Making Errors

Borasi, Raffaella:

Reconceiving Mathematics Instruction
A Focus on Errors

Norwood, NJ: Ablex Publishing, 1996. – 321 p.
(Issues in Curriculum Theory, Policy, and Research)
ISBN 1-56750-168-0

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The elegant, polished face that mathematics offers to the world is meant to be flawless. Mathematical arguments are presented in a rigorous theorem-proof format devised several millennia ago to uncover and eliminate lapses in reasoning. Ingenious and ever vigilant, the community of mathematicians continually develops and applies procedures for checking one's own work and the work of others. The deductive logic that powers mathematical investigation is universally acclaimed as a model of how thought can proceed unerringly from Point A to Point B.

Behind the facade, however, uncertainty and error abound. Mathematicians debate not only the propriety of their assumptions and the adequacy of their arguments but also the very foundations of their discipline. Heroic efforts yield shaky, incomplete, or erroneous results. Counterexamples of established generalizations are uncovered. Trusted conjectures turn out to be groundless. Definitions admit pathological cases. Proofs are found to lack critical assumptions and thus have to be repaired or discarded. Paradox, contradiction, and ambiguity appear to be endemic.

In the school mathematics classroom, too, error lurks everywhere. But because the image of mathematics as a perfect endeavor is so powerful, students and teacher alike ordinarily view error as something to be suppressed, even punished. Error is deficit, weakness, ignorance, calamity, failure, immorality, evil. Students are judged in terms of the errors they make, and they quite reasonably come to view avoidance of error as the hallmark of mathematical thought.

In her new book, which builds on a previous effort (Borasi 1992) to reorient mathematics teaching toward what mathematicians do rather than what they claim, Raffaella Borasi argues that the customary schoolhouse view of error is itself erroneous. Error is valuable; it is how we learn. Error should be sought out and cultivated. Teachers and students should, in her words, "capitalize on errors as springboards for inquiry". Teaching should promote the students' "active participation in error activities".

To illustrate what she means by *error*, Borasi presents 21 "error case studies" drawn in part from the history of mathematics but primarily from her own and her colleagues' experiences in doing, learning, and teaching mathematics. The case studies range from the use of conflicting proposals by middle school students in developing a formula for the probability of disjunction of two events to a discussion of how the common error of summing fractions by adding their numerators and denominators might allow for a reinterpretation of addition in the context of ratios. One case study deals with how secondary school students reacted when they arrived at two different values for 0^0; another recounts the struggles of Galileo, Bolzano, and Cantor in dealing with paradoxes presented by infinite sets.

Borasi introduces the metaphor of getting lost in a city to suggest how errors can be approached productively. Depending on the circumstances, one perceives the problem of getting lost in different ways. Pressed to keep an appointment, the lost traveler asks for directions or uses a map, whereas returning from work to a newly purchased house, a lost commuter might take time to explore the neighborhood as a means of identifying important landmarks or finding a shortcut. Lost in a foreign city, a vacationer may abandon a specific destination to take advantage of opportunities for sightseeing. In school, errors need not always be avoided by determining in advance the most expeditious route to one's goal. Like the lost commuter, students may use errors as opportunities to gain understanding. Like the lost vacationer, students may find that errors can yield valuable and unexpected results.

Because she proposes to use errors as the basis for a new approach to mathematics instruction, Borasi argues that the term *mathematical error* should be interpreted in "the most comprehensive way possible". She offers no definition of the term, contenting herself with examples. She gives several lists of types of error, but they are not the same list and are clearly not meant to be exhaustive. Her ecumenicity leads her to include, along with downright mistakes such as faulty procedures or incorrect results, items such as "incorrect definitions" that signal a departure from standard usage, plus items that few would consider erroneous per se, such as ambiguous expressions in algebra or tentative steps in making a geometric construction. She uses the fact that 0^0 is undefined and the fact that in extending exponentiation from positive to negative bases one encounters imaginary numbers to argue that there are inherent limitations in mathematics – another type of error, in her view. For Borasi, an *error* seems to be essentially an *anomaly*, something unusual or even just incomplete, not something that is necessarily wrong.

For example, Borasi makes much of the "errors" that resulted when college students and in-service mathematics teachers were asked to define *circle*. Many of the definitions were imprecise; some were over-inclusive; some, not inclusive enough. Some were redundant, and a few were circular. There is obvious merit in examining these definitions and using them as vehicles for understanding both what a circle is (as usually defined in mathematics) and what a definition can accomplish. But Borasi seems not to appreciate sufficiently the distinction between definitions as arbitrary constructions derived from primitive mathematical terms and definitions as pedagogical devices that can function to help establish shared meaning; in other words, between definitions in mathematics and definitions in mathematics instruction. Some of the definitions she holds up as erroneous are merely inadequate in some sense from a strictly mathematical point of view or are limited to a specific but unspecified context. Furthermore, she nowhere raises the problem of alternative definitions (e.g., circles as including or excluding their interiors; trapezoids as excluding or including parallelograms), which might be used to illustrate the role and value of convention in mathematics.

To help teachers plan instruction that capitalizes on errors, Borasi asserts that mathematical activities can fall into one of three levels of mathematical discourse: task, content, or mathematics. (*Discourse* seems an inappropriate characterization; she is referring not to representation or communication but to the nature or goal of the activity to which the error might contribute: performing a task, learning technical mathematical content, or learning about mathematics as a discipline.) She also claims that an instructional activity can be informed by any of three stances of learning: remediation (to locate and correct one's errors), discovery (to rely on one's errors in learning or solving something new), and inquiry (to exploit one's errors in formulating and exploring new questions). When the three levels of discourse are crossed with the three stances of learning, the resulting matrix yields what Borasi terms "a taxonomy of uses of errors". She also categorizes error activities according to the level of student involvement: whether the teacher models the inquiry, the students are led by the teacher, or the students engage in independent inquiry. A final means of categorizing the errors in an error activity is according to their source. Ten sources are listed, ranging from a "planned error" that the teacher has selected in advance as appropriate for an activity to a "math-inherent" error that is "due to the limitation of mathematics itself". Borasi applies the various categorization schemes and the taxonomy to the case studies and to other error activities proposed in the book. The categories are intended not only to bring some order to the welter of possible errors and their pedagogical uses, thus aiding teachers who have an instructional goal, but also to demonstrate the range of this approach.

Near the end of the book, Borasi contends that she has offered "both theoretical arguments and anecdotal evidence" for the value of active participation in error activities. That she has provided ample anecdotal evidence there can be no doubt. The case studies, which include numerous excerpts from interviews and from classroom discourse, testify persuasively to the impressive mathematical investigations that can be stimulated simply by taking an error seriously and seeing where it leads. The book offers an extensive collection of activities for teachers and students to pursue.

Theoretical arguments are another matter. Borasi's original contributions tend toward the simple taxonomy, not the subtle distinction or the exhaustive treatment. To provide an appropriate contrast with *inquiry*, which is cast as the most advanced stance of learning (i.e., pedagogical strategy), she limits *discovery* to the "predetermined and unquestioned", although proponents of "discovery learning" have not necessarily taken that view (Shulman & Keislar 1966, p. 10, p. 193). The three types of learning stance may be useful to a teacher seeking a simple scheme to use in generating variety in her or his lessons, but they fail to do justice to the manifold ways lessons can vary in openness, even among the lessons Borasi recounts. She seems less interested in analyzing lesson structure or function than in presenting neat descriptive categories.

Borasi's treatment of epistemology is equally simplistic. She criticizes mathematicians who take a "dualistic" view of knowledge (it's either right or wrong), arguing instead for a humanistic, relativistic stance – thus revealing her own dualistic view of epistemology. She labels as "radical constructivist" the epistemology underlying her inquiry model of teaching and then casts that model in opposition to a "transmission pedagogy". Her portrayal of radical constructivism appears to confound it with social constructivism, and she characterizes as radical constructivists several mathematics educators (e.g., Nicolas Balacheff, Alan Schoenfeld) who would undoubtedly be startled to find themselves so labeled.

The reader may be surprised to find no mention of George Pólya (1945, 1962). Pólya's approach of encouraging students to guess and then test their guess, although not identified as beginning from error, was much the same as Borasi's approach. He too was concerned that students not get the impression that the deductive, "error-free" side of mathematics tells the whole story. He wanted them to see, and to participate in, mathematics in the making. Much of Pólya's spirit, if not his name, can be found in this book.

The book is highly repetitive. Numerous points are repeated, almost verbatim, in several chapters. In the case study on definitions of a circle, 44 definitions are listed, which then appear again and again in different combinations as the definitions are categorized and recategorized. In a later chapter, 37 items from the list are presented once more, reworded and reordered, and then listed several times again to show how teachers classified them. The levels of discourse and stances of learning are first described in the text, then collected in a table, and then – in case the reader has not been paying attention or cannot use an index – they are repeated in an appendix. Also, the book contains quite a few, albeit minor, typographical errors.

Nonetheless, this is a book mathematics educators should read. Borasi candidly recounts her own intellectual journey, relating her successes and failures in using the inquiry model. In particular, she explores the difficulties of convincing students that spending an entire class period discussing one problem has its merits and rewards. Most important, she not only provides a host of ideas for mathematics teachers to use in capitalizing on errors in their instruction but also demonstrates convincingly the power of reflecting on one's practice as a stimulus to inquiry about teaching and about mathematics. Limited in its theoretical power, yet rich in material for the classroom, the book is a notable contribution to the literature of our field.

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