mathbf{M}_t$, $t \ge s$) such that

$$\|A^\theta(u(t) - \tilde{u}(t))\| \le Ce^{-\eta(t-s)}\,, \qquad \eta > 0\,, \quad t > s\,.$$

In this case the solution $\tilde{u}(t)$ is said to be an ***induced trajectory*** for $u(t)$ on the manifold $\mathbf{M}_t$. In particular, the existence of induced trajectories means that the solution to original infinite-dimensional problem (1.1) can be naturally associated with the solution to the system of ordinary differential equations (1.10).

Theorem 3.1.

Assume that spectral gap condition (3.1) *is valid for some* $q < 2-\sqrt{2}$. *Then the manifold* $\{\mathbf{M}_s,\ s\in\mathbb{R}\}$ *given by formula* (3.2) *is inertial for problem* (1.1). *Moreover, for any solution* $u(t)=U(t,\ s)u_0$ *there exists an induced trajectory* $u^*(t)=U(t,\ s)u_0^*$ *such that* $u^*(t)\in\mathbf{M}_t$ *for* $t\ge s$ *and*

$$\|A^\theta(u(t)-u^*(t))\| \le \frac{2(1-q)}{(2-q)^2-2}\, e^{-\gamma(t-s)}\left\|A^\theta(Qu_0-\Phi(P_0u,\ s))\right\|, \tag{3.4}$$

where $\gamma=\lambda_N+\dfrac{2M}{q}\lambda_N^\theta$ *and* $t\ge s$.

Proof.

Obviously it is sufficient just to prove the existence of an induced trajectory $u^*(t)\in\mathbf{M}_t$ possessing property (3.4). Let $u(t)$ be a mild solution to problem (1.1), $u(t)=U(t,\ s)u_0$. We construct the induced trajectory $u^*(t)=U(t,\ s)u_0^*$ for $u(t)$ in the form $u^*(t)=u(t)+w(t)$, where $w(t)$ lies in the space $C_s^+\equiv C_{s,\ \gamma}(s,\ +\infty,\ D(A^\theta))$ of continuous functions on the semiaxis $[s,\ +\infty)$ such that

$$|w|_{s,\ +}\equiv\sup_{t\ge s}\{e^{\gamma(t-s)}\|A^\theta w(t)\|\}<\infty, \tag{3.5}$$

where $\gamma=\lambda_N+(2M/q)\lambda_N^\theta$. We introduce the notation

$$F(w,\ t)=B(u(t)+w,\ t)-B(u(t)) \tag{3.6}$$

and consider the integral equation (cf. (1.15))

$$w(t) = \mathbf{B}_s^+[w](t)\equiv e^{-(t-s)A}q(w)+\int_s^t e^{-(t-\tau)A}QF(w(\tau),\ \tau)\,d\tau - $$
$$-\int_t^{+\infty} e^{-(t-\tau)A}PF(w(\tau),\ \tau)\,d\tau,\qquad t\in[s,\ +\infty), \tag{3.7}$$

in the space C_s^+. Here the value $q(w)\in QD(A^\theta)$ is chosen from the condition

$$u^*(s)=u(s)+w(s)\in\mathbf{M}_s,$$

i.e. such that

$$Qu_0+Qw(s)=\Phi(Pu_0+Pw(s),\ s).$$

Therefore, by virtue of (3.7) we have

$$q(w) = -Qu_0+\Phi\left(Pu_0-\int_s^{+\infty} e^{-(s-\tau)A}PF(w(\tau),\ \tau)\,d\tau,\ s\right). \tag{3.8}$$

Thus, in order to prove inequality (3.4) it is sufficient to prove the solvability of integral equation (3.7) in the space C_s^+ and to obtain the estimate of the solution. The preparatory steps for doing this are hidden in the following exercises.

Exercise *3.1* Assume that $F(w, t)$ has the form (3.6). Show that for any

$$w(t),\ \overline{w}(t) \in C_s^+ = C_{s,\gamma}(s, +\infty; D(A^\theta))$$

and for $t \ge s$ the following inequalities hold:

$$\|F(w(t), t)\| \le e^{-\gamma(t-s)} M |w|_{s,+}, \tag{3.9}$$

$$\|F(w(t), t) - F(\overline{w}(t), t)\| \le e^{-\gamma(t-s)} M |w - \overline{w}|_{s,+}. \tag{3.10}$$

Exercise *3.2* Using (1.8) prove that the equations

$$\left(\int_t^{+\infty} \|A^\theta e^{-(t-\tau)A} P\| e^{-\gamma(\tau - s)} d\tau \right) \le \frac{\lambda_N^\theta}{\gamma - \lambda_N} \cdot e^{-\gamma(t-s)}, \tag{3.11}$$

$$\left(\int_s^t \|A^\theta e^{-(t-\tau)A} Q\| e^{-\gamma(\tau - s)} d\tau \right) \le$$

$$\le \frac{k(\lambda_{N+1} - \gamma)^\theta + \lambda_{N+1}^\theta}{\lambda_{N+1} - \gamma} \cdot e^{-\gamma(t-s)} \tag{3.12}$$

hold for $\lambda_N < \gamma < \lambda_{N+1}$ and $t \ge s$. Here k is defined by formula (1.7).

Lemma 3.1.

Assume that spectral gap condition (3.1) *holds with* $q < 2 - \sqrt{2}$. *Then* B_s^+ *is a continuous contractive mapping of the space* C_s^+ *into itself. The unique fixed point* w *of this mapping satisfies the estimate*

$$|w|_{s,+} \le \frac{2(1-q)}{(2-q)^2 - 2} \left\| A^\theta (Q u_0 - \Phi(P u_0, s)) \right\|. \tag{3.13}$$

Proof.

If we use (3.7), then we find that

$$\left\| A^\theta \mathrm{B}_s^+[w](t) \right\| \le e^{-(t-s)\lambda_{N+1}} \|A^\theta q(w)\| +$$

$$+ \int_s^t \|A^\theta e^{-(t-\tau)A} Q\| \|F(w(\tau), \tau)\| d\tau + \int_t^{+\infty} \|A^\theta e^{-(t-\tau)A} P\| \|F(w(\tau), \tau)\| d\tau$$

for $t > s$. Therefore, (3.9), (3.11), and (3.12) give us that

$$\left\| A^\theta \mathrm{B}_s^+[w](t) \right\| \le e^{-(t-s)\lambda_{N+1}} \|A^\theta q(w)\| +$$

$$+ \left\{ \frac{\lambda_N^\theta}{\gamma - \lambda_N} + \frac{(1+k)\lambda_{N+1}^\theta}{\lambda_{N+1} - \gamma} \right\} M e^{-\gamma(t-s)} |w|_{s,+} \quad .$$

Since $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$, spectral gap condition (3.1) implies that

$$\left\|A^\theta \mathrm{B}_s^+[w](t)\right\| \le e^{-(t-s)\lambda_{N+1}}\|A^\theta q(w)\| + q\, e^{-\gamma(t-s)}|w|_{s,\,+} \quad . \tag{3.14}$$

Similarly with the help of (3.10)–(3.12) we have that

$$\left\|A^\theta(\mathrm{B}_s^+[w](t) - \mathrm{B}_s^+[\overline{w}](t))\right\| \le$$

$$\le e^{-(t-s)\lambda_{N+1}}\|A^\theta(q(w) - q(\overline{w}))\| + q\, e^{-\gamma(t-s)}\,|w-\overline{w}|_{s,\,+} \tag{3.15}$$

for any $w,\ \overline{w} \in C_s^+$. From equations (3.8), (3.9), and (2.15) we obtain that

$$\left\|A^\theta(q(w) + Qu_0 - \Phi(Pu_0,\ s))\right\| \le \frac{qM}{1-q}\int_s^{+\infty}\|A^\theta e^{-(s-\tau)A}P\|\, e^{-\gamma(\tau-s)}\mathrm{d}\tau\cdot|w|_{s,\,+}\,.$$

Therefore, (3.11) implies that

$$\|A^\theta q(w)\| \le \left\|A^\theta(Qu_0 - \Phi(Pu_0,\ s))\right\| + \frac{q^2}{2(1-q)}\,|w|_{s,\,+}\,.$$

Similarly we have that

$$\|A^\theta(q(w) - q(\overline{w}))\| \le \frac{q^2}{2(1-q)}|w-\overline{w}|_{s,\,+}\,. \tag{3.17}$$

It follows from (3.14)–(3.17) that

$$\left|\mathrm{B}_s^+[w]\right|_{s,\,+} \le \left\|A^\theta(Qu_0 - \Phi(Pu_0,\ s))\right\| + \frac{q}{2}\cdot\frac{2-q}{1-q}|w|_{s,\,+}\,, \tag{3.18}$$

$$\left|\mathrm{B}_s^+[w] - \mathrm{B}_s^+[\overline{w}]\right|_{s,\,+} \le \frac{q}{2}\cdot\frac{2-q}{1-q}\,|w-\overline{w}|_{s,\,+} \quad .$$

Therefore, if $q < 2-\sqrt{2}$, then the operator B_s^+ is continuous and contractive in C_s^+. Estimate (3.13) of its fixed point follows from (3.18). Lemma 3.1 is proved.

In order to complete the proof of Theorem 3.1 we must prove that the function

$$u^*(t) = u(t) + w(t)$$

is a mild solution to problem (1.1) lying in $\{\mathrm{M}_t,\ t \ge s\}$ (here $w(t)$ is a solution to integral equation (3.7)). We can do that by using the result of Exercise 1.2, the invariance of the collection $\{\mathrm{M}_t\}$, and the fact that equality (3.8) is equivalent to the equation $u^*(s) \in \mathrm{M}_s$. **Theorem 3.1 is completely proved**.

Exercise *3.3* Show that if the hypotheses of Theorem 3.1 hold, then the induced trajectory $u^*(t)$ is uniquely defined in the following sense: if there exists a trajectory $u^{**}(t)$ such that $u^{**}(t) \in \mathrm{M}_t$ for $t \ge s$ and

$$\|A^\theta(u(t) - u^{**}(t))\| \le C e^{-\gamma(t-s)}$$

with $\gamma \ge \lambda_N + \frac{2M}{q}\lambda_N^\theta$, then $u^{**}(t) \equiv u^*(t)$ for $t \ge s$.

The construction presented in the proof of Theorem 3.1 shows that in order to build the induced trajectory for a solution $u(t)$ with the exponential order of decrease γ given, it is necessary to have the information on the behaviour of the solution $u(t)$ for ***all*** values $t \ge s$. In this connection the following simple fact on the exponential closeness of the solution $u(t)$ to its projection $Pu(t) + \Phi(Pu(t),\ t)$ onto the manifold appears to be useful sometimes.

— Exercise *3.4* Show that if the hypotheses of Theorem 3.1 hold, then the estimate

$$\|A^{\theta}(Qu(t) - \Phi(Pu(t),\ t))\| \le$$
$$\le \frac{2}{(2-q)^2 - 2} e^{-\gamma(t-s)} \|A^{\theta}(Pu_0 - \Phi(Pu_0,\ t))\|$$

is valid for any solution $u(t)$ to problem (1.1). Here $\gamma = \lambda_N + (2M/q)\lambda_N^{\theta}$ and $t \ge s$ (*Hint*: add the value $\Phi((Pu^*(t),\ t) - Qu^*(t)) = 0$ to the expression under the norm sign in the left-hand side. Here $u^*(t)$ is the induced trajectory for $u(t)$).

It is evident that the inertial manifold $\{\mathrm{M}_t\}$ consists of the solutions $u(t)$ to problem (1.1) which are defined for all real t (see Exercises 1.3 and 1.4). These solutions can be characterized as follows.

Theorem 3.2.

Assume that spectral gap condition (3.1) ***holds with*** $q < 2 - \sqrt{2}$ ***and*** $\{\mathrm{M}_t\}$ ***is the inertial manifold for problem*** (1.1) ***constructed in Theorem*** 3.1. ***Then for a solution*** $u(t)$ ***to problem*** (1.1) ***defined for all*** $t \in \mathbb{R}$ ***to lie in the inertial manifold*** $(u(t) \in \mathrm{M}_t)$, ***it is necessary and sufficient that***

$$|u|_s \equiv \sup\{e^{-\gamma(s-t)}\|A^{\theta}u(t)\| : -\infty < t \le s\} < \infty \tag{3.19}$$

for each $s \in \mathbb{R}$, ***where*** $\gamma = \lambda_N + \dfrac{2M}{q}\lambda_N^{\theta}$.

Proof.

If $u(t) \in \mathrm{M}_t$, then $u(t) = Pu(t) + \Phi(Pu(t),\ t)$. Therefore, equation (2.13) implies that

$$\|A^{\theta}u(t)\| \le D_2 + \frac{qD_1}{1-q} + \frac{1}{1-q}\|A^{\theta}Pu(t)\|. \tag{3.20}$$

The function $p(t) = Pu(t)$ satisfies the equation

$$p(t) = e^{-(t-s)A}p(s) + \int_s^t e^{-(t-\tau)A}PB(u(\tau),\ \tau)\,\mathrm{d}\tau$$

for all real t and s. Therefore, we have that

$$\|A^\theta p(t)\| \;\le\; e^{(s-t)\lambda_N}\|A^\theta p(s)\| + M\lambda_N^\theta \int_t^s e^{(\tau-t)\lambda_N}(1+\|A^\theta u(\tau)\|)\,d\tau$$

for $t \le s$. With the help of (3.20) we find that

$$\|A^\theta p(t)\| \;\le\; C(s,\ N,\ q)e^{(s-t)\lambda_N} + \frac{M\lambda_N^\theta}{1-q}\int_t^s e^{(\tau-t)\lambda_N}\|A^\theta p(\tau)\|\,d\tau$$

for $t \le s$, where

$$C(s,\ N,\ q) = \|A^\theta p(s)\| + M\Big(1 + D_2 + \frac{qD_1}{1-q}\Big)\frac{1}{\lambda_N^{1-\theta}}.$$

Hence, the inequality

$$\varphi(t) \;\le\; C(s,\ N,\ q) + \frac{M\lambda_N^\theta}{1-q}\int_t^s \varphi(\tau)\,d\tau$$

holds for the function $\varphi(t) = \|A^\theta p(t)\| e^{(t-s)\lambda_N}$ and $t \le s$. If we introduce the function $\psi(t) = \int_t^s \varphi(\tau)\,d\tau$, then the last inequlity can be rewritten in the form

$$\psi'(t) + \frac{M\lambda_N^\theta}{1-q}\psi(t) \;\ge\; -C(s,\ N,\ q)\,, \qquad t \le s\,,$$

or

$$\frac{d}{dt}\left[\psi(t)\exp\left\{\frac{M\lambda_N^\theta}{1-q}\,t\right\}\right] \;\ge\; -C(s,\ N,\ q)\exp\left\{\frac{M\lambda_N^\theta}{1-q}\,t\right\}, \qquad t \le s\,.$$

After the integration over the segment $[t,\ s]$ and a simple transformation it is easy to obtain the estimate

$$\|A^\theta p(t)\| \;\le\; C(s,\ N,\ q)\exp\left\{\Big(\lambda_N + \frac{M\lambda_N^\theta}{1-q}\Big)(s-t)\right\}. \tag{3.21}$$

Obviously for $q < 2-\sqrt{2}$ we have that

$$\lambda_N + \frac{M\lambda_N^\theta}{1-q} \;<\; \gamma = \lambda_N + \frac{2M}{q}\lambda_N^\theta\,.$$

Therefore, equations (3.21) and (3.20) imply (3.19).

Vice versa, we assume that equation (3.19) holds for the solution $u(t)$. Then

$$\|B(u(t))\| \;\le\; e^{\gamma(s-t)}M(1+|u|_s)\,, \qquad t \le s\,. \tag{3.22}$$

It is evident that $q(t) = e^{-\gamma(s-t)}Qu(t)$ is a bounded (on $(-\infty,\ s]$) solution to the equation

$$\frac{dq}{dt} + (A-\gamma)q = F(t)\,,$$

where $F(t) = \exp\{-\gamma(s-t)\} QB(u(t))$. By virtue of (3.22) the function $F(t)$ is bounded in QH. It is also clear that $A_\gamma = A - \gamma$ is a positive operator with discrete spectrum in QH. Therefore, Lemma 1.1 is applicable. It gives

$$Qu(t) = \int_{-\infty}^{t} e^{-(t-\tau)A} QB(u(\tau))\,d\tau .$$

Using the equation for $Pu(t)$ it is now easy to find that

$$u(t) = \mathrm{B}_p^{s,\,\infty}[u](t), \qquad t \le s ,$$

where $p = Pu(s)$ and $\mathrm{B}_p^{s,\,\infty}[u]$ is the integral operator similar to the one in (2.1). Hence, we have that $Qu(s) = \Phi(Pu(s),\, s)$ accoring to definition (2.12) of the function $\Phi(p,\, s) = \Phi^\infty(p,\, s)$. Thus, **Theorem 3.2 is proved**.

The following assertion shows that IM $\mathrm{M}_s \equiv \mathrm{M}_s^\infty$ can be approximated by the manifolds $\{\mathrm{M}_s^L\}$, $L < \infty$, with the exponential accuracy (see (2.12)).

Theorem 3.3.

Assume that spectral gap condition (3.1) ***is fulfilled with*** $q < 1$. ***We also assume that the function*** $\Phi^L(p,\, s)$ ***is defined by equality*** (2.12) ***for*** $0 < L \le \infty$. ***Then the estimate***

$$\left\| A^\theta(\Phi^{L_1}(p,\, s) - \Phi^{L_2}(p,\, s))\right\| \le$$

$$\le D_2(1-q)^{-1} e^{-\gamma_N L_{\min}} + \frac{1+q}{2(1-q)^2}\{D_1 + \|A^\theta p\|\} e^{-\delta_N L_{\min}} , \tag{3.23}$$

is valid with $L_{\min} = \min(L_1,\, L_2)$, $0 < L_1,\, L_2 \le \infty$***; the constants*** D_1 ***and*** D_2 ***are defined by equations*** (2.6) ***and*** (2.14)***;***

$$\gamma_N = \lambda_N + \frac{2M}{q}\lambda_N^\theta , \qquad \delta_N = \frac{2M(1-q)}{q(1+q)}\lambda_N^\theta .$$

Proof.

Let $0 < L_1 < L_2 < \infty$. Definition (2.12) implies that

$$\Phi^{L_1}(p,\, s) - \Phi^{L_2}(p,\, s) = Q(v_1(s) - v_2(s)), \tag{3.24}$$

where $v_j(t)$ is a solution to integral equation (2.1) with $L = L_j$, $j = 1,\ 2$. The operator $\mathrm{B}_p^{s,\,L_2}$ acting in $C_{\gamma,\,\theta}(s-L_2,\, s)$ (see (2.1)) can be represented in the form

$$B_p^{s,\,L_2}[v](t) = B_p^{s,\,L_1}[v](t) + b(v;\ t,\ s), \qquad t \in [s-L_1,\ s],$$

where

$$b(v;\ t,\ s) = \int_{s-L_2}^{s-L_1} e^{-(t-\tau)A}\, QB(v(\tau),\, \tau)\,d\tau$$

and $v(t)$ is an arbitrary element in $C_{\gamma,\,\theta}(s-L_2,\, s)$. Therefore, if $v_j(t)$ is a solution to problem (2.1) with $L = L_j$, then

$$v_1(t) - v_2(t) = \mathbf{B}_p^{s,\,L_1}[v_1] - \mathbf{B}_p^{s,\,L_1}[v_2] - b(v_2;\; t,\; s) \tag{3.25}$$

for all $s-L_1 \le t \le s$. Let us estimate the value $b(v_2;\; t,\; s)$. As before it is easy to verify that

$$\left\|A^{\theta} b(v_2;\; t,\; s)\right\| \;\le\; e^{-(t-s+L_1)\lambda_{N+1}}\{r_1(s-L_1) + r_2(s-L_1)|v_2|_{s,\,*}\}$$

for all $t \in [s-L_1,\; s]$, where

$$r_1(t) = M\int_{-\infty}^{t} \|A^{\theta} e^{-(t-\tau)A}Q\|\, d\tau\,,$$

$$r_2(t) = M\int_{-\infty}^{t} \|A^{\theta} e^{-(t-\tau)A}Q\|\, e^{\gamma_*(s-\tau)} d\tau\,,$$

and the norm $|v_2|_{s,\,*}$ is defined using the constants $q^* = \dfrac{1+q}{2}$ and $\gamma_* = \lambda_N + \dfrac{2M}{q_*}\lambda_N^{\theta}$ by the formula

$$|v_2|_{s,\,*} = \sup\left\{e^{-\gamma_*(s-t)}\left\|A^{\theta} v_2(t)\right\|:\;\; t \in [s-L_2,\; s]\right\}.$$

Evidently, spectral gap condition (3.1) implies the same equation with the parameter q^* instead of q. Therefore, simple calculations based on (1.8) give us that

$$r_1(t) \le D_2 \quad \text{and} \quad r_2 \le e^{-\gamma_*(s-t)}\frac{q^*}{2}\,,$$

where D_2 is defined by formula (2.14). Using Lemma 2.1 under condition (2.3) with q^* instead of q we obtain that

$$|v_2|_{s,\,*} \;\le\; (1-q^*)^{-1}\{D_1 + \|A^{\theta}p\|\}\,,$$

where D_1 is given by formula (2.6). Therefore, finaly we have that

$$\left\|A^{\theta} b(v_2;\; t,\; s)\right\| \;\le\; e^{-(t-s+L_1)\lambda_{N+1}}\left\{D_2 + \frac{q^*}{2(1-q^*)}\, e^{\gamma_* L_1}(D_1 + \|A^{\theta}p\|)\right\}$$

for all $t \in [s-L_1,\; s]$. Consequently,

$$\sup\left\{e^{\gamma(t-s)}\left\|A^{\theta} b(v_2;\; t,\; s)\right\|:\; t \in [s-L_1,\; s]\right\} \;\le$$

$$\le\; e^{-\gamma L_1}\left\{D_2 + \frac{q^*}{2(1-q^*)}\, e^{\gamma_* L_1}(D_1 + \|A^{\theta}p\|)\right\}.$$

Therefore, since $\mathbf{B}_p^{s,\,L_1}$ is a contractive operator in $C_{\gamma,\,\theta}(s-L_1,\ s)$, equation (3.25) gives us that

$$(1-q)\left|v_1-v_2\right|_{C_{\gamma,\,\theta}(s-L_1,\ s)} \le$$

$$\le\ e^{-\gamma L_1} D_2+\frac{1}{2}\cdot\frac{1+q}{1-q}\ e^{-(\gamma-\gamma_*)L_1}\left(D_1+\|A^\theta p\|\right).$$

Here we also use the equality $q^* = \frac{1}{2}(1+q)$. Hence, estimate (3.23) follows from (3.24). **Theorem 3.3 is proved**.

— Exercise *3.5* Show that in the case when $\|B(u,\ t)\| \le M$ equation (3.23) can be replaced by the inequality

$$\left\|A^\theta(\Phi^{L_1}(p,\ s)-\Phi^{L_2}(p,\ s))\right\| \ \le\ D_2(1-q)^{-1} e^{-\gamma_N L_{\min}}.$$

— Exercise *3.6* Assume that the hypotheses of Theorem 3.1 hold. Then the estimate

$$\|A^\theta(Qu(t)-\Phi^L(Pu(t),\ t))\| \ \le$$

$$\le\ C\left(1+\left\|A^\theta u_0\right\|\right)e^{-\gamma_N(t-s)}+C_R\, e^{-\alpha\lambda_N^\theta L}$$

holds for $t \ge t_*$ and for any solution $u(t)$ to problem (1.1) possessing the dissipativity property: $\|A^\theta u(t)\| \le R$ for $t \ge t_* \ge s$ and for some R and t_*. Here $\gamma_N = \lambda_N + \frac{2M}{q}\lambda_N^\theta$ and the constant $\alpha > 0$ does not depend on N.

Therefore, if the hypotheses of Theorem 3.1 hold, then a bounded solution to problem (1.1) gets into the exponentially small (with respect to λ_N^θ and L) vicinity of the manifold $\{\mathbf{M}_s^L:\ -\infty < s < \infty\}$ at an exponential velocity.

According to (2.12) in order to build an approximation $\{\mathbf{M}_s^L\}$ of the inertial manifold $\{\mathbf{M}_s\}$ we should solve integral equation (2.1) for L large enough. This equation has the same structure both for $L < \infty$ and for $L = \infty$. Therefore, it is impossible to use the surfaces $\{\mathbf{M}_s^L\}$ directly for the effective approximation of $\{\mathbf{M}_s\}$. However, by virtue of contractiveness of the operator $\mathbf{B}_p^{s,\,\infty}$ in the space $C_s^- = C_{\gamma,\,\theta}(-\infty,\ s)$, its fixed point $v_s(t)$ which determines $\mathbf{M}_s$ can be found with the help of iterations. This fact enables us to construct the collection $\{\mathbf{M}_{n,\ s}\}$ of approximations for $\{\mathbf{M}_s\}$ as follows. Let $v_0 = v_{0,\ s}(t;\ p)$ be an element of C_s^-. We take

$$v_n \equiv v_{n,\ s}(t,\ p) = \mathbf{B}_p^{s,\,\infty}[v_{n-1}](t), \qquad n = 1,\ 2,\ \dots,$$

and define the surfaces $\{\mathbf{M}_{n,\ s}\}$ by the formula

$$\mathbf{M}_{n,\ s} = \{p+\Phi_n(p,\ s):\ p\in PH\},$$

where $\Phi_n(p,\ s) = Qv_{n,\,s}(p,\ s), \quad n = 1,\ 2,\ \dots$

— Exercise *3.7* Let $v_0 \equiv p$ and let $B(u,\ t) \equiv B(u)$. Show that

$$\Phi_0(p,\ s) \equiv 0 \quad \text{and} \quad \Phi_1(p,\ s) = A^{-1}QB(p)\,.$$

— Exercise *3.8* Assume that spectral gap condition (3.1) is fulfilled. Show that

$$\left\|A^\theta(\Phi_n(p,\ s) - \Phi(p,\ s))\right\| \le q^n\left(|v_0|_s + (1-q)^{-1}\left[D_1 + \|A^\theta p\|\right]\right),$$

where D_1 is defined by formula (2.6) and $\Phi(p,\ s)$ is the function that determines the inertial manifold.

— Exercise *3.9* Prove the assertion for $\Phi_n(p,\ s)$ similar to the one in Exercise 3.5.

Theorems represented above can also be used in the case when the original system is dissipative and estimates (1.2) and (1.3) are not assumed to be uniform with respect to $u \in D(A^\theta)$. The dissipativity property enables us to restrict ourselves to the consideration of the trajectories lying in a vicinity of the absorbing set when we study the asymptotic behaviour of solutions to problem (0.1). In this case it is convenient to modify the original problem. Assume that the mapping $B(u,\ t)$ is continuous with respect to its arguments and possesses the properties

$$\|B(u,\ t)\| \le C_\rho\,, \qquad \left\|B(u_1,\ t) - B(u_2,\ t)\right\| \le C_\rho\left\|A^\theta(u_1 - u_2)\right\| \qquad (3.26)$$

for any $\rho > 0$ and for all u, u_1, and u_2 lying in the ball $B_\rho = \{v\colon\ \|A^\theta v\| \le \rho\}$. Let $\chi(s)$ be an infinitely differentiable function on $R_+ = [0,\ \infty)$ such that

$$\chi(s) = 1\,,\quad 0 \le s \le 1\,; \qquad \chi(s) = 0\,,\ s \ge 2\,;$$
$$0 \le \chi(s) \le 1\,,\quad |\chi'(s)| \le 2\,,\quad s \in R_+\,.$$

We define the mapping $B_R(u,\ t)$ by assuming that

$$B_R(u,\ t) = \chi(R^{-1}\|A^\theta u\|)B(u,\ t)\,, \quad u \in D(A^\theta)\,. \qquad (3.27)$$

— Exercise *3.10* Show that the mapping $B_R(u,\ t)$ possesses the properties

$$\left\|A^\theta B_R(u,\ t)\right\| \le M\,,$$

$$\left\|B_R(u_1,\ t) - B_R(u_2,\ t)\right\| \le M\left\|A^\theta(u_1 - u_2)\right\|, \qquad (3.28)$$

where $M = C_{2R}(1 + 2/R)$ and C_ρ is a constant from (3.26).

Let us now assume that $B(u,\ t)$ satisfies condition (3.26) and the problem

$$\frac{du}{dt} + Au = B(u,\ t)\,, \qquad u\big|_{t=0} = u_0\,, \qquad (3.29)$$

has a unique mild solution on any segment $[s,\ s+T]$ and possesses the following dissipativity property: there exists $R_0 > 0$ such that for any $R > 0$ the relation

$$\|A^\theta u(t, s; u_0)\| \le R_0 \quad \text{for all} \quad t-s \ge t_0(R) \tag{3.30}$$

holds, provided that $\|A^\theta u_0\| \le R$. Here $u(t, s; u_0)$ is the solution to problem (3.29).

Exercise *3.11* Show that the asymptotic behaviour of solutions to problem (3.29) completely coincides with the asymptotic behaviour of solutions to the problem

$$\frac{du}{dt} + Au = B_{2R_0}(u, t), \qquad u\big|_{t=s} = u_0, \tag{3.31}$$

where B_{2R_0} is defined by formula (3.27) and R_0 is the constant from equation (3.30).

Exercise *3.12* Assume that for a solution to problem (3.29) the invariance property of the absorbing ball is fulfilled: if $\|A^\theta u_0\| \le R_0$, then $\|A^\theta u(t, s; u_0)\| \le R$ for all $t \le s$. Let $\mathbf{M}_t$ be the invariant manifold of problem (3.31). Then the set $\mathbf{M}_t^{R_0} = \mathbf{M}_t \cap \{u : \|A^\theta u\| \le R_0\}$ is invariant for problem (3.29): if $u_0 \in \mathbf{M}_s^{R_0}$, then $u(t, s; u_0) \in \mathbf{M}_s^{R_0}$, $t \ge s$.

Thus, if the appropriate spectral gap condition for problem (3.29) is fulfilled, then there exists a finite-dimensional surface which is a locally invariant exponentially attracting set.

In conclusion of this section we note that the version of the Lyapunov-Perron method represented here can also be used for the construction (see [13]) of inertial manifolds for retarded semilinear parabolic equations similar to the ones considered in Section 8 of Chapter 2. In this case both the smallness of retardation and the fulfilment of the spectral gap condition of the form (3.1) are required.

§ 4 *Continuous Dependence of Inertial Manifold on Problem Parameters*

Let us consider the Cauchy problem

$$\frac{du}{dt} + Au = B^*(u, t), \qquad u\big|_{t=s} = u_0, \quad s \in \mathbb{R} \tag{4.1}$$

in the space H together with problem (1.1). Assume that $B^*(u, t)$ is a nonlinear mapping from $D(A^\theta) \times \mathbb{R}$ into H possessing properties (1.2) and (1.3) with the same constant M as in problem (1.1). If spectral gap condition (3.1) is fulfilled, then problem (4.1) (as well as (1.1)) possesses an invariant manifold

$$\mathbf{M}_s^* = \{p + \Phi^*(p,\ s):\ p \in PH\}, \qquad s \in \mathbb{R}. \tag{4.2}$$

The aim of this section is to obtain an estimate for the distance between the manifolds $\mathbf{M}_s$ and $\mathbf{M}_s^*$. The main result is the following assertion.

Theorem 4.1.

Assume that conditions (1.2), (1.3), ***and*** (3.1) ***are fulfilled both for problems*** (1.1) ***and*** (4.1). ***We also assume that***

$$\|B(v,\ t) - B^*(v,\ t)\| \ \le\ \rho_1 + \rho_2 \|A^\theta v\| \tag{4.3}$$

for all $v \in D(A^\theta)$ ***and*** $t \in \mathbb{R}$, ***where*** ρ_1 ***and*** ρ_2 ***are positive numbers. Then the equation***

$$\sup_{s\ \in R} \|A^\theta(\Phi(p,\ s) - \Phi^*(p,\ s))\| \ \le\ C_1(q,\ \theta)\frac{\rho_1 + \rho_2}{\lambda_N^{1-\theta}} + C_2(q,\ \theta,\ M)\,\rho_2 \|A^\theta p\|$$

is valid for the functions $\Phi(p,\ s)$ ***and*** $\Phi^*(p,\ s)$ ***which give the invariant manifolds for problems*** (1.1) ***and*** (4.1) ***respectively. Here the numbers*** $C_1(q,\ \theta)$ ***and*** $C_2(q,\ \theta,\ M)$ ***do not depend on*** N ***and*** ρ_j.

Proof.

Equation (2.12) with $L = \infty$ implies that

$$\|A^\theta(\Phi(p,\ s) - \Phi^*(p,\ s))\| \ \le \int_{-\infty}^{s} \|A^\theta e^{-(s-\tau)A} Q\| \, \|B(v(\tau),\ \tau) - B^*(v^*(\tau),\ \tau)\| \, d\tau\ ,$$

where $v(\tau)$ and $v^*(\tau)$ are solutions to the integral equations of the type (2.1) corresponding to problems (1.1) and (4.1) respectively. Equations (1.3) and (4.3) give us that

$$\|B(v(\tau),\ \tau) - B^*(v^*(\tau),\ \tau)\| \ \le\ M\|A^\theta(v(\tau) - v^*(\tau))\| + \left(\rho_1 + \rho_2 \|A^\theta v(\tau)\|\right) \ \le$$

$$\le\ e^{\gamma(s-\tau)}(M|v - v^*|_s + \rho_2 |v|_s) + \rho_1 \tag{4.4}$$

for $\tau \le s$, where

$$|w|_s = \operatorname*{ess\ sup}_{t \le s} \left\{ e^{-\gamma(s-t)} \|A^\Theta w(t)\| \right\} \tag{4.5}$$

and $\gamma = \lambda_N + \dfrac{2M}{q}\lambda_N^\theta$ as before. Hence, after simple calculations as in Section 2 we find that

$$\|A^\theta(\Phi(p,\ s) - \Phi^*(p,\ s))\| \ \le\ \frac{q}{2}\left(|v - v^*|_s + \frac{\rho_2}{M}|v|_s\right) + \rho_1 \frac{1+k}{\lambda_{N+1}^{1-\theta}}\,. \tag{4.6}$$

Let us estimate the value $|v - v^*|_s$. Since v and v^* are fixed points of the corresponding operator $\mathbf{B}_p^{s,\ \infty}$, we have that

$$\|A^\theta(v(t)-v^*(t))\| \le \int_t^s \|A^\theta e^{-(t-\tau)A}P\|\,\|B(v(\tau),\tau)-B^*(v^*(\tau),\tau)\|\,d\tau + $$
$$+\int_{-\infty}^s \|A^\theta e^{-(t-\tau)A}Q\|\,\|B(v(\tau),\tau)-B^*(v^*(\tau),\tau)\|\,d\tau\ .$$

Therefore, by using spectral gap condition (3.1) and estimate (4.4) as above it is easy to find that

$$|v-v^*|_s \le q|v-v^*|_s + \frac{q\rho_2}{M}\cdot|v|_s + \rho_1\left(\frac{1}{\lambda_N^{1-\theta}} + \frac{1+k}{\lambda_{N+1}^{1-\theta}}\right).$$

Consequently,

$$|v-v^*|_s \le \frac{q}{1-q}\cdot\frac{\rho_2}{M}\cdot|v|_s + \frac{\rho_1}{1-q}\cdot\frac{2+k}{\lambda_N^{1-\theta}}.$$

Therefore, equation (4.6) implies that

$$\|A^\theta(\Phi(p,s)-\Phi^*(p,s))\| \le \frac{q}{1-q}\cdot\frac{\rho_2}{2M}\cdot|v|_s + \rho_1\cdot\frac{2-q}{2-2q}\cdot\frac{2+k}{\lambda_N^{1-\theta}}.$$

Hence, estimate (2.5) gives us the inequality

$$\|A^\theta(\Phi(p,s)-\Phi^*(p,s))\| \le \frac{\rho_1+\rho_2}{(1-q)^2}\cdot\frac{2+k}{\lambda_N^{1-\theta}} + \frac{q}{2(1-q)^2}\cdot\frac{\rho_2}{M}\|A^\theta p\|\,.$$

This **implies the assertion of Theorem 4.1**.

Let us now consider the Galerkin approximations $u_m(t)$ of problem (1.1). We remind (see Chapter 2) that the Galerkin approximation of the order m is defined as a function $u_m(t)$ with the values in P_mH, this function being a solution to the problem

$$\frac{du_m}{dt} + Au_m = P_mB(u_{m_0}), \qquad u_m\big|_{t=s} = u_{0m}\ . \tag{4.7}$$

Here P_m is the orthoprojector onto the span of elements $\{e_1, \dots, e_m\}$ in H.

Exercise *4.1* Assume that spectral gap condition (3.1) holds and $m \ge N+1$. Show that problem (4.7) possesses an invariant manifold of the form

$$\mathbf{M}_s^{(m)} = \{p + \Phi^{(m)}(p,s):\ p \in PH\}$$

in P_mH, where the function $\Phi^{(m)}(p,s):\ PH \to (P_m-P)H$ is defined by equation similar to (2.12).

The following assertion holds.

Theorem 4.2.

Assume that spectral gap condition (3.1) ***holds. Let*** $\Phi(p, s)$ ***and*** $\Phi^{(m)}(p, s)$ ***be the functions defined by the formulae of the type*** (2.12) ***and let these functions give invariant manifolds for problems*** (1.1) ***and*** (4.7) ***for*** $m \ge N+1$ ***respectively. Then the estimate***

$$\|A^\theta(\Phi(p, s) - \Phi^{(m)}(p, s))\| \le \frac{C(q, M, \theta)}{\lambda_{m+1}^{1-\theta}} \left\{ 1 + \frac{D_1 + \|A^\theta p\|}{1 - \frac{\lambda_{N+1}}{\lambda_{m+1}}} \right\} \tag{4.8}$$

is valid, where the constant D_1 ***is defined by formula*** (2.6).

Proof.

It is evident that

$$(\Phi(p, s) - \Phi^{(m)}(p, s)) = Q[v(s;\, p) - v^{(m)}(s;\, p)], \tag{4.9}$$

where $v(t, p)$ and $v^{(m)}(t, p)$ are solutions to the integral equations

$$v(t) = \mathrm{B}_p^{s,\,\infty}[v](t), \qquad -\infty < t \le s,$$

and

$$v^{(m)}(t) = P_m \mathrm{B}_p^{s,\,\infty}[v^{(m)}](t), \qquad -\infty < t \le s.$$

Here $\mathrm{B}_p^{s,\,\infty}$ is defined as in (2.1). Since

$$v(t) - v^{(m)}(t) = (I - P_m)v(t) + P_m[\mathrm{B}_p^{s,\,\infty}[v](t) - \mathrm{B}_p^{s,\,\infty}[v^{(m)}](t)],$$

we have

$$\|A^\theta(v(t) - v^{(m)}(t))\| = \|A^\theta(1 - P_m)v(t)\| + \|A^\theta[\mathrm{B}_p^{s,\,\infty}[v](t) - \mathrm{B}_p^{s,\,\infty}[v^{(m)}](t)]\|.$$

The contractiveness property of the operator $\mathrm{B}_p^{s,\,\infty}$ leads to the equation

$$\|A^\theta(v(t) - v^{(m)}(t))\| = \|A^\theta(1 - P_m)v(t)\| + q \cdot |v - v^{(m)}|_s\, e^{\gamma(s-t)}.$$

In particular, this implies that

$$|v - v^{(m)}|_s \equiv \sup_{t<s} e^{-\gamma(s-t)} \|A^\theta(v(t) - v^{(m)}(t))\| \le (1-q)^{-1} |(1 - P_m)v|_s .$$

Hence, with the help of (4.9) we find that

$$\|A^\theta(\Phi(p, s) - \Phi^{(m)}(p, s))\| \le \|A^\theta(v(s) - v^{(m)}(s))\| \le |v - v^{(m)}|_s \le \tag{4.10}$$

$$\le (1-q)^{-1} |(1 - P_m)v|_s .$$

Let us estimate the value $|(1-P_m)v|_s$. It is clear that

$$(1-P_m)v(t) = \int_{-\infty}^{t} e^{-(t-\tau)A}(1-P_m)B(v(\tau))\,d\tau\,.$$

Therefore, Lemma 2.1.1 (see also (1.8)) gives us that

$$\|A^\theta(1-P_m)v(t)\| \;\le\; M\int_{-\infty}^{t}\left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right]e^{-(t-\tau)\lambda_{m+1}}\,d\tau\,+$$

$$+M\int_{-\infty}^{t}\left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right]e^{-\lambda_{m+1}(t-\tau)}\,e^{(t-\tau)\gamma}\,d\tau\;|v|_s\cdot e^{\gamma(s-t)}\;,$$

where

$$\gamma \;\equiv\; \lambda_N + \frac{2M}{q}\lambda_N^\theta < \;\lambda_{N+1} < \lambda_{m+1}$$

as above. Simple calculations analogous to the ones in Lemma 2.1 imply that

$$\|A^\theta(1-P_m)v(t)\| \;\le\; \frac{M(1+k)}{\lambda_{m+1}^{1-\theta}} + \frac{M(1+k)\lambda_{m+1}^\theta}{\lambda_{m+1}-\gamma}\;e^{\gamma(s-t)}\,|v|_s\,,$$

where the constant k has the form (1.7). Consequently, using (2.5) we obtain

$$|(1-P_m)v|_s \;\le\; \frac{M(1+k)}{\lambda_{m+1}^{1-\theta}}\left(1+\left(1-\frac{\lambda_{N+1}}{\lambda_{m+1}}\right)^{-1}|v|_s\right) \;\le$$

$$\le\; \frac{M(1+k)}{\lambda_{m+1}^{1-\theta}}\left(1+\left(1-\frac{\lambda_{N+1}}{\lambda_{m+1}}\right)^{-1}(1-q)^{-1}(D_1+\|A^\theta p\|)\right).$$

This and (4.10) imply estimate (4.8). **Theorem 4.2 is proved**.

E x e r c i s e *4.2* In addition assume that the hypotheses of Theorem 4.2 hold and $\|B(u,\,t)\| \le M$. Show that in this case estimate (4.8) has the form

$$\|A^\theta(\Phi(p,\,s)-\Phi^{(m)}(p,\,s))\| \;\le\; C(q;\,M,\,\theta)\lambda_{m+1}^{-1+\theta}\,.$$

§ 5 *Examples and Discussion*

Example *5.1*

Let us consider the nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial x^2} + f(x,\ u,\ t), \quad 0 < x < l,\ t > 0\ , & (5.1)\\ u\big|_{x=0} = u\big|_{x=l} = 0\ , & (5.2)\\ u\big|_{t=0} = u_0(x)\ . & (5.3) \end{cases}$$

Assume that ν is a positive parameter and $f(x,\ u,\ t)$ is a continuous function of its variables which possesses the properties

$$\left|f(x,\ u_1,\ t) - f(x,\ u_2,\ t)\right| \le M\left|u_1 - u_2\right|, \quad |f(x,\ 0,\ t)| \le \frac{M}{\sqrt{l}}\ .$$

Problem (5.1)–(5.3) generates a dynamical system in $L^2(0,\ l)$ (see Section 3 of Chapter 2). Therewith

$$A = -\nu\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad D(A) = H_0^1(0,\ L) \cap H^2(0,\ l),$$

where $H^s(0,\ l)$ is the Sobolev space of the order s. The mapping $B(\cdot,\ t)$ given by the formula $u(x) \to f(x,\ u(x),\ t)$ satisfies conditions (1.2) and (1.3) with $\theta = 0$. In this case spectral gap condition (2.3) has the form

$$\nu\frac{\pi^2}{l^2}\left((N+1)^2 - N^2\right) \ge \frac{4M}{q}\ .$$

Thus, problem (5.1)–(5.3) possesses an inertial manifold of the dimension N, provided that

$$N > -\frac{1}{2} + \frac{2M}{\nu q}\frac{l^2}{\pi^2} \tag{5.4}$$

for some $q < 2 - \sqrt{2}$.

Exercise *5.1* Find the conditions under which the inertial manifold of problem (5.1)–(5.3) is one-dimensional. What is the structure of the corresponding inertial form?

Exercise *5.2* Consider problem (5.1) and (5.3) with the Neumann boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \tag{5.5}$$

Show that problem (5.1), (5.3), and (5.5) has an inertial manifold of the dimension $N+1$, provided condition (5.4) holds for some

$N \geq 0$. (*Hint*: $A = -\nu(d^2/dx^2) + \varepsilon$ with condition (5.5), $B(u, t) = -\varepsilon u + f(x, u, t)$, where $\varepsilon > 0$ is small enough).

Exercise *5.3* Find the conditions on the parameters of problem (5.1), (5.3), and (5.5) under which there exists a one-dimensional inertial manifold. Show that if $f(x, u, t) \equiv f(u, t)$, then the corresponding inertial form is of the type

$$\dot{p}(t) = f(p(t), t), \qquad p|_{t=0} = p_0 .$$

Example *5.2*

Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} + f\left(x, u, \dfrac{\partial u}{\partial x}, t\right), \quad 0 < x < l, \ t > 0 , \\ u|_{x=0} = u|_{x=l} = 0, \quad u|_{t=0} = u_0(x) . \end{cases} \tag{5.6}$$

Here $\nu > 0$ and $f(x, u, \xi, t)$ is a continuous function of its variables such that

$$|f(x, u_1, \xi, t) - f(x, u_2, \xi, t)| \leq L_1|u_1 - u_2| + L_2|\xi_1 - \xi_2| \tag{5.7}$$

for all $x \in (0, l)$, $t \geq 0$ and

$$\int_0^l [f(x, 0, 0, t)]^2 \, dx \leq L_3^2 ,$$

where L_j are nonnegative numbers. As in Example 5.1 we assume that

$$A = -\nu \frac{d^2}{dx^2}, \quad D(A) = H_0^1(0, l) \cap H^2(0, l), \quad B(u, t) = f\left(x, u, \frac{\partial u}{\partial x}, t\right).$$

It is evident that

$$\|B(u_1, t) - B(u_2, t)\| \leq L_1\|u_1 - u_2\| + L_2\left\|\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right\| .$$

Here $\|\cdot\|$ is the norm in $L^2(0, l)$. By using the obvious inequality

$$\left\|\frac{\partial u}{\partial x}\right\|^2 \geq \left(\frac{\pi}{l}\right)^2 \|u\|^2 , \qquad u \in H_0^1(0, l),$$

we find that

$$\|B(u_1, t) - B(u_2, t)\| \leq \frac{1}{\sqrt{\nu}}\left(L_1 \frac{l}{\pi} + L_2\right)\|A^{1/2}(u_1 - u_2)\| .$$

Hence, conditions (1.2) and (1.3) are fulfilled with

$$\theta = \frac{1}{2}, \qquad M = \max\left\{L_3; \ \frac{1}{\sqrt{\nu}}\left(\frac{l}{\pi}L_1 + L_2\right)\right\} .$$

Therewith spectral gap condition (2.3) acquires the form

$$\sqrt{\nu}\,\frac{\pi}{l}(2N+1) \;\ge\; \frac{2M}{q}(2N+1+k(N+1)),$$

where

$$k = \frac{1}{\sqrt{2}}\int_0^x \xi^{-1/2}e^{-\xi}\,d\xi = \sqrt{\frac{\pi}{2}}\,.$$

Thus, the equation

$$1+\sqrt{\frac{\pi}{2}}\cdot\frac{N+1}{2N+1} \;\le\; \frac{\pi q\sqrt{\nu}}{2lM}$$

or

$$2+\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\pi}{2}}\cdot\frac{1}{2N+1} \;\le\; \frac{\pi q\sqrt{\nu}}{lM}$$

must be valid for some $0 < q < 2-\sqrt{2}$. We can ensure the fulfilment of this condition only in the case when

$$2+\sqrt{\frac{\pi}{2}} < \frac{\pi q_0\sqrt{\nu}}{lM}, \qquad q_0 = 2-\sqrt{2},$$

i.e. if

$$M \equiv \max\left\{L_3;\; \frac{1}{\sqrt{\nu}}\left(\frac{l}{\pi}L_1+L_2\right)\right\} \;<\; 2\pi\frac{\sqrt{\nu}}{l}\cdot\frac{\sqrt{2}-1}{2\sqrt{2}+\sqrt{\pi}}\,. \tag{5.9}$$

Thus, in order to apply the above-presented theorems to the construction of the inertial manifold for problem (5.6) one should pose some additional conditions (see (5.7) and (5.9)) on the nonlinear term $f(x, u, \partial u/\partial x, t)$ or require that the diffusion coefficient ν be large enough.

— Exercise *5.4* Assume that $f(x, u, \xi, t) = \varepsilon\bar{f}(x, u, \xi, t)$ in (5.6), where the function $\bar{f}$ possesses properties (5.7) and (5.8) with arbitrary $L_j \ge 0$. Show that problem (5.6) has an inertial manifold for any $0 < \varepsilon < \varepsilon_0$, where

$$\varepsilon_0 = 2\pi\frac{\sqrt{\nu}}{l}\cdot\frac{\sqrt{2}-1}{2\sqrt{2}+\sqrt{\pi}}\cdot\left[\max\left\{L_3;\; \frac{1}{\nu}\left(\frac{l}{\sqrt{\pi}}L_1+L_2\right)\right\}\right]^{-1}.$$

Characterize the dependence of the dimension of inertial manifold on ε.

— Exercise *5.5* Study the question on the existence of an inertial manifold for problem (5.6) in which the Dirichlet boundary condition is replaced by the Neumann boundary condition (5.5).

It should be noted that

$$\lambda_n = C_d n^{2/d}(1 + o(1)), \quad n \to \infty, \quad d = \dim \Omega,$$

where λ_n are the eigenvalues of the linear part of the equation of the type

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(x, u, \nabla u, t), \quad x \in \Omega, \quad t > 0,$$

in a multidimensional bounded domain Ω. Therefore, we can not expect that Theorem 3.1 is directly applicable in this case. In this connection we point out the paper [3] in which the existence of IM for the nonlinear heat equation is proved in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \le 3)$ that satisfies the so-called "principle of spatial averaging" (the class of these domains contains two- and three-dimensional cubes).

It is evident that the most severe constraint that essentially restricts an application of Theorem 3.1 is spectral gap condition (3.1). In some cases it is possible to weaken or modify it a little. In this connection we mention papers [6] and [7] in which spectral gap condition (3.1) is given with the parameters $q = 2$ and $k = 0$ for $0 \le \theta < 1$. Besides it is not necessary to assume that the spectrum of the operator A is discrete. It is sufficient just to require that the selfadjoint operator A possess a gap in the positive part of the spectrum such that for its edges the spectral condition holds. We can also assume the operator A to be sectorial rather than selfadjoint (for example, see [6]).

Unfortunately, we cannot get rid of the spectral conditions in the construction of the inertial manifold. One of the approaches to overcome this difficulty runs as follows: let us consider the regularization of problem (0.1) of the form

$$\frac{du}{dt} + Au + \varepsilon A^m u = B(u, t), \quad u\big|_{t=0} = u_0. \tag{5.10}$$

Here $\varepsilon > 0$ and the number $m > 0$ is chosen such that the operator $\tilde{A} = A + \varepsilon A^m$ possesses spectral gap condition (3.1). Therewith IM for problem (5.10) should be naturally called an approximate IM for system (0.1). Other approaches to the construction of the approximate IM are presented below.

It should also be noted that in spite of the arising difficulties the number of equations of mathematical physics for which it is possible to prove the existence of IM is large enough. Among these equations we can name the Cahn-Hillard equations in the domain $\Omega = (0, L)^d$, $d = \dim \Omega \le 2$, the Ginzburg-Landau equations ($\Omega = (0, L)^d$, $d \le 2$), the Kuramoto-Sivashinsky equation, some equations of the theory of oscillations ($d = 1$), a number of reaction-diffusion equations, the Swift-Hohenberg equation, and a non-local version of the Burgers equation. The corresponding references and an extended list of equations can be found in survey [8].

In conclusion of this section we give one more interesting application of the theorem on the existence of an inertial manifold.

Example 5.3

Let us consider the system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(u, \nabla u), \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial \Omega} = 0, \tag{5.11}$$

in a bounded domain $\Omega \subset R^d$. Here $u = (u_1, \ldots, u_m)$ and the function $f(u, \xi)$ satisfies the global Lipschitz condition:

$$|f(u, \xi) - f(v, \eta)| \le L\{|u - v|^2 + |\xi - \eta|^2\}^{1/2}, \tag{5.12}$$

where $u, v \in \mathbb{R}^m$, $\xi, \eta \in \mathbb{R}^{md}$, and $L > 0$. We also assume that $|f(0, 0)| \le L$. Problem (5.11) can be rewritten in the form (0.1) in the space $H = [L^2(\Omega)]^m$ if we suppose

$$Au = -\nu \Delta u + u, \qquad B(u) = u + f(u, \nabla u).$$

It is clear that the operator A is positive in its natural domain and it has a discrete spectrum. Equation (5.12) implies that the relation

$$\|B(u) - B(v)\| \le L\left\{\|u - v\|^2 + \|\nabla(u - v)\|^2\right\}^{1/2} + \|u - v\| \le$$

$$\le \left(1 + L\max\left\{1; \frac{1}{\sqrt{\nu}}\right\}\right)\left\{\|u - v\|^2 + \nu\|\nabla(u - v)\|^2\right\}^{1/2}$$

is valid for $B(u)$. Thus,

$$\|B(u) - B(v)\| \le M\|A^{1/2}(u - v)\|,$$

where

$$M = 1 + L\max\left\{1; \frac{1}{\sqrt{\nu}}\right\}.$$

Therefore, problem (5.11) generates an evolutionary semigroup S_t (see Chapter 2) in the space $D(A^{1/2})$. An important property of S_t is the following: the subspace $\mathbf{L}$ which consists of constant vectors is invariant with respect to this semigroup. The dimension of this subspace is equal to m. The action of the semigroup in this subspace is generated by a system of ordinary differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u, 0), \qquad u(t) \in \mathbf{L}. \tag{5.13}$$

Exercise *5.6* Assume that equation (5.12) holds for $\xi = \eta = 0$. Show that equation (5.13) is uniquely solvable on the whole time axis for any initial condition and the equation

$$\sup_{t \le s}\left\{e^{-L(s - t)}|u(s)|\right\} < \infty \tag{5.14}$$

holds for any $s \in \mathbb{R}$.

The subspace $\mathbf{L}$ consists of the eigenvectors of the operator A corresponding to the eigenvalue $\lambda_1 = 1$. The next eigenvalue has the form $\lambda_2 = \nu\mu_1 + 1$, where μ_1 is the first nonzero eigenvalue of the Laplace operator with the Neumann boundary condition on $\partial\Omega$. Therefore, spectral gap equation (3.1) can be rewritten in the form

$$\nu\mu_1 \geq \frac{2}{q}\left(1 + L\max\left\{1;\ \frac{1}{\sqrt{\nu}}\right\}\right)\left(\left(1+\sqrt{\frac{\pi}{2}}\right)\sqrt{\nu\mu_1+1}+1\right) \tag{5.15}$$

for $N = m$ and $\theta = 1/2$, where $0 < q < 2 - \sqrt{2}$. It is clear that there exists $\nu_0 > 0$ such that equation (5.15) holds for all $\nu \geq \nu_0$. Therefore, we can apply Theorem 3.1 to find that if ν is large enough, then there exists IM of the type

$$\mathbf{M} = \left\{p + \Phi(p):\ p \in \mathbf{L},\ \Phi:\ \mathbf{L} \to H \ominus \mathbf{L}\right\}.$$

The invariance of the subspace $\mathbf{L}$ and estimate (5.14) enable us to use Theorem 3.2 and to state that $\mathbf{L} \subset \mathbf{M}$. This easily implies that $\Phi(p) \equiv 0$, i.e. $\mathbf{M} = \mathbf{L}$. Thus, Theorem 3.1 gives us that for any solution $u(t)$ to problem (5.11) there exists a solution $\tilde{u}(t)$ to the system of ordinary differential equations (5.13) such that

$$\|u(t) - \tilde{u}(t)\|_1 \leq Ce^{-\gamma t}, \qquad t \geq 0,$$

where the constant $\gamma > 0$ does not depend on $u(t)$ and $\|\cdot\|_1$ is the Sobolev norm of the first order.

— Exercise *5.7* Consider the problem

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} + f(x,\ u),\ \ 0 < x < \pi;\quad u\big|_{x=0} = u\big|_{x=\pi} = 0, \tag{5.16}$$

where the function $f(x,\ u)$ has the form

$$f(x,\ u) = g_1(u_1,\ u_2)\sin x + g_2(u_1,\ u_2)\sin 2x.$$

Here

$$u_j = \frac{2}{\pi}\int_0^\pi u(x)\sin jx\,dx, \quad j = 1,\ 2,$$

and $g_j(u_1,\ u_2)$ are continuous functions such that

$$|g_j(u_1,\ u_2) - g_j(v_1,\ v_2)| \leq$$

$$\leq L_j\left(|u_1 - v_1|^2 + |u_2 - v_2|^2\right)^{1/2}; \qquad g_j(0,\ 0) = 0.$$

Show that if

$$\nu > \frac{2}{5(\sqrt{2}-1)}\sqrt{\pi(L_1^2 + L_2^2)},$$

then the dynamical system generated by problem (5.16) has the two-dimensional (flat) inertial manifold

$$M = \{p_1 \sin x + p_2 \sin 2x : \ p_1, \ p_2 \in \mathbb{R}\}$$

and the corresponding inertial form is:

$$\dot{p}_1 + \nu p_1 = g_1(p_1, \ p_2) \,, \qquad \dot{p}_2 + 4\nu p_2 = g_2(p_1, \ p_2) \,.$$

— Exercise *5.8* Study the question on the existence of an inertial manifold for the Hopf model of turbulence appearance (see Section 7 of Chapter 2).

§ 6 *Approximate Inertial Manifolds for Semilinear Parabolic Equations*

Even in the cases when the existence of IM can be proved, the question concerning the effective use of the inertial form

$$\partial_t p + Ap = PB(p + \Phi(p, \ t), \ t) \tag{6.1}$$

is not simple. The fact is that it is not practically possible to find a more or less explicit solution to the integral equation for $\Phi(p, \ t)$ even in the finite-dimensional case. In this connection we face the problem of approximate or asymptotic construction of an invariant (inertial) manifold. Various aspects of this problem related to finite-dimensional systems are presented in the book by Ya. Baris and O. Lykova [14].

For infinite-dimensional systems the problem of construction of an approximate IM can be interpreted as a problem of reduction, i.e. as a problem of constructive description of finite-dimensional projectors P and functions $\Phi(\,\cdot\,, \ t) : PH \to (1-P)H$ such that an equation of form (6.1) "inherits" (of course, this needs to be specified) all the peculiarities of the long-time behaviour of the original system (0.1). It is clear that the manifolds arising in this case have to be close in some sense to the real IM (in fact, the dynamics on IM reproduces all the essential features of the qualitative behaviour of the original system). Under such a formulation a problem of construction of IM acquires secondary importance, so one can directly construct a sequence of approximate IMs. Usually (see the references in survey [8]) the problem of the construction of an approximate IM can be formulated as follows: find a surface of the form

$$\mathbf{M}_t = \{p + \Phi(p, \ t) : \ p \in PH\}, \tag{6.2}$$

which attracts all the trajectories of the system in its small vicinity. The character of closeness is determined by the parameter λ_{N+1}^{-1} related to the decomposition

$$\begin{cases} \dfrac{dp}{dt} + Ap = PB(p+q,\ t)\ , \\ \\ \dfrac{dq}{dt} + Aq = (1-P)B(p+q,\ t)\ . \end{cases} \tag{6.3}$$

We obtain the trivial approximate IM $\mathbf{M}_t^{(0)}$ if we put $\Phi(p,\ t) = \Phi_0(p,\ t) \equiv 0$ in (6.2). In this case $\mathbf{M}_t^{(0)}$ is a finite-dimensional subspace in $\mathcal{H}$ whereas inertial form (6.1) turns into the standard Galerkin approximation of problem (0.1) corresponding to this subspace. One can find the simplest non-trivial approximation $\mathbf{M}_t^{(1)}$ using formula (6.2) and assuming that

$$\Phi(p,\ t) = \Phi_1(p,\ t) \equiv A^{-1}(1-P)B(p,\ t)\,. \tag{6.4}$$

The consideration of system (0.1) on $\mathbf{M}_t^{(1)}$ leads to the second equation of equations (6.3) being replaced by the equality $Aq = (1-P)B(p,\ t)$. The results of the computer simulation (see the references in survey [8]) show that the use of just the first approximation to IM has a number of advantages in comparison with the traditional Galerkin method (some peculiarities of the qualitative behaviour of the system can be observed for a smaller number of modes).

There exist several methods of the construction of an approximate IM. We present the approach based on Lemma 2.1 which enables us to construct an approximate IM of the exponential order, i.e. the surfaces in the phase space H such that their exponentially small (with respect to the parameter λ_{N+1}) vicinities uniformly attract all the trajectories of the system. For the first time this approach was used in paper [15] for a class of stochastic equations in the Hilbert space. Here we give its deterministic version.

Let us consider the integral equation (see(2.1))

$$v(t) = \mathbf{B}_p^{s,\,L}[v](t), \qquad s-L \le t \le s$$

and assume that $L = \rho\lambda_{N+1}^{-\theta}$, where the parameter ρ possesses the property

$$q \equiv M\left(\frac{\theta^\theta}{1-\theta}\,\lambda_1^{-(1-\theta)}\,\rho^{1-\theta} + \rho\right) < 1\,. \tag{6.5}$$

In this case equations (2.2) hold. Hence, Lemma 2.1 enables us to construct a collection of manifolds $\{\mathbf{M}_s^L\}$ for $L = \rho\lambda_{N+1}^{-\theta}$ with the help of the formula

$$\mathbf{M}_s^L = \{p + \Phi^L(p,\ s):\ p \in PH\}, \tag{6.6}$$

where

$$\Phi^L(p,\ s) = \int_{s-L}^{s} e^{-(s-\tau)A}QB(v(\tau),\ \tau)\,d\tau \equiv Qv(s,\ p)\,. \tag{6.7}$$

Here $v(t) = v(t,\ p)$ is a solution to integral equation (2.1) and $L = \rho\lambda_{N+1}^{-\theta}$.

Exercise *6.1* Show that both the function $\Phi^L(p, s)$ and the surface $\mathbf{M}_s^L$ do not depend on s in the autonomous case $(B(u, t) \equiv B(t))$.

The following assertion is valid.

Theorem 6.1.

There exist positive numbers $\rho_1 = \rho_1(M, \theta, \lambda_1)$ ***and*** $\Lambda = \Lambda(M, \theta, \lambda_1)$ ***such that if***

$$\lambda_{N+1}^{1-\theta} \geq \Lambda \rho^{-1}, \qquad L = \rho \lambda_{N+1}^{-\theta}, \quad 0 < \rho \leq \rho_1, \tag{6.8}$$

then the mappings $\Phi^L(\cdot, s): PH \to QH$ ***defined by equation*** (6.7) ***possess the property***

$$\|A^\theta(Qu(t) - \Phi^L(Pu(t), t))\| \leq$$

$$\leq C_R^{(1)} \exp\left\{-\frac{\sigma_0}{\rho} \lambda_{N+1}^{\theta}(t - t_*)\right\} + C_R^{(2)} \exp\left\{-\frac{\rho}{2} \lambda_{N+1}^{1-\theta}\right\} \tag{6.9}$$

for all $t \geq t_* + L/2$. ***Here*** $\sigma_0 > 0$ ***is an absolute constant and*** $u(t)$ ***is a mild solution to problem*** (1.1) ***such that***

$$\|A^\theta u(t)\| \leq R \quad \textit{for} \quad t \in [t_*, +\infty). \tag{6.10}$$

If $\|B(u, t)\| \leq M$, ***then estimate*** (6.9) ***can be rewritten as follows:***

$$\|A^\theta(Qu(t) - \Phi^L(Pu(t), t))\| \leq$$

$$\leq C_R \exp\left\{-\frac{\sigma_0}{\rho} \lambda_{N+1}^{\theta}(t - t_*)\right\} + D_2 \exp\{-\rho \lambda_{N+1}^{1-\theta}\}, \tag{6.11}$$

where D_2 ***is defined by equality*** (2.14).

Proof.

Let

$$\hat{u}(t) = U(t, s; Pu(s) + \Phi^L(Pu(s), s)), \qquad t_* \leq s \leq t,$$

where $U(t, s; v)$ is a mild solution to problem (1.1) with the initial condition $v \in D(A^\theta)$ at the moment s. Therewith $u(t) = U(t, 0; u_0)$. It is evident that

$$Qu(t) - \Phi^L(Pu(t), t) = Q(u(t) - \hat{u}(t)) +$$

$$+ \left[Q\hat{u}(t) - \Phi^L(P\hat{u}(t), t)\right] + \left[\Phi^L(P\hat{u}(t), t) - \Phi^L(Pu(t), t)\right]. \tag{6.12}$$

Let us estimate each term in this decomposition. Equation (1.6) implies that

$$\|A^\theta Q(u(t) - \hat{u}(t))\| \leq \overline{\alpha}_N(t - s)\|A^\theta(Qu(s) - \Phi^L(Pu(s), s))\|, \tag{6.13}$$

where

$$\overline{\alpha}_N(\tau) = e^{-\lambda_{N+1}\tau} + M(1+k)a_1\, \lambda_{N+1}^{-1+\theta}\, e^{a_2\tau} .$$

Using (2.16) we find that

$$\|A^{\theta}(Q\hat{u}(s) - \Phi^L(P\hat{u}(s),\ s))\| \ \le\ \beta_N(L,\ t-s) \tag{6.14}$$

where

$$\beta_N(L,\ \tau) = D_2(1-q)^{-1}e^{-\gamma L} + q(1-q)^{-2}(R+D_1)\,e^{-\gamma\tau} ,$$

moreover, the second term in $\beta_N(L,\ \tau)$ can be omitted if $\|B(u,\ t)\| \le M$ (see Exercise 2.1). At last equations (2.15) and (1.5) imply that

$$\|A^{\theta}(\Phi^L(P\hat{u}(t),\ t)) - \Phi^L(Pu(t),\ t)\| \ \le$$

$$\le\ a_1\frac{q}{1-q}\,e^{a_2(t-s)}\|A^{\theta}(Qu(s) - \Phi^L(Pu(s),\ s))\| \ . \tag{6.15}$$

Thus, equations (6.12)–(6.15) give us the inequality

$$d(t)\ \le\ \alpha_N(t-s)d(s) + \beta_N(L,\ t-s) \tag{6.16}$$

for $t \ge s \ge t_*$, where

$$d(t) = \|A^{\theta}(Qu(t) - \Phi^L(Pu(t),\ t))\|$$

and

$$\alpha_N(\tau)\ = \overline{\alpha}_N(\tau) + a_1\frac{q}{1-q}\,e^{a_2\tau}\ =$$

$$=\ e^{-\lambda_{N+1}\tau} + a_1\left[M(1+k)\,\lambda_{N+1}^{-1+\theta} + q(1-q)^{-1}\right]e^{a_2\tau}.$$

It follows from (6.16) that under the condition $s + L/2\ \le\ t\ \le s+L$ the equation

$$d(t)\ \le\ \alpha_{N,\,L}\,d(s) + \beta_N(L,\ L/2) \tag{6.17}$$

holds with

$$\alpha_{N,\,L} = e^{-\lambda_{N+1}\frac{L}{2}} + a_1\left[M(1+k)\,\lambda_{N+1}^{-1+\theta} + \frac{q}{1-q}\right]e^{a_2 L} .$$

It is clear that $\alpha_{N,\,L} \le 1/2$ if

$$\lambda_{N+1}L > 4\ln 2 , \qquad \lambda_{N+1}^{-1+\theta} > 16\,a_1M(1+k)$$

and

$$a_2L < \ln 2 , \qquad q \le (1+16\,a_1)^{-1} . \tag{6.18}$$

Let $\rho_1 = \rho_1(M,\ \theta,\ \lambda_1)$ be such that equation (6.18) holds for $L = \rho\lambda_{N+1}^{-\theta}$ and for the parameter q of the form (6.5) with $0 < \rho \le \rho_1$. Then equation (6.8) with $\Lambda = 4(1+4a_1M(1+k)\rho_1)$ implies that $\alpha_{N,\,L} \le 1/2$. Let $t_n = t_* + (1/2)nL$. Then it follows from (6.17) that

$$d(t_{n+1}) \le \frac{1}{2}d(t_n) + \beta_N(L,\ L/2)\,, \qquad n = 0,\ 1,\ 2,\ \ldots$$

After iterations we find that

$$d(t_n) \le 2^{-n}d(t_0) + 2\,\beta_N\Big(L,\ \frac{L}{2}\Big), \qquad n = 0,\ 1,\ 2,\ \ldots \tag{6.19}$$

Equation (6.17) also gives us that

$$d(t) \le \frac{1}{2}d(t_n) + \beta_N\Big(L,\ \frac{L}{2}\Big), \qquad t_n + \frac{L}{2} \le t \le t_n + L\,.$$

Therefore, it follows from (6.19) that

$$d(t) \le 2\exp\Big\{-\frac{2}{L}(t - t_*)\ln 2\Big\}d(t_*) + 2\,\beta_N\Big(L,\ \frac{L}{2}\Big)$$

for all $t \ge t_* + L/2$. This implies (6.9) and (6.11) if we take $\gamma = \lambda_{N+1}$ in the equation for $\beta_N(L,\ L/2)$. Thus, **Theorem 6.1 is proved**.

In particular, it should be noted that relations (6.9) and (6.11) also mean that a solution to problem (0.1) possessing the property (6.10) reaches the layer of the thickness $\varepsilon_N = c_1\exp\{-c_2\,\lambda_{N+1}^{1-\theta}\}$ adjacent to the surface $\{\mathbf{M}_t^L\}$ given by equation (6.6) for t large enough. Moreover, it is clear that if problem (0.1) is autonomous $(B(u,\ t) \equiv B(u))$ and if it possesses a global attractor, then the attractor lies in this layer. In the autonomous case $\mathbf{M}^L$ does not depend on t (see Exercise 6.1). These observations give us some information about the position of the attractor in the phase space. Sometimes they enable us to establish the so-called localization theorems for the global attractor.

Exercise *6.2* Let $\|B(u,\ t)\| \le M$. Use equations (1.4) and (1.8) to show that

$$\|A^\theta u(t)\| \le e^{-\lambda_1(t-s)}\|A^\theta u_0\| + R_0\,,$$

where $R_0 = M(1+k)\,\lambda_1^{-1+\theta}$.

In particular, the result of this exercise means that assumption (6.10) holds for any $R > R_0$ and for t_* large enough under the condition $\|B(u,\ t)\| \le M$. In the general case equation (6.10) is a variant of the dissipativity property.

Exercise *6.3* Let $v_0 = v_{0,\,s}^L(t,\ s)$ be a function from $C_{\gamma,\,\theta}(s-L,\ s)$. Assume that

$$v_n^L \equiv v_{n,\,s}^L(t,\ s) = \mathbf{B}_p^{s,\,L}[v_{n-1}](t)\,, \qquad n = 1,\ 2,\ \ldots$$

and

$$\Phi_n^L(p,\ s) = Q\,v_{n,\,s}^L(s,\ p)\,, \qquad n = 0,\ 1,\ 2,\ \ldots$$

Show that the assertions of Theorem 6.1 remain true for the function $\Phi_n^L(p,\ s)$ if we add the term

$$q^n(|v_0|_s + c_0(D_1+R))$$

to the right-hand sides of equations (6.9) and (6.11). Here q is defined by equality (6.5) and $|v_0|_s$ is the norm of the function v_0 in the space $C_{\gamma,\,\theta}(s-L,\, s)$.

Therefore, the function $\Phi_n^L(p,\, s)$ generates a collection of approximate inertial manifolds of the exponential (with respect to λ_{N+1}) order for n large enough.

E x a m p l e *6.1*

Let us consider the nonlinear heat equation in a bounded domain $\Omega \subset \mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u,\, \nabla u), \quad x \in \Omega,\ t > s\ , \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=s} = u_0(x)\ . \end{cases} \tag{6.20}$$

Assume that the function $f(w,\, \xi)$ possesses the properties

$$|f(u_1,\, \xi_1) - f(u_2,\, \xi_2)| \le C_1(|u_1 - u_2| + |\xi_1 - \xi_2|), \qquad |f(u,\, \xi)| \le C_2 .$$

We use Theorem 6.1 and the asymptotic formula

$$\lambda_N \sim c_0 N^{2/d}, \qquad N \to \infty,$$

for the eigenvalues of the operator $-\Delta$ in $\Omega \subset \mathbb{R}^d$ to obtain that in the Sobolev space $H_0^1(\Omega)$ for any N there exists a finite-dimensional Lipschitzian surface $\mathbf{M}_N$ of the dimension N such that

$$\operatorname{dist}_{H_0^1(\Omega)}(u(t),\, \mathbf{M}_N) \le C_1 \exp\{-\sigma_1 N^{1/d}(t-t_*)\} + C_2 \exp\{-\sigma_2 N^{1/d}\}$$

for $t \ge t_*$ and for any mild (in $H_0^1(\Omega)$) solution $u(t)$ to problem (6.20). Here t_* is large enough, C_j and σ_j are positive constants.

Exercise *6.4* Consider the abstract form of the two-dimensional system of the Navier-Stokes equations

$$\frac{du}{dt} + \nu A u + b(u,\, u) = f(t), \qquad u\big|_{t=0} = u_0 \tag{6.21}$$

(see Example 3.5 and Exercises 4.10 and 4.11 of Chapter 2). Assume that $\|A^{1/2} f(t)\| < C$ for $t \ge 0$. Use the dissipativity property for (6.21) and the formula

$$c_0 k \le \frac{\lambda_k}{\lambda_1} \le c_1 k$$

for the eigenvalues of the operator A to show that there exists a collection of functions $\{\Phi(p,\, t) \colon t \ge 1\}$ from $PD(A)$ into $(1-P)D(A)$ possessing the properties

a) $\|A\Phi(p,\ t)\| \le c_1 N^{-1/2}$;

$\|A(\Phi(p_1,\ t) - \Phi(p_2,\ t))\| \le c_2 \|A(p_1 - p_2)\|$

for any $p,\ p_1,\ p_2 \in PD(A)$;

b) for any solution $u(t) \in D(A)$ to problem (6.21) there exists $t^* > 1$ such that

$$\|AQ(u(t) - \Phi(Pu(t),\ t))\| \le$$

$$\le\ c_3 \exp\{-\sigma_0 N^{1/2}(t - t_*)\} + c_4 \exp\{-\sigma_1 N^{1/2}\}.$$

Here P is the orthoprojector onto the first N eigenelements of the operator A.

Exercise *6.5* Use Theorem 6.1 to construct approximate inertial manifolds for (a) the nonlocal Burgers equation, (b) the Cahn-Hilliard equation, and (c) the system of reaction-diffusion equations (see Sections 3 and 4 of Chapter 2).

In conclusion of the section we note (see [8], [9]) that in the autonomous case the approximate IM can also be built using the equation

$$\langle \Phi'(p);\ -Ap + PB(p + \Phi(p))\rangle\ + A\Phi(p)\ =\ QB(p + \Phi(p))\,. \qquad (6.22)$$

Here $p \in PH$, $Q = I - P$, $\Phi'(p)$ is the Frechét derivative and $\langle \Phi'(p),\ w\rangle$ is its value at the point p on the element w. At least formally, equation (6.22) can be obtained if we substitute the pair $\{p(t);\ \Phi(p(t))\}$ into equation (6.3). The second of equations (6.3) implicitly contains a small parameter λ_{N+1}^{-1}. Therefore, using (6.22) we can suggest an iteration process of calculation of the sequence $\{\Phi_m\}$ giving the approximate IM:

$$A\Phi_k(p) = QB(p + \Phi_{\nu_1(k)}(p)) +$$

$$+ \langle \Phi'_{\nu_2(k)}(p)\ ;\ \ Ap - PB(p + \Phi_{\nu_3(k)}(p))\rangle\ , \qquad k \ge 1\,, \qquad (6.23)$$

where the integers $\nu_i(k)$ are such that

$$0\ \le\ \nu_i(k)\ \le\ k - 1\,, \qquad \lim_{k\to\infty} \nu_i(k) = \infty\,, \qquad i = 1,\ 2,\ 3\,.$$

One should also choose the zeroth approximation and concretely define the form of the values $\nu_i(k)$ (for example, we can take $\Phi_0(v) \equiv 0$ and $\nu_i(k) = k-1$, $i = 1,\ 2,\ 3$). When constructing a sequence of approximate IMs one has to solve only a linear stationary problem on each step. From the point of view of concrete calculations this gives certain advantages in comparison with the construction used in Theorem 6.1. However, these manifolds have the power order of approximation only (for detailed discussion of this construction and for proofs see [9]).

Exercise *6.6* Prove that the mapping $\Phi_1(v)$ has the form (6.4) under the condition $\Phi_0(v) \equiv 0$. Write down the equation for $\Phi_2(v)$ when $\nu_1(2) = 1$, $\nu_2(2) = \nu_3(2) = 0$.

§ 7 Inertial Manifold for Second Order in Time Equations

The approach to the construction of IM given in Sections 2–4 is essentially based on the fact that the system has form (0.1) with a selfadjoint positive operator A. However, there exists a wide class of problems which cannot be reduced to this form. From the point of view of applications the important representatives of this class are second order in time systems arising in the theory of nonlinear oscillations:

$$\begin{cases} \dfrac{d^2 u}{dt^2} + 2\varepsilon \dfrac{du}{dt} + A u = B(u, t), \quad t > s, \ \varepsilon > 0, \\ u\big|_{t=s} = u_0, \quad \dfrac{du}{dt}\Big|_{t=s} = u_1 . \end{cases} \tag{7.1}$$

In this section we study the existence of IM for problem (7.1). We assume that A is a selfadjoint positive operator with discrete spectrum (μ_k and e_k are the corresponding eigenvalues and eigenelements) and the mapping $B(u, t)$ possesses the properties of the type (1.2) and (1.3) for $0 \le \theta \le 1/2$, i.e. $B(u, t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H such that

$$\|B(0, t)\| \le M_0,$$

$$\|B(u_1, t) - B(u_2, t)\| \le M_1 \|A^\theta(u_1 - u_2)\|, \tag{7.2}$$

where $0 \le \theta \le 1/2$ and $u_1, u_2 \in D(\mathcal{A}^\theta) \equiv \mathcal{F}_\theta$.

The simplest example of a system of the form (7.1) is the following nonlinear wave equation with dissipation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2\varepsilon \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + f\left(x, t; u, \dfrac{\partial u}{\partial x}\right) = 0, \quad 0 < x < L, \ t > s, \\ u\big|_{x=0} = u\big|_{x=L} = 0, \\ u\big|_{t=s} = u_0(x), \quad \dfrac{\partial u}{\partial t}\Big|_{t=s} = u_1(x) . \end{cases} \tag{7.3}$$

Let $\mathcal{H} = D(A^{1/2}) \times H$. It is clear that $\mathcal{H}$ is a separable Hilbert space with the inner product

$$(U, V) = (Au_0, v_0) + (u_1, v_1), \tag{7.4}$$

where $U = (u_0; u_1)$ and $V = (v_0; v_1)$ are elements of $\mathcal{H}$. In the space $\mathcal{H}$ problem (7.1) can be rewritten as a system of the first order:

$$\frac{d}{dt} U(t) + \mathbf{A} U(t) = \mathbf{B}(U(t), t), \quad t > s; \qquad U\big|_{t=s} = U_0 . \tag{7.5}$$

Here

$$U(t) = \left(u(t);\ \frac{\partial u(t)}{\partial t}\right), \qquad U_0 = (u_0;\ u_1) \in \mathcal{H}.$$

The linear operator $\mathbf{A}$ and the mapping $\mathbf{B}(U,\ t)$ are defined by the equations:

$$\mathbf{A}U = (-u_1;\ Au_0 + 2\varepsilon u_1), \qquad D(\mathbf{A}) = D(A) \times D(A^{1/2}), \tag{7.6}$$

$$\mathbf{B}(U,\ t) = (0;\ B(u_0,\ t)), \quad U = (u_0;\ u_1).$$

— Exercise *7.1* Prove that the eigenvalues and eigenvectors of the operator $\mathbf{A}$ have the form:

$$\lambda_n^{\pm} = \varepsilon \pm \sqrt{\varepsilon^2 - \mu_n}, \quad f_n^{\pm} = (e_n;\ -\lambda_n^{\pm} e_n), \quad n = 1,\ 2,\ \ldots, \tag{7.7}$$

where μ_n and e_n are the eigenvalues and eigenvectors of A.

— Exercise *7.2* Display graphically the spectrum of the operator A on the complex plane.

These exercises show that although problem (7.1) can be represented in the form (7.5) which is formally identical to (0.1) we cannot use Theorem 3.1 here. Nevertheless, after a small modification the reasoning of Sections 2–4 enables us to prove the existence of IM for problem (7.1). Such a modification based on an idea from [16] is given below.

First of all we prove the solvability of problem (7.1). Let us first consider the linear problem

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + 2\varepsilon\dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = h(t), \quad t > s\ , \\[2ex] u\big|_{t=s} = u_0, \quad \dfrac{\mathrm{d}u}{\mathrm{d}t}\Big|_{t=s} = u_1\ . \end{cases} \tag{7.8}$$

These equations can also be rewritten in the form (cf. (7.5))

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) + \mathbf{A}U(t) = H(t), \qquad U\big|_{t=s} = U_0, \tag{7.9}$$

where $U(t) = (u(t);\ \dot{u}(t))$ and $H(t) = (0;\ h(t))$. We define a ***mild solution*** to problem (7.8) (or (7.9)) on the segment $[s,\ s+T]$ as a function $u(t)$ from the class

$$\mathbf{L}_{s,\ T} \equiv C(s,\ s+T;\ \mathcal{F}_{1/2}) \cap C^1(s,\ s+T;\ H) \cap C^2(s,\ s+T;\ \mathcal{F}_{-1/2})$$

which satisfies equations (7.8). Here $\mathcal{F}_\theta = D(A^\theta)$ as before (see Chapter 2). One can prove the existence and uniqueness of mild solutions to (7.8) using the Galerkin method, for example. The ***approximate Galerkin solution*** of the order m is defined as a function

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k$$

satisfying the equations

$$\begin{cases} (\ddot{u}_m(t),\ e_j) + 2\varepsilon(\dot{u}_m(t),\ e_j) + (Au_m(t),\ e_j) = (h(t),\ e_j), \quad t > s\ , \\ (u_m(s),\ e_j) = (u_0,\ e_j), \quad (\dot{u}_m(s),\ e_j) = (u_1,\ e_j) \end{cases} \tag{7.10}$$

for $j = 1,\ 2,\ \ldots,\ m$. Moreover, we assume that $g_j(t) \in C^1(s,\ s+T)$ and $\dot{g}_j(t)$ is absolutely continuous. Hereinafter we use the notation $\dot{v}(t) = \mathrm{d}v/\mathrm{d}t$. Evidently equations (7.10) can be rewritten in the form

$$\begin{cases} \ddot{u}_m(t) + 2\,\varepsilon\,\dot{u}_m(t) + A\,u_m(t) = p_m h(t)\ , \\ u_m\big|_{t=s} = p_m u_0\,, \quad \dot{u}_m\big|_{t=s} = p_m u_1\ , \end{cases} \tag{7.11}$$

where p_m is the orthoprojector onto $\mathrm{Lin}\{e_1,\ \ldots\ e_m\}$ in H.

In the exercises given below it is assumed that

$$h(t) \in L^\infty(\mathbb{R},\ H)\,, \qquad u_0 \in D(A^{1/2})\,, \qquad u_1 \in H\,. \tag{7.12}$$

— Exercise *7.3* Show that problem (7.10) is uniquely solvable on any segment $[s,\ s+T]$ and $u_m(t) \in \mathbf{L}_{s,\ T}$.

— Exercise *7.4* Show that the energy equality

$$\frac{1}{2}\Big(\|\dot{u}_m(t)\|^2 + \|A^{1/2} u_m(t)\|^2\Big) + 2\,\varepsilon \int_s^t \|\dot{u}_m(\tau)\|^2\,\mathrm{d}\tau =$$

$$= \frac{1}{2}\Big(\|p_m u_1\|^2 + \|A^{1/2} p_m u_0\|^2\Big) + \int_s^t (h(\tau),\ \dot{u}_m(\tau))\,\mathrm{d}\tau \tag{7.13}$$

holds for any solution to problem (7.10).

— Exercise *7.5* Using (7.11) and (7.13) prove the a priori estimate

$$\|A^{-1/2}\ddot{u}_m(t)\|^2 + \|\dot{u}_m(t)\|^2 + \|A^{1/2} u_m(t)\| \ \le\ C(T,\ u_0,\ u_1)$$

for the approximate Galerkin solution $u_m(t)$ to problem (7.8).

— Exercise *7.6* Using the linearity of problem (7.11) show that for every two approximate solutions $u_m(t)$ and $u_{m'}(t)$ the estimate

$$\|A^{-1/2}(\ddot{u}_m(t) - \ddot{u}_{m'}(t))\|^2 +$$

$$+ \|\dot{u}_m(t) - \dot{u}_{m'}(t)\| + \|A^{1/2}(u_m(t) - u_{m'}(t))\|^2 \ \le$$

$$\leq C_T \left\| (p_m - p_{m'}) u_1 \right\|^2 +$$

$$+ \left\| A^{1/2} (p_m - p_{m'}) u_0 \right\|^2 + \operatorname*{ess\ sup}_{\tau \in [s,\ s+T]} \left\| (p_m - p_{m'}) h(\tau) \right\|^2$$

holds for all $t \in [s,\ s+T]$, where $T > 0$ is an arbitrary number.

— Exercise *7.7* Using the results of Exercises 7.5 and 7.6 show that we can pass to the limit $n \to \infty$ in equations (7.11) and prove the existence and uniqueness of mild solutions to problem (7.8) on every segment $[s,\ s+T]$ under the condition (7.12).

— Exercise *7.8* For a mild solution $u(t)$ to problem (7.8) prove the energy equation:

$$\frac{1}{2}\left(\|\dot{u}(t)\|^2 + \|A^{1/2} u(t)\|^2 \right) + 2\varepsilon \int_s^t \|\dot{u}(\tau)\|^2 \, d\tau =$$

$$= \frac{1}{2}\left(\|u_1\|^2 + \|A^{1/2} u_0\|^2 \right) + \int_s^t (h(\tau),\ \dot{u}(\tau))\, d\tau \ . \qquad (7.14)$$

In particular, the exercises above show that for $h(t) \equiv 0$ problem (7.8) generates a linear evolutionary semigroup $e^{-t\mathbf{A}}$ in the space $\mathcal{H} = D(A^{1/2}) \times H$ by the formula

$$e^{-t\mathbf{A}}(u_0;\ u_1) = (u(t);\ \dot{u}(t)), \qquad (7.15)$$

where $u(t)$ is a mild solution to problem (7.8) for $h(t) \equiv 0$. Equation (7.14) implies that the semigroup $e^{-t\mathbf{A}}$ is contractive for $\varepsilon \geq 0$.

— Exercise *7.9* Assume that conditions (7.12) are fulfilled. Show that the mild solution to problem (7.8) can be presented in the form

$$(\dot{u}(t);\ u(t)) \ = \ e^{-(t-s)\mathbf{A}}(u_0;\ u_1) + \int_s^t e^{-(t-s)\mathbf{A}}(0;\ h(\tau))\, d\tau, \quad (7.16)$$

where the semigroup $e^{-t\mathbf{A}}$ is defined by equation (7.15).

Let us now consider nonlinear problem (7.1) and define its ***mild solution*** as a function $U(t) \equiv (u(t);\ \dot{u}(t)) \in C(s,\ s+T;\ \mathcal{H})$ satisfying the integral equation

$$U(t) = e^{-(t-s)\mathbf{A}} U_0 + \int_s^t e^{-(t-s)\mathbf{A}} \mathbf{B}(U(\tau),\ \tau)\, d\tau \qquad (7.17)$$

on $[s,\ s+T]$. Here $\mathbf{B}(U(t),\ t) = (0;\ B(u(t),\ t))$ and $U_0 = (u_0;\ u_1)$.

— Exercise *7.10* Show that the estimates

$$\|\mathbf{B}(U,\ t)\|_{\mathscr{H}} \le M(1+\|U\|_{\mathscr{H}}),$$

$$\|\mathbf{B}(U_1,\ t)-\mathbf{B}(U_2,\ t)\|_{\mathscr{H}} \le M\|U_1-U_2\|_{\mathscr{H}}$$

hold in the space $\mathscr{H} = D(A^{1/2})\times H$. Here M is a positive constant.

— Exercise *7.11* Follow the reasoning used in the proof of Theorems 2.1 and 2.3 of Chapter 2 to prove the existence and uniqueness of a mild solution to problem (7.1) on any segment $[s,\ s+T]$.

Thus, in the space $\mathscr{H}$ there exists a continuous evolutionary family of operators $S(t,\ s)$ possessing the properties

$$S(t,\ t) = I, \qquad S(t,\ \tau)\circ S(\tau,\ s) = S(t,\ s),$$

and

$$S(t,\ s)U_0 = (u(t);\ \dot{u}(t)),$$

where $u(t)$ is a mild solution to problem (7.1) with the initial condition $U_0 = (u_0;\ u_1)$.

Let condition $\varepsilon^2 > \mu_{N+1}$ hold for some integer N. We consider the decomposition of the space $\mathscr{H}$ into the orthogonal sum

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2\ ,$$

where

$$\mathscr{H}_1 = \operatorname{Lin}\{(e_k;\ 0),\ (0;\ e_k):\ \ k = 1,\ 2,\ \dots N\}$$

and $\mathscr{H}_2$ is defined as the closure of the set

$$\operatorname{Lin}\{(e_k;\ 0),\ (0;\ e_k):\ k \ge N+1\}.$$

— Exercise *7.12* Show that the subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are invariant with respect to the operator $\mathbf{A}$. Find the spectrum of the restrictions of the operator $\mathbf{A}$ to each of these spaces.

Let us introduce the following inner products in the spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ (the purpose of this introduction will become apparent further):

$$\begin{aligned} \langle U,\ V\rangle_1 &= \varepsilon^2(u_0,\ v_0)-(Au_0,\ v_0)+(\varepsilon u_0+u_1,\ \varepsilon v_0+v_1)\ , \\ \langle U,\ V\rangle_2 &= (Au_0,\ v_0)+(\varepsilon^2-2\mu_{N+1})(u_0,\ v_0)+(\varepsilon u_0+u_1,\ \varepsilon v_0+v_1)\ . \end{aligned} \tag{7.18}$$

Here $U = (u_0;\ u_1)$ and $V = (v_0;\ v_1)$ are elements from the corresponding subspace $\mathscr{H}_i$. Using (7.18) we define a new inner product and a norm in $\mathscr{H}$ by the equalities:

$$\langle U,\ V\rangle = \langle U_1,\ V_1\rangle_1 + \langle U_2,\ V_2\rangle_2, \qquad |U| = \langle U,\ U\rangle^{1/2},$$

where $U = U_1 + U_2$ and $V = V_1 + V_2$ are decompositions of the elements U and V into the orthogonal terms $V_i,\ U_i \in \mathscr{H}_i,\ \ i = 1,\ 2$.

Lemma 7.1.

The estimates

$$|U|_1 \ge \frac{1}{\mu_N^\theta}\sqrt{\varepsilon^2-\mu_N}\,\|A^\theta u_0\|, \qquad U=(u_0;\ u_1)\in\mathcal{H}_1; \tag{7.19}$$

$$|U|_2 \ge \frac{1}{\mu_{N+1}^\theta}\,\delta_{N,\,\varepsilon}\|A^\theta u_0\|, \qquad U=(u_0;\ u_1)\in\mathcal{H}_2 \tag{7.20}$$

hold for $0\le\theta\le 1/2$. *Here*

$$\delta_{N,\,\varepsilon}=\sqrt{\mu_{N+1}}\,\min\left(1,\ \sqrt{\frac{\varepsilon^2-\mu_{N+1}}{\mu_{N+1}}}\right). \tag{7.21}$$

Proof.

Let $U=(u_0;\ u_1)\in\mathcal{H}_1$. It is evident that in this case $\|A^\beta u_0\|\le\mu_N^\beta\|u_0\|$ for any $\beta>0$. Therefore,

$$|U|_1^2\ \ge\ \varepsilon^2\|u_0\|^2-\|A^{1/2}u_0\|^2\ \ge\ \mu_N^{-2\theta}(\varepsilon^2-\mu_N)\|A^\theta u_0\|^2,$$

i.e. equation (7.19) holds. Let $U\in\mathcal{H}_2$. Then using the inequality

$$\|A^\beta u_0\|\ \ge\ \mu_{N+1}^\beta\|u_0\|, \qquad \beta>0, \qquad u_0\in\overline{\mathrm{Lin}\{e_k:\ k\ge N+1\}} \tag{7.22}$$

for $0<\delta\le 1$ we find that

$$|U|_2^2\ \ge\ \delta^2\|A^{1/2}u_0\|^2+(\varepsilon^2-(1+\delta^2)\mu_{N+1})\|u_0\|^2.$$

If we take $\delta=\delta_{N,\,\varepsilon}\,\mu_{N+1}^{-1/2}$ and use (7.22), then we obtain estimate (7.20). The lemma is proved.

In particular, this lemma implies the estimate

$$\|A^\theta u_0\|\le\mu_{N+1}^\theta\,\delta_{N,\,\varepsilon}^{-1}|U| \tag{7.23}$$

for any $U=(u_0;\ u_1)\in\mathcal{H}$, where $0\le\theta\le 1/2$ and $\delta_{N,\,\varepsilon}$ has the form (7.21).

— Exercise *7.13* Prove the equivalence of the norm $|\cdot|$ and the norm generated by the inner product (7.4).

— Exercise *7.14* Show that we can take $\delta_{N,\varepsilon}=\sqrt{\varepsilon^2-\mu_{N+1}}$ for $\theta=0$ in (7.20) and (7.23).

— Exercise *7.15* Prove that the eigenvectors $\{f_k^\pm\}$ of the operator $\mathbf{A}$ (see (7.7)) possess the following orthogonal properties:

$$\langle f_n^+,f_k^+\rangle=\langle f_n^-,f_k^-\rangle=\langle f_n^+,f_k^-\rangle=0, \qquad k\ne n,$$

$$\langle f_k^+,\ f_k^-\rangle=0, \qquad 1\le k\le N. \tag{7.24}$$

Note that the last of these equations is one of the reasons of introducing a new inner product.

Let $P_{\mathcal{H}_i}$ be the orthoprojector onto the subspace $\mathcal{H}_i$ in $\mathcal{H}$, $i = 1, 2$.

Lemma 7.2.

The equality

$$\left|e^{-\mathrm{A}t} P_{\mathcal{H}_2}\right| = e^{-\bar{\lambda}_{N+1} t}, \qquad t \geq 0, \tag{7.25}$$

is valid. Here $|\cdot|$ *is the operator norm which is induced by the corresponding vector norm.*

Proof.

Let $U \in \mathcal{H}_2$. We consider the function $\psi(t) = |e^{-\mathrm{A}t} U|^2$. Since $\mathcal{H}_2$ is invariant with respect to $e^{-\mathrm{A}t}$, the equation

$$\psi(t) = (Au(t),\ u(t)) + (\varepsilon^2 - 2\mu_{N+1})(u(t),\ u(t)) + \|\dot{u} + \varepsilon u\|^2$$

holds, where $u(t)$ is a solution to problem (7.8) for $h(t) \equiv 0$. After simple calculations we obtain that

$$\frac{d\psi}{dt} + 2\varepsilon\psi = 4(\varepsilon^2 - \mu_{N+1})(\dot{u} + \varepsilon u,\ u).$$

It is evident that

$$2\sqrt{\varepsilon^2 - \mu_{N+1}}\,(\dot{u} + \varepsilon u,\ u) \leq (\varepsilon^2 - \mu_{N+1})\|u\|^2 + \|\dot{u} + \varepsilon u\|^2 \leq \psi(t).$$

Therefore,

$$\frac{d\psi}{dt} + 2\varepsilon\psi \leq 2\sqrt{\varepsilon^2 - \mu_{N+1}}\ \psi.$$

Consequently,

$$\psi(t) \leq e^{-\bar{\lambda}_{N+1} t}\ \psi(0), \qquad t > 0. \tag{7.26}$$

If we now notice that

$$\exp\{-\mathrm{A}t\} f_{N+1}^{-} = e^{-\bar{\lambda}_{N+1} t} f_{N+1}^{-},$$

then equation (7.26) implies (7.25). Thus, Lemma 7.2 is proved.

Let us consider the subspaces

$$\mathcal{H}_1^{\pm} = \mathrm{Lin}\{f_k^{\pm} \ :\ k \leq N\}.$$

Equation (7.24) gives us that the subspaces are orthogonal to each other and therefore $\mathcal{H}_1 = \mathcal{H}_1^{+} \oplus \mathcal{H}_1^{-}$. Using (7.24) it is easy to prove (do it yourself) that

$$\left|e^{\mathrm{A}t}P_{\mathcal{H}_1^-}\right| \le e^{\lambda_N^- |t|}, \qquad t \in \mathbb{R}, \tag{7.27}$$

$$\left|e^{-\mathrm{A}t}P_{\mathcal{H}_1^+}\right| \le e^{-\lambda_N^+ |t|}, \qquad t > 0. \tag{7.28}$$

We use the following pair of orthogonal (with respect to the inner product $\langle \cdot, \cdot \rangle$) projectors in the space $\mathcal{H}$

$$P = P_{\mathcal{H}_1^-}, \qquad Q = I - P = P_{\mathcal{H}_1^+} + P_{\mathcal{H}_2}$$

to construct the inertial manifold of problem (7.1) (or (7.5)). Lemma 7.2 and equations (7.27) and (7.28) imply the dichotomy equations

$$\left|e^{\mathrm{A}t}P\right| \le e^{\lambda_N^- |t|}, \quad t \in \mathbb{R}; \qquad \left|e^{-\mathrm{A}t}Q\right| \le e^{-\lambda_{N+1}^- |t|}, \quad t > 0. \tag{7.29}$$

We remind that $\lambda_N^- = \varepsilon - \sqrt{\varepsilon^2 - \mu_k}$ and $\varepsilon^2 > \mu_{N+1}$.

The assertion below plays an important role in the estimates to follow.

Lemma 7.3.

Let $\mathrm{B}(U, t) = (0; B(u_0, t))$, *where* $U = (u_0; u_1) \in \mathcal{H}$ *and* $B(u_0)$ *possesses properties* (7.2). *Then*

$$\left|\mathrm{B}(U, t)\right| \le M_0 + K_N|U|, \qquad U \in \mathcal{H},$$

$$\left|\mathrm{B}(U_1, t) - \mathrm{B}(U_2, t)\right| \le K_N\left|U_1 - U_2\right|, \qquad U_1, U_2 \in \mathcal{H}, \tag{7.30}$$

where

$$K_N = M_1 \mu_{N+1}^{\theta - 1/2} \max\left(1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}}\right). \tag{7.31}$$

The proof of this lemma follows from the structure of the mapping $\mathrm{B}(U, t)$ and from estimates (7.2) and (7.23).

Exercise *7.16* Show that one can take $K_N = M_1(\varepsilon^2 - \mu_{N+1})^{-1/2}$ for $\theta = 0$ in (7.30) (*Hint*: see Exercise 7.14).

Let us now consider the integral equation (cf. (2.1) for $L = \infty$)

$$V(t) = \mathfrak{B}_p^s[V](t) \equiv$$

$$\equiv e^{-(t-s)\mathrm{A}}p - \int_t^s e^{-(t-\tau)\mathrm{A}}P\mathrm{B}(V(\tau), \tau)\,d\tau + \int_{-\infty}^t e^{-(t-\tau)\mathrm{A}}Q\mathrm{B}(V(\tau), \tau)\,d\tau \tag{7.32}$$

in the space C_s of continuous vector-functions $U(t)$ on $(-\infty, s]$ with the values in $\mathcal{H}$ such that the norm

$$\| U \| \equiv \sup_{t<s} e^{-\gamma(s-t)}|U(t)| < \infty\,, \qquad \gamma = \frac{1}{2}(\lambda^-_{N+1} + \lambda^-_N)\,,$$

is finite. Here $p \in P\mathcal{H}$ and $t \in (-\infty, s)$.

Exercise *7.17* Show that the right-hand side of equation (7.32) is a continuous function of the variable t with the values in $\mathcal{H}$.

Lemma 7.4.

The operator $\mathfrak{B}^s_p$ maps the space C_s into itself and possesses the properties

$$\| \mathfrak{B}^s_p[V] \| \le |p| + M_0\left(\frac{1}{\lambda^-_N} + \frac{1}{\lambda^-_{N+1}}\right) + \frac{4K_N}{\lambda^-_{N+1} - \lambda^-_N} \| V \| \tag{7.33}$$

and

$$\| \mathfrak{B}^s_p[V_1] - \mathfrak{B}^s_p[V_2] \| \le \frac{4K_N}{\lambda^-_{N+1} - \lambda^-_N} \| V_1 - V_2 \|\,. \tag{7.34}$$

Proof.

Let us prove (7.34). Evidently, equations (7.29) and (7.30) imply that

$$\left|\mathfrak{B}^s_p[V_1(t)] - \mathfrak{B}^s_p[V_2(t)]\right| \le K_N \int_t^s e^{\lambda^-_N(\tau - t)}\left|V_1(\tau) - V_2(\tau)\right| d\tau + $$
$$+ K_N \int_{-\infty}^t e^{-\lambda^-_{N+1}(t-\tau)}\left|V_1(\tau) - V_2(\tau)\right| d\tau\,.$$

Since

$$\left|V_1(\tau) - V_2(\tau)\right| \le e^{\gamma(s-\tau)} \| V_1 - V_2 \|\,,$$

it is evident that

$$\left|\mathfrak{B}^s_p[V_1](t) - \mathfrak{B}^s_p[V_2](t)\right| \le q\, e^{\gamma(s-t)} \| V_1 - V_2 \|$$

with

$$q = K_N\left\{\int_t^s e^{(\lambda^-_N - \gamma)(\tau - t)} d\tau + \int_{-\infty}^t e^{-(\lambda^-_{N+1} - \gamma)(t-\tau)} d\tau\right\}.$$

Simple calculations show that $q \le 4K_N(\lambda^-_{N+1} - \lambda^-_N)^{-1}$. Consequently, equation (7.34) holds. Equation (7.33) can be proved similarly. Lemma 7.4 is proved.

Thus, if for some $q < 1$ the condition

$$\lambda^-_{N+1} - \lambda^-_N \geq \frac{4K_N}{q} \tag{7.35}$$

holds, then equation (7.32) is uniquely solvable in C_s and its solution V can be estimated as follows:

$$\| V \| \leq (1-q)^{-1}\left(|p| + M_0\left(\frac{1}{\lambda^-_N} + \frac{1}{\lambda^-_{N+1}}\right)\right). \tag{7.36}$$

Therefore, we can define a collection of manifolds $\{\mathrm{M}_s\}$ in the space $\mathcal{H}$ by the formula

$$\mathrm{M}_s = \{p + \Phi(p, s) \colon p \in P\mathcal{H}\}, \tag{7.37}$$

where

$$\Phi(p, s) = \int_{-\infty}^{s} e^{-(t-\tau)\mathrm{A}} Q \mathrm{B}(V(\tau), \tau)\, d\tau . \tag{7.38}$$

Here $V(\tau)$ is a solution to integral equation (7.32). The main result of this section is the following assertion.

Theorem 7.1.

Assume that

$$\varepsilon^2 > \mu_{N+1} \quad \textit{and} \quad \lambda^-_{N+1} - \lambda^-_N \geq \frac{4K_N}{q} \tag{7.39}$$

for some $0 < q < 1$, ***where*** $\lambda^-_k = \varepsilon - \sqrt{\varepsilon^2 - \mu_k}$ ***and*** K_N ***is defined by formula*** (7.31). ***Then the function*** $\Phi(p, s)$ ***given by equality*** (7.38) ***satisfies the Lipschitz condition***

$$|\Phi(p_1, s) - \Phi(p_2, s)| \leq \frac{q}{2(1-q)} |p_1 - p_2| \tag{7.40}$$

and the manifold M_s ***is invariant with respect to the evolutionary operator*** $S(t, \tau)$ ***generated by the formula***

$$S(t, \tau)U_0 = (u(t); \dot{u}(t)), \qquad t \geq s,$$

in $\mathcal{H}$, ***where*** $u(t)$ ***is a solution to problem*** (7.1) ***with the initial condition*** $U_0 = (u_0; u_1)$. ***Moreover, if*** $0 < q < 2 - \sqrt{2}$, ***then there exist initial conditions*** $U_0^* = (u_0^*; u_1^*) \in \mathrm{M}_s$ ***such that***

$$|S(t, s)U_0 - S(t, s)U_0^*| \leq C_q e^{-\gamma(t-s)} |QU_0 - \Phi(PU_0, s)|$$

for $t \geq s$, ***where*** $\gamma = \frac{1}{2}(\lambda^-_N + \lambda^-_{N+1})$.

The proof of the theorem is based on Lemma 7.4 and estimates (7.29) and (7.30). It almost entirely repeats the corresponding reasonings in Sections 2 and 3. We give the reader an oppotunity to recover the details of the reasonings as an exercise.

Let us analyse condition (7.39). Equation (7.31) implies that (7.39) holds if

$$\varepsilon^2 \ge 2\mu_{N+1}, \qquad \frac{\mu_{N+1}-\mu_N}{2\sqrt{\varepsilon^2-\mu_N}} \ge \frac{4}{q} M_1\, \mu_{N+1}^{\theta-1/2}. \tag{7.41}$$

However, if we assume that

$$\mu_{N+1}-\mu_N \ge \frac{8\sqrt{2}}{q} M_1\, \mu_{N+1}^{\theta}, \tag{7.42}$$

then for condition (7.41) to be fulfilled it is sufficient to require that

$$2\mu_{N+1} \le \varepsilon^2 \le 2\mu_{N+1}+\mu_N. \tag{7.43}$$

Thus, if for some N conditions (7.42) and (7.43) hold, then the assertions of Theorem 7.1 are valid for system (7.1). This enables us to formulate the assertion on the existence of IM as follows.

Theorem 7.2.

Assume that the eigenvalues μ_N ***of the operator*** A ***possess the properties***

$$\inf_N \frac{\mu_N}{\mu_{N+1}} > 0 \quad \textbf{\textit{and}} \quad \mu_{N(k)+1} = c_0 k^{\rho}(1+o(1)), \quad \rho > 0, \quad k \to \infty, \tag{7.44}$$

for some sequence $\{N(k)\}$ ***which tends to infinity and satisfies the estimate***

$$\mu_{N(k)+1}-\mu_{N(k)} \ge \frac{8\sqrt{2}}{q} M_1\, \mu_{N(k)+1}^{\theta}, \qquad 0 < q < 2-\sqrt{2}\,.$$

Then there exists $\varepsilon_0 > 0$ ***such that the assertions of Theorem*** 7.1 ***hold for all*** $\varepsilon \ge \varepsilon_0$.

Proof.

Equation (7.44) implies that there exists k_0 such that the intervals

$$[2\mu_{N(k)+1},\ 2\mu_{N(k)+1}+\mu_{N(k)}], \qquad k \ge k_0,$$

cover some semiaxis $[\varepsilon_0, +\infty)$. Indeed, otherwise there would appear a subsequence $\{N(k_j)\}$ such that

$$\mu_{N(k_j)} < 2(\mu_{N(k_j+1)+1} - \mu_{N(k_j)+1})$$

But that is impossible due to (7.44). Consequently, for any $\varepsilon \ge \varepsilon_0$ there exists $N = N_\varepsilon$ such that equations (7.42), (7.43) as well as (7.39) hold.

Exercise *7.18* Consider problem (7.3) with the function $f(x, t, u, \frac{\partial u}{\partial x}) = f(x, t, u)$ possessing the property

$$|f(x, t, u_1) - f(x, t, u_2)| \le L|u_1 - u_2|.$$

Use Theorem 7.1 to find a domain in the plane of the parameters (ε, L) for which one can guarantee the existence of an inertial manifold.

§ 8 Approximate Inertial Manifolds for Second Order in Time Equations

As seen from the results of Section 7, in order to guarantee the existence of IM for a problem of the type

$$\frac{d^2u}{dt^2} + \gamma \frac{du}{dt} + A\,u = B(u)\ ,$$
$$u\big|_{t=0} = u_0\,, \quad \frac{du}{dt}\bigg|_{t=0} = u_1\ , \tag{8.1}$$

we have to require that the parameter $\gamma = 2\varepsilon > 0$ be large enough and the spectral gap condition (see (7.41)) be valid for the operator A. Therefore, as in the case with parabolic equations there arises a problem of construction of an approximate inertial manifold without any assumptions on the behaviour of the spectrum of the operator A and the parameter $\gamma > 0$ which characterizes the resistance force.

Unfortunately, the approach presented in Section 6 is not applicable to the equation of the type (8.1) without any additional assumptions on γ. First of all, it is connected with the fact that the regularizing effect which takes place in the case of parabolic equations does not hold for second order equations of the type (8.1) (in the parabolic case the solution at the moment $t > 0$ is smoother than its initial condition).

In this section (see also [17]) we suggest an iteration scheme that enables us to construct an approximate IM as a solution to a class of linear problems. For the sake of simplicity, we restrict ourselves to the case of autonomous equations $(B(u, t) \equiv \equiv B(u))$. The suggested scheme is based on the equation in functional derivatives such that the function giving the original true IM should satisfy it. This approach was developed for the parabolic equation in [9] (see also [8]). Unfortunately, this approach has two defects. First, approximate IMs have the power order (not the exponential one as in Section 6) and, second, we cannot prove the convergence of approximate IMs to the exact one when the latter exists.

Thus, in a separable Hilbert space H we consider a differential equation of the type (8.1) where γ is a positive number, A is a positive selfadjoint operator with discrete spectrum and $B(\cdot)$ is a nonlinear mapping from the domain $D(A^{1/2})$ of the operator $A^{1/2}$ into H such that for some integer $m \ge 2$ the function $B(u)$ lies in C^m as a mapping from $D(A^{1/2})$ into H and for every $\rho > 0$ the following estimates hold:

$$\left\|\langle B^{(k)}(u);\ w_1,\ \dots,\ w_k\rangle\right\| \ \le\ C_\rho \prod_{j=1}^{k}\left\|A^{1/2} w_j\right\|\ , \tag{8.2}$$

$$\left\| \langle B^{(k)}(u) - B^{(k)}(u^*);\ w_1, \ldots, w_k \rangle \right\| \le C_\rho \|A^{1/2}(u-u^*)\| \prod_{j=1}^{k} \left\| A^{1/2} w_j \right\|, \quad (8.3)$$

where $k = 0, 1, \ldots, m$, $\|\cdot\|$ is a norm in the space H, $\|A^{1/2}u\| \le \rho$, $\|A^{1/2}u^*\| \le \rho$, and $w_j \in D(A^{1/2})$. Here $B^{(k)}(u)$ is the Frechét derivative of the order k of $B(u)$ and $\langle B^{(k)}(u);\ w_1, \ldots, w_k \rangle$ is its value on the elements $w_1, \ldots, w_k$.

Let $L_{m,R}$ be a class of solutions to problem (8.1) possessing the following properties of regularity:

I) for $k = 0, 1, \ldots, m-1$ and for all $T > 0$

$$u^{(k)}(t) \in C(0, T; D(A))$$

and

$$u^{(m)}(t) \in C(0, T; D(A^{1/2})), \qquad u^{(m+1)}(t) \in C(0, T; H),$$

where $C(0, T; V)$ is the space of strongly continuous functions on $[0, T]$ with the values in V, hereinafter $u^{(k)}(t) = \partial_t^k u(t)$;

II) for any $u \in L_{m,R}$ the estimate

$$\|u^{(k+1)}(t)\|^2 + \|A^{1/2}u^{(k)}(t)\|^2 + \|Au^{(k-1)}(t)\|^2 \le R^2 \quad (8.4)$$

holds for $k = 1, \ldots, m$ and for $t \ge t^*$, where t^* depends on u_0 and u_1 only.

In fact, the classes $L_{m,R}$ are studied in [18]. This paper contains necessary and sufficient conditions which guarantee that a solution belongs to a class $L_{m,R}$. It should be noted that in [18] the nonlinear wave equation of the type

$$\begin{aligned} &\partial_t^2 u + \gamma \partial_t u - \Delta u + g(u) = f(x), \quad x \in \Omega,\ t > 0, \\ &u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x), \end{aligned} \quad (8.5)$$

serves as the main example. Here $\gamma > 0$, $f(x) \in C^\infty(\overline{\Omega})$ and the conditions set on the function $g(s)$ from $C^\infty(\mathbb{R})$ are such that we can take $g(u) = \sin u$ or $g(u) = u^{2p+1}$, where $p = 0, 1, 2, \ldots$ for $d = \dim \Omega \le 2$ and $p = 0, 1$ for $d = 3$. In this example the classes $L_{m,R}$ are nonempty for all m. Other examples will be given in Chapter 4.

We fix an integer N and assume $P = P_N$ to be the projector in H onto the subspace generated by the first N eigenvectors of the operator A. Let $Q = I - P$. If we apply the projectors P and Q to equation (8.1), then we obtain the following system of two equations for $p(t) = PU(t)$ and $q(t) = Qu(t)$:

$$\begin{aligned} &\partial_t^2 p + \gamma \partial_t p + Ap = PB(p+q), \\ &\partial_t^2 q + \gamma \partial_t q + Aq = QB(p+q). \end{aligned} \quad (8.6)$$

The reasoning below is formal. Its goal is to obtain an iteration scheme for the determination of an approximate IM. We assume that system (8.6) has an invariant manifold of the form

$$\mathbf{M} = \{(p + h(p, \dot{p});\ \dot{p} + l(p, \dot{p})):\ p, \dot{p} \in PH\} \quad (8.7)$$

in the phase space $D(A^{1/2}) \times H$. Here h and l are smooth mappings from $PH \times PH$ into $QD(A)$. If we substitute $q(t) = h(p(t),\ \partial_t p(t))$ and $\partial_t q(t) = l(p(t),\ \partial_t p(t))$ in the second equality of (8.6), then we obtain the following equation:

$$\langle \delta_p l;\ \dot{p}\rangle + \langle \delta_{\dot{p}} l;\ -\gamma\dot{p} - Ap + PB(p + h(p,\ \dot{p}))\rangle +$$
$$+\gamma l(p,\ \dot{p}) + Ah(p,\ \dot{p}) = QB(p + h(p,\ \dot{p}))\ .$$

The compatibility condition

$$l(p(t),\ \partial_t p(t)) = \partial_t h(p(t),\ \partial_t p(t))$$

gives us that

$$l(p,\ \dot{p}) = \langle \delta_p h;\ \dot{p}\rangle + \langle \delta_{\dot{p}} h;\ -\gamma\dot{p} - Ap + PB(p + h(p,\ \dot{p}))\rangle\ .$$

Hereinafter $\delta_p f$ and $\delta_{\dot{p}} f$ are the Frechét derivatives of the function $f(p, \dot{p})$ with respect to p and $\dot{p}$; $\langle \delta_p f;\ w\rangle$ and $\langle \delta_{\dot{p}} f;\ w\rangle$ are values of the corresponding derivatives on an element w.

Using these formal equations, we can suggest the following iteration process to determine the class of functions $\{h_k;\ l_k\}$ giving the sequence of approximate IMs with the help of (8.7):

$$Ah_k(p,\ \dot{p}) = QB(p + h_{k-1}(p,\ \dot{p})) - \gamma l_{\nu(k)}(p,\ \dot{p}) - \langle \delta_p l_{k-1};\ \dot{p}\rangle -$$
$$- \langle \delta_{\dot{p}} l_{k-1};\ -\gamma\dot{p} - Ap + PB(p + h_{k-1}(p,\ \dot{p}))\rangle\ , \tag{8.8}$$

where $k = 1,\ 2,\ 3,\ \ldots$ and the integers $\nu(k)$ should be choosen such that $k - 1 \le$ $\le \nu(k) \le k$. Here $l_k(p,\ \dot{p})$ is defined by the formula

$$l_k(p,\ \dot{p}) = \langle \delta_p h_{k-1};\ \dot{p}\rangle + \langle \delta_{\dot{p}} h_{k-1};\ -\gamma\dot{p} - Ap + PB(p + h_{k-1}(p,\ \dot{p}))\rangle\ , \tag{8.9}$$

where $k = 1,\ 2,\ 3,\ \ldots$. We also assume that

$$h_0(p,\ \dot{p}) \equiv l_0(p,\ \dot{p}) \equiv 0\ . \tag{8.10}$$

Exercise *8.1* Find the form of $h_1(p,\ \dot{p})$ and $l_1(p,\ \dot{p})$ for $\nu(1) = 0$ and for $\nu(1) = 1$.

The following assertion contains information on the smoothness properties of the functions h_n and l_n which will be necessary further.

Theorem 8.1.

Assume that the class of functions $\{h_n;\ l_n\}$ ***is defined according to*** (8.8)–(8.10). ***Then for each*** n ***the functions*** h_n ***and*** l_n ***belong to the class*** C^m ***as mappings from*** $PH \times PH$ ***into*** QH ***and for all integers*** $\alpha,\ \beta \ge 0$ ***such that*** $\alpha + \beta \le m$ ***the estimates***

$$\left\|A\langle D^{\alpha,\,\beta} h_n(p,\,\dot p);\;w_1,\,\dots,\,w_\alpha;\;\dot w_1,\,\dots,\,\dot w_\beta\rangle\right\| \le$$

$$\le C_{\alpha,\,\beta,\,R}\prod_{i=1}^{\alpha}\|Aw_i\|\cdot\prod_{i=1}^{\beta}\|A^{1/2}\dot w_i\|\,, \qquad (8.11)$$

$$\left\|A^{1/2}\langle D^{\alpha,\,\beta} l_n(p,\,\dot p);\;\;w_1,\,\dots,\,w_\alpha;\;\;\dot w_1,\,\dots,\,\dot w_\beta\rangle\right\| \le$$

$$\le C_{\alpha,\,\beta,\,R}\prod_{i=1}^{\alpha}\|Aw_i\|\cdot\prod_{i=1}^{\beta}\|A^{1/2}\dot w_i\| \qquad (8.12)$$

are valid for all p ***and*** $\dot p$ ***from*** PH ***such that*** $\|Ap\|\le R$ ***and*** $\|A^{1/2}\dot p\|\le R$. ***Hereinafter*** $D^{\alpha,\,\beta}f$ ***is the mixed Frechét derivative of the function*** f ***of the order*** α ***with respect to*** p ***and of the order*** β ***with respect to*** $\dot p$***; the values*** w_j ***and*** $\dot w_j$ ***are from*** PH. ***Moreover, if*** $\alpha=0$ ***or*** $\beta=0$***, then the corresponding products in*** (8.11) ***and*** (8.12) ***should be omitted.***

Proof.

We use induction with respect to n. It follows from (8.10) and (8.2) that estimates (8.11) and (8.12) are valid for $n=0,\ 1$. Assume that (8.11) and (8.12) hold for all $n\le k-1$. Then the following lemma holds.

Lemma 8.1.

Let $F_\nu(p,\,\dot p)=B(p+h_\nu(p,\,\dot p))$ *and let*

$$F_\nu^{\alpha,\,\beta}(w)=\langle D^{\alpha,\,\beta}F_\nu(p,\,\dot p);\;w_1,\,\dots,\,w_\alpha;\;\dot w_1,\,\dots,\,\dot w_\beta\rangle.$$

Then for $\nu\le k-1$ *and for all integers* $\alpha,\ \beta\ge 0$ *such that* $\alpha+\beta\le m$ *the estimate*

$$F_\nu^{\alpha,\,\beta}(w)\;\le\;C\prod_{i=1}^{\alpha}\|Aw_i\|\cdot\prod_{i=1}^{\beta}\|A^{1/2}\dot w_i\| \qquad (8.13)$$

holds, where $w_j,\ \ \dot w_j,\ \ p,\ \ \dot p\in PH$ *and* $\|Ap\|\le R,\ \|A^{1/2}\dot p\|\le R$.

Proof.

It is evident that $F_\nu^{\alpha,\,\beta}(w)$ is the sum of terms of the type

$$B_\nu^s(y)=\langle B^{(s)}(p+h_\nu(p,\,\dot p));\;\;y_1,\,\dots,\,y_s\rangle\,,\qquad s\ge 0\,.$$

Here y_σ is one of the values of the form:

$$y_*=w_\sigma+\langle\delta_p h_\nu;\;\;w_\sigma\rangle\,,$$

$$y_{**}=\langle D^{\sigma,\,\tau}h_\nu;\;\;w_1,\,\dots,\,w_\alpha;\;\;\dot w_1,\,\dots,\,\dot w_\beta\rangle\,.$$

Equation (8.2) implies that

$$\left\|B_\nu^s(y)\right\| \le C_R \prod_{j=1}^{s} \left\|A^{1/2} y_j\right\| .$$

Therefore, the induction hypothesis gives us (8.13).

Let us prove (8.12). The induction hypothesis implies that it is sufficient to estimate the derivatives of the second term in the right-hand side of (8.9). It has the form

$$\langle \delta_{\dot p} h_{k-1};\ \ D_k(p,\, \dot p)\rangle \,, \tag{8.14}$$

where

$$D_k(p,\, \dot p) = -\gamma \dot p - A p + PB(p + h_{k-1}(p,\, \dot p)) .$$

The Frechét derivatives of value (8.14)

$$\langle D^{\alpha,\,\beta} \langle \delta_{\dot p} h_{k-1};\ D_k(p,\, \dot p)\rangle;\ w_1,\, \ldots,\, w_\alpha;\ \dot w_1,\, \ldots,\, \dot w_\beta\rangle$$

are sums of the terms of the type

$$G(\sigma,\, \tau) \equiv \langle D^{\sigma,\, \tau+1} h_{k-1}(p,\, \dot p);\ w_{j_1},\, \ldots,\, w_{j_\sigma};\ \dot w_{i_1},\, \ldots,\, \dot w_{i_\tau},\ y_{\sigma,\,\tau}\rangle \,,$$

where

$$y_{\sigma,\,\tau} = \langle D^{\alpha-\sigma,\ \beta-\tau} D_k(p,\, \dot p);\ w_{\omega_1},\, \ldots,\, w_{\omega_{\alpha-\sigma}};\ \dot w_{\rho_1},\, \ldots,\, \dot w_{\rho_{\beta-\tau}}\rangle .$$

Here $0 \le \sigma \le \alpha$, $0 \le \tau \le \beta$ and the sets of indices possess the following properties:

$$\{j_1,\, \ldots,\, j_\sigma\} \cap \{\omega_1,\, \ldots,\, \omega_{\alpha-\sigma}\} = \varnothing \,,$$

$$\{j_1,\, \ldots,\, j_\sigma\} \cup \{\omega_1,\, \ldots,\, \omega_{\alpha-\sigma}\} = \{1,\ 2,\ \ldots,\ \alpha\} ;$$

$$\{i_1,\, \ldots,\, i_\tau\} \cap \{\rho_1,\, \ldots,\, \rho_{\beta-\tau}\} = \varnothing \,,$$

$$\{i_1,\, \ldots,\, i_\tau\} \cup \{\rho_1,\, \ldots,\, \rho_{\beta-\tau}\} = \{1,\ 2,\ \ldots,\ \beta\} .$$

The induction hypothesis implies that

$$\|AG(\sigma,\, \tau)\| \le C \prod_{\theta=1}^{\sigma} \left\|A w_{j_\theta}\right\| \cdot \prod_{\theta=1}^{\tau} \left\|A^{1/2} \dot w_{i_\theta}\right\| \cdot \left\|A^{1/2} y_{\sigma,\,\tau}\right\| .$$

Using the induction hypothesis again as well as Lemma 8.1 and the inequality

$$\left\|A^{1/2} P h\right\| \le \lambda_N^{1/2} \|Ph\| \,,$$

we obtain an estimate of the following form (if $\sigma = \alpha$ or $\tau = \beta$, then the corresponding product should be considered to be equal to 1):

$$\left\|A^{1/2} y_{\sigma,\,\tau}\right\| \le C(1 + \lambda_N^{1/2}) \prod_{\theta=1}^{\alpha-\sigma} \left\|A w_{\omega_\theta}\right\| \cdot \prod_{\theta=1}^{\beta-\tau} \left\|A^{1/2} \dot w_{\rho_\theta}\right\| .$$

Hereinafter λ_k is the k-th eigenvalue of the operator A. Thus, it is possible to state that

$$\|AG(\sigma,\tau)\| \le C(1+\lambda_N^{1/2})\prod_i \|Aw_i\| \cdot \prod_i \|A^{1/2}\dot{w}_i\| . \qquad (8.15)$$

Using the inequality

$$\|Qh\| \le \lambda_{N+1}^{-s}\|A^s Qh\| , \qquad s>0 , \qquad (8.16)$$

and equation (8.15) it is easy to find that estimates (8.12) are valid for $n=k$. If we use (8.8), (8.12) and follow a similar line of reasoning, we can easily obtain (8.11). **Theorem 8.1 is proved**.

Theorem 8.1 and equation (8.4) imply the following lemma.

Lemma 8.2.

Assume that $u(t)$ is a solution to problem (8.1) *lying in $L_{m,R}$,* $m \ge 1$. *Let $p(t)=Pu(t)$ and let*

$$q_s(t) = h_s(p(t), \partial_t p(t)) , \qquad \bar{q}_s(t) = l_s(p(t), \partial_t p(t)) . \qquad (8.17)$$

Then the estimates

$$\left\|A^{1/2}\bar{q}_s^{(j)}(t)\right\|^2 + \left\|Aq_s^{(j)}(t)\right\|^2 \le C_{R,m}$$

with $0 \le j \le m-1$ *and*

$$\left\|\bar{q}_s^{(m)}(t)\right\|^2 + \left\|A^{1/2}q_s^{(m)}(t)\right\|^2 \le C_{R,m}$$

are valid for t large enough.

Proof.

It should be noted that $q_s^{(j)}(t)$ is the sum of terms of the form

$$\langle D^{\alpha,\beta} h_s(p, \partial_t p), p^{(i_1)}(t), \ldots, p^{(i_\alpha)}(t); \;\; p^{(\tau_1+1)}(t), \ldots, p^{(\tau_\beta+1)}(t)\rangle ,$$

where α, β, $i_1, \ldots, i_\alpha$, $\tau_1, \ldots, \tau_\beta$ are nonnegative integers such that

$$1 \le \alpha+\beta \le j , \qquad i_1+\ldots+i_\alpha+\tau_1+\ldots+\tau_\beta = j .$$

Similar equation also holds for $\bar{q}_s^{(i)}(t)$. Further one should use Theorem 8.1 and the estimates

$$\|p^{(k+1)}(t)\|^2 + \|A^{1/2}p^{(k)}(t)\|^2 + \|Ap^{(n-1)}(t)\|^2 \le R^2 , \qquad t \ge t_* , \quad 1 \le k \le m ,$$

which follow from (8.4).

Let us define the induced trajectories of the system by the formula

$$U_s(t) = (u_s(t);\ \bar{u}_s(t)) ,$$

where $s = 0, 1, 2, \ldots$ and

$$u_s(t) = p(t) + q_s(t) , \qquad \bar{u}_s(t) = \partial_t p(t) + \bar{q}_s(t) . \qquad (8.18)$$

Here $p(t) = Pu(t)$, $u(t)$ is a solution to problem (8.1); $q_s(t)$ and $\bar{q}_s(t)$ are defined with the help of (8.17). Assume that $u(t)$ lies in $L_{m,\,R}$. Then Lemma 8.2 implies that the induced trajectories can be estimated as follows:

$$\|A^{1/2}\,\bar{u}_s^{(j)}(t)\| + \|Au_s^{(j)}(t)\| \le C_{R,\,s}\,, \qquad 0 \le j \le m-1\,;$$

$$\|\bar{u}_s^{(m)}(t)\|^2 + \|A^{1/2}u_s^{(m)}(t)\|^2 \le C_{R,\,s}$$

for t large enough. Using (8.3), (8.4), and the last estimates, it is easy to prove the following assertion (do it yourself).

Lemma 8.3.

Let

$$E_s(t) = B(p(t)+q(t)) - B(p(t)+q_s(t))\,.$$

Then

$$\|E_s^{(j)}(t)\| \le C_{R,\,j}\sum_{i=0}^{j}\|A^{1/2}(q^{(i)}(t) - q_s^{(i)}(t))\|$$

for $j = 0,\ 1,\ \ldots,\ m$ *and for* t *large enough.*

The main result of this section is the following assertion.

Theorem 8.2.

Let $u(t)$ ***be a solution to problem*** (8.1) ***lying in*** $L_{m,\,R}$ ***with*** $m \ge 2$. ***Assume that*** $h_n(p,\,\dot{p})$ ***and*** $l_n(p,\,\dot{p})$ ***are defined by*** (8.8)–(8.10). ***Then the estimates***

$$\|A\partial_t^j(u(t) - u_n(t))\| \le C_{n,\,R}\,\lambda_{N+1}^{-n/2}\,, \tag{8.19}$$

$$\|A^{1/2}\,\partial_t^j(\partial_t u(t) - \bar{u}_n(t))\| \le C_{n,\,R}\,\lambda_{N+1}^{-n/2}\,, \tag{8.20}$$

are valid for $n \le m-1$ ***and for*** t ***large enough. Here*** $0 \le j \le m-n-1$, $u_n(t)$ ***and*** $\bar{u}_n(t)$ ***are defined by*** (8.18)***, and*** λ_{N+1} ***is the*** $(N+1)$***-th eigenvalue of the operator*** A.

Proof.

Let us consider the difference between the solution $u(t)$ and the trajectory induced by this solution:

$$\chi_s(t) = u(t) - u_s(t)\,, \qquad \bar{\chi}_s(t) = \partial_t u(t) - \bar{u}_s(t)\,, \qquad s \ge 0\,,$$

where $\bar{u}_s(t)$ and $u_s(t)$ are defined by formula (8.18). Since $\chi_0(t) = q(t)$, equation (8.4) implies that

$$\|A^{1/2}\chi_0^{(j+1)}(t)\| + \|A\chi_0^{(j)}(t)\| \le C\,, \qquad j = 0,\ 1,\ 2,\ \ldots,\ m-1\,, \tag{8.21}$$

for t large enough. Equations (8.8)–(8.10) also give us that

$$A\chi_1(t) = -\chi_0''(t) - \gamma\chi_0'(t) + QE_0(t) .$$

We use Lemma 8.3 and equation (8.21) to find that

$$\left\| A\chi_1^{(j)}(t) \right\| \le C\lambda_{N+1}^{-1/2}, \quad j = 0,\ 1,\ \dots,\ m-2 ,$$

for t large enough. Therefore, equation (8.19) holds for $n = 0,\ 1$ and for t large enough. From equations (8.6), (8.8), and (8.9) it is easy to find that

$$A\chi_k = -\partial_t\bar{\chi}_{k-1} - \gamma\partial_t\bar{\chi}_{\nu(k)-1} - \gamma\langle\delta_{\dot{p}} h_{\nu(k)-1};\ PE_{\nu(k)-1}\rangle - \\ - \langle\delta_{\dot{p}} l_{k-1};\ PE_{k-1}\rangle + QE_{k-1}$$

and

$$\bar{\chi}_k = \partial_t\chi_{k-1} + \langle\delta_{\dot{p}} h_{k-1};\ PE_{k-1}\rangle . \tag{8.22}$$

Lemma 8.4.

The estimates

$$\left\| A\,\partial_t^j\langle\delta_{\dot{p}} h_\nu;\ PE_\nu\rangle \right\| \le C\lambda_N^{1/2}\sum_{s=0}^{j}\left\| A^{1/2}\chi_\nu^{(s)}(t) \right\| \tag{8.23}$$

and

$$\left\| A^{1/2}\,\partial_t^j\langle\delta_{\dot{p}} l_\nu;\ PE_\nu\rangle \right\| \le C\lambda_N^{1/2}\sum_{s=0}^{j}\left\| A^{1/2}\chi_\nu^{(s)}(t) \right\| \tag{8.24}$$

are valid for t large enough and for each $\nu \ge 0$, where $j = 0, 1, \dots, m-1$.

Proof.

Let $f_\nu = h_\nu$ or $f_\nu = l_\nu$. It is clear that the value $\partial_t^j\langle\delta_{\dot{p}} f_\nu;\ PE_\nu\rangle$ is the algebraic sum of terms of the form:

$$\langle D^{\alpha,\,\beta+1} f_\nu;\ p^{\gamma_1}, \dots, p^{\gamma_\alpha};\ p^{\sigma_1}, \dots, p^{\sigma_\beta},\ \partial_t^s PE_\nu\rangle .$$

Therefore, Theorem 8.1 and Lemma 8.3 imply (8.23) and (8.24). Lemma 8.4 is proved.

We use Lemmata 8.3 and 8.4 as well as inequality (8.16) to obtain that

$$\left\| A\chi_k^{(j)}(t) \right\| \le c_{k,j}\,\lambda_{N+1}^{-1}\left\{ \sum_{s=0}^{j+2}\left\| A\chi_{k-1}^{(s)}(t) \right\| + \sum_{s=0}^{j+1}\left\| A\chi_{\nu(k)-1}^{(s)}(t) \right\| \right\} + \\ + d_{kj}\,\lambda_{N+1}^{-1/2}\sum_{s=0}^{j}\left\| A\chi_{k-1}^{(s)}(t) \right\| , \tag{8.25}$$

where $j = 0,\ 1,\ \dots,\ m-2$ and the numbers $c_{k,\,j}$ and $d_{k,\,j}$ do not depend on N.

If we now assume that (8.19) holds for $n \le k-1$, then equation (8.25) implies (8.19) for $n = k$ and for $k \le m-1$. Using (8.22) and (8.23) we obtain equation (8.20). **Theorem 8.2 is proved**.

Corollary 8.1

Let the manifold $\mathbf{M}_n$ *have the form* (8.7) *with* $h(p, \dot p) = h_n(p, \dot p)$ *and* $l(p, \dot p) = l_n(p, \dot p)$. *We also assume that* $U(t) = (u(t);\ \dot u(t))$, *where* $u(t)$ *is the solution to problem* (0.1) *from the class* $L_{m,\,R}$. *Then*

$$\operatorname{dist}_{D(A)\times D(A^{1/2})}(U(t), \mathbf{M}_n) \le C_n \lambda_{N+1}^{-n/2}, \qquad n = 0,\ 1,\ 2 \ldots m-1 .$$

Thus, the thickness of the layer that attracts the trajectories in the phase space has the power order with respect to λ_{N+1} unlike the semilinear parabolic equations of Section 6.

Example *8.1*

Let us consider the nonlinear wave equation (8.5). Let $d = \dim\Omega \le 2$. We assume the following (cf. [18]) about the function $g(s)$:

$$\lim_{|s| \to \infty} s^{-1}\int_0^s g(\sigma)\mathrm{d}\sigma \ge 0 ;$$

there exists $C_1 > 0$ such that

$$\lim_{|s| \to \infty} s^{-1}\left(sg(s) - C_1\int_0^s g(\sigma)\mathrm{d}\sigma\right) \ge 0 ;$$

for any m there exists $\beta(m) > 0$ such that

$$|g^{(m)}(s)| \le C_2(1 + |s|^{\beta(m)}) . \tag{8.26}$$

Under these assumptions the solution $u(t)$ lies in $L_{m,\,R}$ for $R > 0$ large enough if and only if the initial data satisfy some compatibility conditions [18]. Moreover, the global attractor $\mathcal{A}$ of system (8.5) exists and any trajectory lying in $\mathcal{A}$ possesses properties (8.4) for all $t \in \mathbb{R}$ and $k = 1,\ 2,\ \ldots$, [18]. It is easy to see that Theorem 8.2 is applicable here (the form of A, $B(\cdot)$ and H is evident in this case). In particular, Theorem 8.2 gives us that for a trajectory $U(t) = (u(t);\ \partial_t u(t))$ of problem (8.5) which lies in the global attractor $\mathcal{A}$ the estimate

$$\left\{\left\|A\partial_t^j(u(t) - u_n(t))\right\|^2 + \left\|A^{1/2}\partial_t^j(\partial_t u(t) - \overline{u}_n(t))\right\|^2\right\}^{1/2} \le C_{n,\,R,\,j}\,\lambda_{N+1}^{-n/2}$$

holds for all $n = 1,\ 2,\ \ldots$, all $j = 1,\ 2,\ \ldots$, and all $t \in \mathbb{R}$. Here $\overline{u}_n(t)$ and $u_n(t)$ are defined with the help of (8.18). Therewith

$$\sup\{\operatorname{dist}(U, \mathbf{M}_n) \colon U \in \mathcal{A}\} \le c_n \lambda_{N+1}^{-n/2}, \qquad n = 1, 2, \ldots, \tag{8.27}$$

where $\mathbf{M}_n$ is a manifold of the type (8.7) with $h = h_n(p, \dot{p})$ and $l = l_n(p, \dot{p})$. Here $\operatorname{dist}(U, \mathbf{M}_n)$ is the distance between U and $\mathbf{M}_n$ in the space $D(A) \times D(A^{1/2})$. Equation (8.27) gives us some information on the location of the global attractor in the phase space.

Other examples of usage of the construction given here can be found in papers [17] and [19] (see also Section 9 of Chapter 4).

§ 9 Idea of Nonlinear Galerkin Method

Approximate inertial manifolds have proved to be applicable to the computational study of the asymptotic behaviour of infinite-dimensional dissipative dynamical systems (for example, see the discussion and the references in [8]). Their usage leads to the appearance of the so-called nonlinear Galerkin method [20] based on the replacement of the original problem by its approximate inertial form. In this section we discuss the main features of this method using the following example of a second order in time equation of type (8.1):

$$\frac{d^2u}{dt^2} + \gamma \frac{du}{dt} + Au = B(u), \qquad u\big|_{t=0} = u_0, \qquad \left.\frac{du}{dt}\right|_{t=0} = u_1. \tag{9.1}$$

If all conditions on A and $B(\cdot)$ given in the previous section are fulfilled, then Theorem 8.2 is valid. It guarantees the existence of a family of mappings $\{h_k;\ l_k\}$ from $PH \times PH$ into QH possessing the properties:

1) there exist constants $M_j \equiv M_j(n, \rho)$ and $L_j \equiv L_j(n, \rho)$, $j = 1, 2$, such that

$$\|Ah_n(p_0, \dot{p}_0)\| \le M_1, \qquad \|A^{1/2} l_n(p_0, \dot{p}_0)\| \le M_2 \mathrm{e}, \tag{9.2}$$

$$\|A(h_n(p_1, \dot{p}_1) - h_n(p_2, \dot{p}_2))\| \le L_1\Big(\|A(p_1 - p_2)\| + \|A^{1/2}(\dot{p}_1 - \dot{p}_2)\|\Big), \tag{9.3}$$

$$\|A^{1/2}(l_n(p_1, \dot{p}_1) - l_n(p_2, \dot{p}_2))\| \le L_1\Big(\|A(p_1 - p_2)\| + \|A^{1/2}(\dot{p}_1 - \dot{p}_2)\|\Big) \tag{9.4}$$

for all p_j and $\dot{p}_j$ from PH such that

$$\|Ap_j\|^2 + \|A^{1/2}\dot{p}_j\|^2 \le \rho^2, \qquad j = 0, 1, \quad \rho > 0;$$

2) for any solution $u(t)$ to problem (9.1) which lies in $L_{m,R}$ for $m \ge 2$ the estimate

$$\left\{\|A(u(t)-u_n(t))\|^2+\|A^{1/2}(\partial_t u(t)-\bar{u}_n(t))\|^2\right\}^{1/2} \le C_{n,R}\,\lambda_{N+1}^{-n/2} \tag{9.5}$$

is valid (see Theorem 8.2) for all $n \le m-1$ and t large enough. Here

$$u_n(t)=p(t)+h_n(p(t),\ \partial_t p(t))\ ,$$
$$\bar{u}_n(t)=\partial_t p(t)+l_n(p(t),\ \partial_t p(t))\ , \tag{9.6}$$

λ_{N+1} is the $(N+1)$-th eigenvalue of A, and R is the constant from (8.4).

The family $\{h_k;\ l_k\}$ is defined with the help of a quite simple procedure (see (8.8) and (8.9)) which can be reduced to the process of solving of stationary equations of the type $Av=g$ in the subspace QH. Moreover,

$$h_0(p,\dot{p})\equiv l_0(p,\dot{p})\equiv 0\,,\qquad h_1(p,\dot{p})=A^{-1}QB(p)\,,\qquad l_1(p,\dot{p})\equiv 0\,. \tag{9.7}$$

In particular, estimates (9.5) and (9.6) mean (see Corollary 8.1) that trajectories $U(t)=(u(t);\ \partial_t u(t))$ of system (9.1) are attracted by a small (for N large enough) vicinity of the manifold

$$M_n=\{(p+h_n(p,\dot{p});\ \dot{p}+l_n(p,\dot{p})):\ p,\ \dot{p}\in PH\}\,. \tag{9.8}$$

The sequence of mappings $\{h_n(p,\dot{p})\}$ generates a family of approximate inertial forms of problem (9.1):

$$\partial_t^2 p+\gamma\partial_t p+Ap=PB(p+h_n(p,\ \partial_t p))\,. \tag{9.9}$$

A finite-dimensional dynamical system in PH which approximates (in some sense) the original system corresponds to each form. For $n=0$ equation (9.9) transforms into the standard Galerkin approximation of problem (9.1) (due to (9.7)). If $n>0$, then we obtain a class of numerical methods which can be naturally called the non-linear Galerkin methods. However, we cannot use equation (9.9) in the computational study directly. The point is that, first, in the calculation of $h_n(p,\dot{p})$ we have to solve a linear equation in the infinite-dimensional space QH and, second, we can lose the dissipativity property. Therefore, we need additional regularization. It can be done as follows. Assume that $f_n(p,\dot{p})$ stands for one of the functions $h_n(p,\dot{p})$ or $l_n(p,\dot{p})$. We define the value

$$f_n^*(p,\dot{p})\equiv f_{N,M,n}(p,\dot{p})=\chi\left(R^{-1}\left(\|Ap\|^2+\|A^{1/2}\dot{p}\|^2\right)^{1/2}\right)P_M f_n(p,\dot{p})\,, \tag{9.10}$$

where $\chi(s)$ is an infinitely differentiable function on $\mathbb{R}_+$ such that a) $0\le\chi(s)\le 1$; b) $\chi(s)=1$ for $0\le s\le 1$; c) $\chi(s)=0$ for $s\ge 2$; R is the radius of dissipativity (see (8.4) for $k=0$) of system (9.1); P_M is the orthoprojector in H onto the subspace generated by the first M eigenvectors of the operator A, $M>N$. We consider the following N-dimensional evolutionary equation in the subspace $P_N H$:

$$\begin{aligned} &\partial_t^2 p^* + \gamma \partial_t p^* + A p^* = P_N B(p^* + h_n^*(p^*, \partial_t p^*)) , \\ &p^*|_{t=0} = P_N u_0 , \quad \partial_t p^*|_{t=0} = P_N u_1 . \end{aligned} \tag{9.11}$$

— Exercise *9.1* Prove that problem (9.11) has a unique solution for $t > 0$ and the corresponding dynamical system is dissipative in $P_N H \times P_N H$.

We call problem (9.11) a nonlinear Galerkin (n, N, M)-approximation of problem (9.1). The following assertion is valid.

Theorem 9.1.

Assume that the mappings $h_n(p, \dot{p})$ ***and*** $l_n(p, \dot{p})$ ***satisfy equations*** (9.2)–(9.5) ***for*** $n \le m-1$ ***and for some*** $m \ge 2$. ***Moreover, we assume that*** (9.5) ***is valid for all*** $t > 0$. ***Let*** h_n^* ***and*** l_n^* ***be defined by*** (9.10) ***with the help of*** h_n ***and*** l_n ***and let***

$$u_n^*(t) = p^*(t) + h_n^*(p^*(t), \partial_t p^*(t)),$$

$$\overline{u}_n^*(t) = \partial_t p^*(t) + l_n^*(p^*(t), \partial_t p^*(t)),$$

where $p^*(t)$ ***is a solution to problem*** (9.11). ***Then the estimate***

$$\left\{ \left\| A^{1/2}(u(t) - u_n^*(t)) \right\|^2 + \left\| \partial_t u(t) - \overline{u}_n^*(t) \right\|^2 \right\}^{1/2} \le$$

$$\le (\alpha_1 \lambda_{N+1}^{-(n+1)/2} + \alpha_2 \lambda_{M+1}^{-1/2}) \exp(\beta t) \tag{9.12}$$

holds, where $u(t)$ ***is a solution to problem*** (9.1) ***which lies in*** $L_{m, R}$ ***for*** $m \ge 2$ ***and possesses property*** (8.4) ***for*** $k = 1$ ***and for all*** $t > 0$. ***Here*** $n \le$ $\le m-1$, α_1, α_2 ***and*** β ***are positive constants independent of*** M ***and*** N, λ_k ***is the*** k***-th eigenvalue of the operator*** A.

Proof.

Let $p(t) = P_N u(t)$. We consider the values

$$u(t) - u_n^*(t) = p(t) - p^*(t) + [Q_N u(t) - h_n(p(t), \partial_t p(t))] +$$
$$+ [h_n(p(t), \partial_t p(t)) - h_n^*(p^*(t), \partial_t p^*(t))]$$

and

$$\partial_t u(t) - \overline{u}_n^*(t) = \partial_t (p(t) - p^*(t)) + [Q_N \partial_t u(t) - l_n(p(t), \partial_t p(t))] +$$
$$+ [l_n(p(t), \partial_t p(t)) - l_n^*(p^*(t), \partial_t p^*(t))] .$$

The equalities

$$P_M h_n(p(t), \partial_t p(t)) = h_n^*(p(t), \partial_t p(t))$$

and

$$P_M l_n(p(t),\ \partial_t p(t)) = l_n^*(p(t),\ \partial_t p(t))$$

are valid for the class of solutions under consideration. Therefore, we use (9.5) to find that

$$\begin{aligned}&\left\|A^{1/2}(u(t)-u_n^*(t))\right\| \le \\ &\le C_1\Big(\|A^{1/2}(p(t)-p^*(t))\| + \|\partial_t p(t) - \partial_t p^*(t)\|\Big) + \frac{C_2}{\lambda_{M+1}^{1/2}} + \frac{C_{n,\,R}}{\lambda_{N+1}^{(n+1)/2}}\end{aligned} \tag{9.13}$$

and

$$\begin{aligned}&\left\|\partial_t u(t) - \partial_t u_n^*(t)\right\| \le \\ &\le C_3\Big(\|A^{1/2}(p(t)-p^*(t))\| + \|\partial_t p(t) - \partial_t p^*(t)\|\Big) + \frac{C_4}{\lambda_{M+1}^{1/2}} + \frac{C_{n,\,R}}{\lambda_{n+1}^{(n+1)/2}}\ .\end{aligned} \tag{9.14}$$

Therefore, we must compare the solution $p^*(t)$ to problem (9.11) with the value $p(t) = P_N u(t)$ which satisfies the equation

$$\partial_t^2 p + \gamma\partial_t p + A p \;=\; Q_N B(p + Q_N u) \tag{9.15}$$

with the same initial conditions as the function $p^*(t)$. Let $r(t) = p(t) - p^*(t)$. Then it follows from (9.11) and (9.15) that

$$\begin{aligned}&\partial_t^2 r(t) + \gamma\partial_t r(t) + A\,r(t) \;=\; F(t,\ p^*,\ u)\ ,\\ &r(0) = 0, \quad \partial_t r(0) = 0\ ,\end{aligned} \tag{9.16}$$

where

$$F(t,\ p^*,\ u) = Q_N\big[B(u(t)) - B(u_n^*(t))\big]\,.$$

Due to the dissipativity of problems (9.11) and (9.15) we use (9.13) to obtain

$$\|F(t,\ p^*,\ u)\| \;\le\; C_R\Big(\|A^{1/2}r(t)\|^2 + \|\dot r(t)\|^2\Big)^{1/2} + C_{n,\,R}\,\lambda_{N+1}^{-(n+1)/2} + C\,\lambda_{M+1}^{-1/2}$$

for the class of solutions under consideration. Therefore, equation (9.16) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\dot r(t)\|^2 + \|A^{1/2}r(t)\|^2\Big) \le\; \overline{C}_R\Big(\|\dot r(t)\|^2 + \|A^{1/2}r(t)\|^2\Big) + \overline{C}_{n,\,R}\,\lambda_{N+1}^{-(n+1)} + C\,\lambda_{M+1}^{-1}\,.$$

Hence, Gronwall's lemma gives us that

$$\|\dot r(t)\|^2 + \|A^{1/2}r(t)\|^2 \;\le\; (C_{n,\,R}\,\lambda_{N+1}^{-(n+1)} + C\,\lambda_{M+1}^{-1})\,e^{\overline{C}_R t}\,.$$

This and equations (9.13) and (9.14) imply estimate (9.12). **Theorem 9.1 is proved**.

If we take $n = 0$ and $N = M$ in Theorem 9.1, then estimate (9.12) changes into the accuracy estimate of the standard Galerkin method of the order N. Therefore, if the parameters N, M, and n are compatible such that $\lambda_{M+1} \le \lambda_{N+1}^{n+1}$, then the error of the corresponding nonlinear Galerkin method has the same order of smallness as in the standard Galerkin method which uses M basis functions. However, if we use the nonlinear method, we have to solve a number of linear algebraic systems of the order $M-N$ and the Cauchy problem for system (9.11) which consists of N equations. In particular, in order to determine the value $h_1(p, \dot{p})$ we must solve the equation

$$A h_1(p, \dot{p}) = (P_M - P_N) Q B(p)$$

for $n = 1$ and choose the numbers N and M such that $\lambda_{M+1} \le \lambda_{N+1}^2$. Moreover, if $\lambda_k \cong c_0 k^\sigma (1 + o(1))$, $\sigma > 0$, as $k \to \infty$, then the values N and M must be compatible such that $M \le c_\sigma N^2$.

We note that Theorem 9.1 as well as the corresponding variant of the nonlinear Galerkin method can be used in the study of the asymptotic properties of solutions to the nonlinear wave equation (8.5) under some conditions on the nonlinear term $g(u)$. Other applications of Theorem 9.1 can also be pointed out.

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