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EXISTENCE RESULTS FOR FUNCTIONAL PERTURBED DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH STATE-DEPENDENT DELAY IN BANACH SPACES

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Abstract. In this paper, we provide sufficient conditions for the existence of solutions of initial value problem, for perturbed partial functional hyperbolic differential equations of fractional order involving Caputo fractional derivative with state-dependent delay by reducing the research to the search of the existence and the uniqueness of fixed points of appropriate operators. Our main result for this problem is based on a nonlinear alternative fixed point theorem for the sum of a completely continuous operator and a contraction one in Banach spaces due to Burton and Kirk and a fractional version of Gronwall's inequality. We should observe the structure of the space and the properties of the operators to obtain existence results. To our knowledge, there are very few papers devoted to fractional differential equations with finite and/or infinite constant delay on bounded domains. Many other questions and issues can be investigated regarding the existence in the space of weighted continuous functions, the uniqueness, the structure of the solutions set and also whether or not the condition satisfied by the operators are optimal. This paper can be considered as a contribution in this setting case. Examples are given to illustrate this work.

Key words: partial differential equation, fractional order, solution, left-sided mixed Riemann–Liouville integral, Caputo fractional-order derivative, state-dependent delay, fixed point.

AMS Subject Classification: 26A33, 34K30, 34K37, 35R11.

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1. Introduction

It is well known that differential equations of fractional order play a very important role in describing some real world problems. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [1]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Miller and Ross [2], Podlubny [3], the papers of Abbas and Benchohra [4, 5], Benchohra et al. [6] and the references therein.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay, see for instance the books [7, 8] and the paper [9].

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years (see for instance [10] and the references therein). These equations are frequently called equations with state-dependent delay. Existence results, among other things, were derived recently for various classes of functional differential equations when the delay is depending on the solution. We refer the reader to the papers by Anguraj et al. [11], Hartung [12], and Hernandez et al. [13]. In [14], the authors considered a class of semilinear functional fractional order differential equations with state-dependent delay.

The first result of this paper deals with the existence of solutions to fractional order initial value problems (IVP for short), for the system

$$\begin{aligned}\left({}^cD_0^r u\right)(t,x) &= f\big(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}\big)\\ &\quad + g\big(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}\big), \quad \text{if } (t,x)\in J,\end{aligned} \tag{1}$$

$$u(t,x)=\phi(t,x), \quad \text{if } (t,x)\in \widetilde{J}, \tag{2}$$

$$\begin{cases} u(t,0)=\varphi(t), \\ u(0,x)=\psi(x), \end{cases} \quad (t,x)\in J, \tag{3}$$

where $\varphi(0)=\psi(0)$, $J:=[0,a]\times[0,b]$, $a,b,\alpha,\beta>0$, $\widetilde{J}:=[-\alpha,a]\times[-\beta,b]\backslash[0,a]\times[0,b]$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $f,g: J\times C\to\mathbb{R}^n$, $\rho_1: J\times C\to[-\alpha,a]$, $\rho_2: J\times C\to[-\beta,b]$ are given functions, $\phi\in C:=C(\widetilde{J},\mathbb{R}^n)$ is a given continuous function with $\phi(t,0)=\varphi(t)$, $\phi(0,x)=\psi(x)$ for each $(t,x)\in J$, $\varphi:[0,a]\to\mathbb{R}^n$, $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions and C is the space of continuous functions on $\widetilde{J}$. We denote by $u_{(t,x)}$ the element of C defined by $u_{(t,x)}(s,\tau)=u(t+s,x+\tau)$, $(s,\tau)\in\widetilde{J}$, here $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state u.

The second result deals with the existence of solutions to fractional order partial differential equations

$$\begin{aligned}\left({}^cD_0^r u\right)(t,x) &= f\big(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}\big)\\ &\quad + g\big(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}\big), \quad \text{if } (t,x)\in J,\end{aligned} \tag{4}$$

$$u(t,x)=\phi(t,x), \quad \text{if } (t,x)\in \widetilde{J}', \tag{5}$$

$$\begin{cases} u(t,0)=\varphi(t), \\ u(0,x)=\psi(x), \end{cases} \quad (t,x)\in J, \tag{6}$$

where φ,ψ are as in problem (1)–(3), $\widetilde{J}':=(-\infty,a]\times(-\infty,b]\backslash[0,a]\times[0,b]$, $f,g: J\times\mathscr{B}\to\mathbb{R}^n$, $\rho_1: J\times\mathscr{B}\to(-\infty,a]$, $\rho_2: J\times\mathscr{B}\to(-\infty,b]$ are given functions, $\phi:\widetilde{J}'\to\mathbb{R}^n$ is a given continuous function with $\phi(t,0)=\varphi(t)$, $\phi(0,x)=\psi(x)$ for each $(t,x)\in J$ and $\mathscr{B}$ is called a phase space that will be specified in Section 4.

Motivated by the previous papers, we consider the existence result for each of our problems (1)–(3) and (4)–(6). Our analysis is based upon on a fixed point theorem due to Burton and Kirk for the sum of contraction and completely continuous operators and a fractional version of Gronwall's inequality. We look for sufficient conditions ensuring existence of solutions for each of our problems. The present results extend those considered with integer order derivative and those with finite and/or infinite constant delay on bounded domains in [15–18].

As far as we know, no papers exist in the literature related to fractional order hyperbolic functional differential equations with state-dependent delay. The aim of this paper is to initiate this study.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J,\mathbb{R}^n)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm $\|u\|_\infty = \sup_{(t,x)\in J}\|u(t,x)\|$, where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. As usual, by $AC(J,\mathbb{R}^n)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J,\mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm $\|u\|_{L^1} = \int_0^a\int_0^b \|u(t,x)\|dxdt$.

Now, we give some definitions and properties of fractional calculus.

Definition 2.1 [19]. Let $r = (r_1, r_2) \in (0,\infty)\times(0,\infty), \theta = (0,0)$ and for $u \in L^1(J,\mathbb{R}^n)$, the expression

$$(I_0^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}u(s,\tau)\,d\tau ds,$$

where $\Gamma(\cdot)$ is the gamma function, is called the left-sided mixed Riemann–Liouville integral of order r of u.

In particular, $(I_\theta^\theta u)(t,x) = u(t,x)$, $(I_\theta^\sigma u)(t,x) = \int_0^t\int_0^x u(s,\tau)d\tau ds$ for almost all $(t,x)\in J$, where $\sigma = (1,1)$, For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)\times(0,\infty)$, when $u \in L^1(J,\mathbb{R}^n)$. Note also that when $u\in C(J,\mathbb{R}^n)$, then $(I_\theta^r u)\in C(J,\mathbb{R}^n)$, moreover $(I_\theta^r u)(t,0)=(I_\theta^r u)(0,x)=0$, $(t,x)\in J$. By $1-r$ we mean $(1-r_1, 1-r_2)\in(0,1]\times(0,1]$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t\partial x}$, the mixed second order partial derivative.

Definition 2.2 [19]. Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J,\mathbb{R}^n)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^cD_0^r u)(t,x) = \left(I_\theta^{1-r}\frac{\partial^2}{\partial t\partial x}u\right)(t,x).$$

The case $\sigma = (1,1)$ is included and we have $(D_\theta^\sigma u)(t,x) = ({}^cD_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x)$ for almost all $(t,x)\in J$.

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.1 [20]. *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \leqslant \omega(t,x) + c\int_0^t\int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}\,d\tau ds,$$

then there exists a constant $\delta = \delta(r_1,r_2)$ such that

$$\upsilon(t,x) \leqslant \omega(t,x) + \delta c\int_0^t\int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}\,d\tau ds,$$

for every $(t,x)\in J$.

Theorem 2.1 (Burton–Kirk [21]). *Let X be a Banach space, and $A, B : X \to X$ two operators satisfying: (i) B is a contraction, and (ii) A is completely continuous. Then either*

(a) the operator equation $u = A(u) + B(u)$ has a solution, or

(b) the set $\mathscr{E} = \left\{u \in X : u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right)\right\}$ is unbounded for $\lambda \in (0,1)$.

3. Existence Results for the Finite Delay Case

In this section, we give our main existence result for problem (1)–(3).

Before starting and proving this result, we give what we mean by a solution of this problem. Let the space $C_{(a,b)} := C([-\alpha, a] \times [-\beta, b], \mathbb{R}^n)$, $a, b > 0$.

DEFINITION 3.1. A function $u \in C_{(a,b)}$ is said to be a solution of (1)–(3) if u satisfies equations (1) and (3) on J and the condition (2) on $\widetilde{J}$.

Let $f, g \in L^1(J, \mathbb{R}^n)$ and consider the following problem

$$\begin{cases} ({}^cD_0^r u)(t,x) = f(t,x) + g(t,x), & (t,x) \in J, \\ u(t,0) = \varphi(t), \quad u(0,x) = \psi(x), \ \varphi(0) = \psi(0). \end{cases} \tag{1}$$

For the existence of solutions for the problem (1)–(3), we need the following lemma.

Lemma 3.1. *A function $u \in C(J, \mathbb{R}^n)$ is a solution of problem* (1) *if and only if $u(t,x)$ satisfies*

$$u(t,x) = z(t,x) + (I_0^r f)(t,x) + (I_0^r g)(t,x), \quad (t,x) \in J, \tag{2}$$

where $z(t,x) = \varphi(t) + \psi(x) - \varphi(0)$.

$\lhd$ Let $u(t,x)$ be a solution of problem (1). Then, taking into account the definition of the fractional Caputo derivative $({}^cD_0^r u)(t,x)$, we have $I_0^{1-r}(D_{tx}^2 u)(s,\tau) = f(t,x) + g(t,x)$. Hence, we obtain $I_0^r(I_0^{1-r} D_{tx}^2 u)(s,\tau) = I_0^r f(t,x) + I_0^r g(t,x)$, then $I_0^1(D_{tx}^2 u)(s,\tau) = I_0^r f(t,x) + I_0^r g(t,x)$. Since $I_0^1(D_{tx}^2 u)(s,\tau) = u(t,x) - u(t,0) - u(0,x) + u(0,0)$, we have $u(t,x) = z(t,x) + (I_0^r f)(t,x) + (I_0^r g)(t,x)$. By the definition of the left-sided mixed Riemann–Liouville integral of order r of f and g, we have

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} f(s,\tau)\, d\tau ds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} g(s,\tau)\, d\tau ds,$$

so

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} [f(s,\tau) + g(t,x)]\, d\tau ds,$$

where $z(t,x) = \varphi(t) + \psi(x) - \varphi(0)$. Now let $u(t,x)$ satisfy (2). It is clear that $u(t,x)$ satisfies (1). $\rhd$

As a consequence of Lemma 3.1 we have the following auxiliary result.

Corollary 3.1. *The function $u \in C_{(a,b)}$ is a solution of problem* (1)–(3) *if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} f\big(s,\tau,u_{(s,\tau)}\big)\, d\tau ds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} g\big(s,\tau,u_{(s,\tau)}\big)\, d\tau ds,$$

for all $(t,x) \in J$ and the condition (2) *on $\widetilde{J}$.*

Set $\mathscr{R} := \mathscr{R}_{(\rho_1^-, \rho_2^-)} = \{(\rho_1(s,\tau,u), \rho_2(s,\tau,u)) : (s,\tau,u) \in J \times C,\ \rho_i(s,\tau,u) \leqslant 0;\ i = 1,2\}$. We always assume that $\rho_1 : J \times C \to [-\alpha, a]$, $\rho_2 : J \times C \to [-\beta, b]$ are continuous and the function $(s,\tau) \longmapsto u_{(s,\tau)}$ is continuous from $\mathscr{R}$ into C.

Our main existence result in this section is based upon the fixed point theorem due to Burton–Kirk. We will need to introduce the following hypothesis:

$(H1)$ The functions $f, g : J \times C \to \mathbb{R}^n$ are continuous.

$(H2)$ There exists $k > 0$ such that $\|g(t,x,u) - g(t,x,v)\| \leqslant k\|u - v\|_C$ for any $u, v \in C$ and $(t,x) \in J$.

$(H3)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that $\|f(t,x,u)\| \leqslant p(t,x) + q(t,x)\|u\|_C$ for $(t,x) \in J$ and each $u \in C$.

Theorem 3.1. *Assume that hypotheses* $(H1)$–$(H3)$ *hold. If*

$$\frac{Ka^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{3}$$

then the IVP (1)–(3) *has at least one solution on* $[-\alpha, a] \times [-\beta, b]$.

$\lhd$ Transform the problem (1)–(3) into a fixed point problem. Consider the operators $F, G : C_{(a,b)} \to C_{(a,b)}$ defined by,

$$(Fu)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \widetilde{J}, \\ z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^t \int\limits_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \qquad \times f\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\, d\tau ds, & (t,x) \in J, \end{cases}$$

and

$$(Gu)(t,x) = \begin{cases} 0, & (t,x) \in \widetilde{J}, \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^t \int\limits_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \qquad \times g\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\, d\tau ds, & (t,x) \in J. \end{cases}$$

The problem of finding the solutions of the IVP (1)–(3) is reduced to finding the solutions of the operator equation $(Fu)(t,x) + (Gu)(t,x) = u(t,x)$, $(t,x) \in J$. We shall show that the operators F and G satisfies all the conditions of Theorem 2.1. The proof will be given in several steps.

STEP 1. First, we show that F is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$. Let $\eta > 0$ be such that $\|u_n\| \leqslant \eta$. Then

$$\big\|(Fu_n)(t,x) - (Fu)(t,x)\big\| \leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^t \int\limits_0^x \big|(t-s)^{r_1-1}(x-\tau)^{r_2-1}\big|$$

$$\times \big\|f\big(s,\tau,u_{n(\rho_1(s,\tau,u_{n(s,\tau)}),\rho_2(s,\tau,u_{n(s,\tau)}))}\big) - f\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\big\|\, d\tau ds$$

$$\leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^t \int\limits_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \sup_{(s,\tau)\in J} \big\|f\big(s,\tau,u_{n(\rho_1(s,\tau,u_{n(s,\tau)}),\rho_2(s,\tau,u_{n(s,\tau)}))}\big)$$

$$- f\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\big\|\, d\tau ds \leqslant \frac{\|f(\cdot,\cdot,u_{n(\cdot,\cdot)}) - f(\cdot,\cdot,u)\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^t \int\limits_0^x (t-s)^{r_1-1}$$

$$\times (x-\tau)^{r_2-1} d\tau ds \leqslant \frac{t^{r_1}x^{r_2}\|f(\cdot,\cdot,u_{n(\cdot,\cdot)}) - f(\cdot,\cdot,u_{(\cdot,\cdot)})\|_\infty}{r_1 r_2 \Gamma(r_1)\Gamma(r_2)} \leqslant \frac{a^{r_1}b^{r_2}\|f(\cdot,\cdot,u_{n(\cdot,\cdot)}) - f(\cdot,\cdot,u_{(\cdot,\cdot)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Since f is a continuous function, we have

$$\big\|(Fu_n) - (Fu)\big\|_\infty \leqslant \frac{a^{r_1}b^{r_2}\|f(\cdot,\cdot,u_{n(\cdot,\cdot)}) - f(\cdot,\cdot,u_{(\cdot,\cdot)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus F is continuous.

STEP 2. F maps bounded sets into bounded sets in $C_{(a,b)}$.

Indeed, it is enough show that, for any $\eta > 0$, there exists a positive constant ℓ^* such that, for each $u \in B_\eta = \{u \in C_{(a,b)} : \|u\|_\infty \leqslant \eta\}$, we have $\|F(u)\|_\infty \leqslant \ell^*$. By (H_3) we have for each $(t,x) \in J$,

$$\|(Fu)(t,x)\| \leqslant \|z(t,x)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times \left\| f\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\right\| d\tau ds \leqslant \|z(t,x)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times\, p(s,\tau)\, d\tau ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} q(s,\tau) \left\| u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right\|_C d\tau ds$$

$$\leqslant \|z(t,x)\| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\, d\tau ds + \frac{\|q\|_\infty \eta}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\, d\tau ds \leqslant \|z(t,x)\| + \frac{\|p\|_\infty + \|q\|_\infty \eta}{\Gamma(r_1+1)\Gamma(r_2+1)}\, a^{r_1} b^{r_2}.$$

Thus

$$\|(Fu)\|_\infty \leqslant \|z\|_\infty + \frac{\|p\|_\infty + \|q\|_\infty \eta}{\Gamma(r_1+1)\Gamma(r_2+1)}\, a^{r_1} b^{r_2} := \ell^*.$$

STEP 3. F maps bounded sets into equicontinuous sets in $C_{(a,b)}$.

Let $(t_1,x_1),(t_2,x_2) \in (0,a] \times (0,b]$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set of $C_{(a,b)}$ as in Step 2, and let $u \in B_\eta$. Then

$$\left\|(Fu)(t_2,x_2) - (Fu)(t_1,x_1)\right\| \leqslant \left\|z(t_1,x_1) - z(t_2,x_2)\right\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int_0^{t_1} \int_0^{x_1} \left[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}\right] \left\| f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\right\| d\tau ds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1} \left\| f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\right\| d\tau ds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1} \left\| f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\right\| d\tau ds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1} \left\| f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\right\| d\tau ds$$

$$\leqslant \left\|z(t_1,x_1) - z(t_2,x_2)\right\| + \frac{\|p\|_\infty + \|q\|_\infty\, \eta}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} \left[(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}\right.$$

$$\left. - (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\right] d\tau ds + \frac{\|p\|_\infty + \|q\|_\infty\, \eta}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\, d\tau ds$$

$$+\frac{\|p\|_\infty+\|q\|_\infty\,\eta}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^{t_1}\int\limits_{x_1}^{x_2}(t_2-s)^{r_1-1}\,(x_2-\tau)^{r_2-1}\,d\tau ds+\frac{\|p\|_\infty+\|q\|_\infty\,\eta}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times\int\limits_{t_1}^{t_2}\int\limits_0^{x_1}(t_2-s)^{r_1-1}\,(x_2-\tau)^{r_2-1}\,d\tau ds\leqslant\big\|z(t_1,x_1)-z(t_2,x_2)\big\|+\frac{\|p\|_\infty+\|q\|_\infty\,\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$\times\Big[x_2^{r_2}(t_2-t_1)^{r_1}+t_2^{r_1}(x_2-x_1)^{r_2}-(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}\Big]$$

$$+\frac{\|p\|_\infty+\|q\|_\infty\,\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}\,(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}+\frac{\|p\|_\infty+\|q\|_\infty\,\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}\,[t_2^{r_1}-(t_2-t_1)^{r_1}](x_2-x_1)^{r_2}$$

$$+\frac{\|p\|_\infty+\|q\|_\infty\,\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}\,(t_2-t_1)^{r_1}\big[x_2^{r_2}-(x_2-x_1)^{r_2-1}\big]\leqslant\big\|z(t_1,x_1)-z(t_2,x_2)\big\|$$

$$+\frac{\|p\|_\infty+\|q\|_\infty\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}\big[2x_2^{r_2}(t_2-t_1)^{r_1}+2t_2^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}-2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}\big].$$

As $t_1\to t_2$, $x_1\to x_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1<t_2<0$, $x_1<x_2<0$ and $t_1\leqslant 0\leqslant t_2$, $x_1\leqslant 0\leqslant x_2$ is obvious. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we can conclude that $F:C_{(a,b)}\to C_{(a,b)}$ is continuous and completely continuous.

STEP 4. We show that G is a contraction.

Let $v,w\in C([-\alpha,a]\times[-\beta,b],\mathbb{R}^n)$. Then, for $(t,x)\in[-\alpha,a]\times[-\beta,b]$,

$$\big\|(Gv)(t,x)-(Gw)(t,x)\big\|\leqslant\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^{t}\int\limits_0^{x}\big|(t-s)^{r_1-1}\big|\,\big|(x-\tau)^{r_2-1}\big|$$

$$\times\big\|f\big(s,\tau,v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)-f\big(s,\tau,w_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\big\|\,d\tau ds\leqslant\frac{k}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times\int\limits_0^{t}\int\limits_0^{x}(t-s)^{r_1-1}\,(x-\tau)^{r_2-1}\,\big\|v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}-w_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big\|_C\,d\tau ds$$

$$\leqslant\frac{k}{\Gamma(r_1)\Gamma(r_2)}\,\|v-w\|_C\int\limits_0^{t}\int\limits_0^{x}(t-s)^{r_1-1}\,(x-\tau)^{r_2-1}\,d\tau ds\leqslant\frac{k\,t^{r_1}x^{r_2}}{\Gamma(r_1+1)\,\Gamma(r_2+1)}\,\|v-w\|_C.$$

Consequently,

$$\big\|(Gv)-(Gw)\big\|_{C_{(a,b)}}\leqslant\frac{k\,a^{r_1}b^{r_2}}{\Gamma(r_1+1)\,\Gamma(r_2+1)}\,\|v-w\|_C.$$

Since by (3), G is a contraction.

STEP 5 (a priori bounds). Now it remains to show that the set $\mathscr{E}=\{u\in C(J,\mathbb{R}):u=\lambda F(u)+\lambda G(\frac{u}{\lambda})\}$ for some $\lambda\in(0,1)$ is bounded.

Let $u\in\mathscr{E}$, then and $u=\lambda F(u)+\lambda G\big(\frac{u}{\lambda}\big)$ for some $0<\lambda<1$. Thus for each $(t,x)\in J$, we have

$$u(t,x)=\lambda z(t,x)+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^{t}\int\limits_0^{x}(t-s)^{r_1-1}(x-\tau)^{r_2-1}f\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\,d\tau ds$$

$$+\frac{\lambda}{\Gamma(r_1)\,\Gamma(r_2)}\int\limits_0^{t}\int\limits_0^{x}(t-s)^{r_1-1}\,(x-\tau)^{r_2-1}\,g\left(s,\tau,\frac{u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}}{\lambda}\right)d\tau ds.$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x)\in J$, we have

$$\|u(t,x)\| \leqslant \lambda\|z(t,x)\| + \frac{1}{\Gamma(r_1)\,\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times\Big[p(s,\tau)+q(s,\tau)\big\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big\|_C\Big]\,d\tau ds + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\left|g\left(s,\tau,\frac{u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}}{\lambda}\right)-g(s,\tau,0)\right|d\tau ds$$

$$+\frac{\lambda}{\Gamma(r_1)\,\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|g(s,\tau,0)|\,d\tau ds \leqslant \|z(t,x)\| + \frac{a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_1+1)\,\Gamma(r_2+1)}$$

$$+\frac{a^{r_1}b^{r_2}g^*}{\Gamma(r_1+1)\Gamma(r_2+1)}+\frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\big\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big\|_C\,d\tau ds$$

$$+\frac{k}{\Gamma(r_1)\,\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\big\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big\|_C\,d\tau ds \leqslant \|z(t,x)\|$$

$$+\frac{a^{r_1}b^{r_2}(\|p\|_\infty+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}+\frac{\|q\|_\infty+k}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\big\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big\|_C\,d\tau ds,$$

where $g^*=\sup_{(s,\tau)\in J}|g(s,\tau,0)|$.

Consider the function y defined by $y(t,x)=\sup\{\|u(s,\tau)\| : -\alpha\leqslant s\leqslant t, -\beta\leqslant\tau\leqslant x\}$, $0\leqslant t\leqslant a$, $0\leqslant x\leqslant b$.

Let $(t^*,x^*)\in[-\alpha,t]\times[-\beta,x]$ be such that $y(t,x)=\|u(t^*,x^*)\|$. If $(t^*,x^*)\in J$, then by the previous inequality, we have for $(t,x)\in J$,

$$y(t,x)\leqslant\|z(t,x)\|+\frac{a^{r_1}b^{r_2}(\|p\|_\infty+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}+\frac{\|q\|_\infty+k}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\,y(s,\tau)\,d\tau ds.$$

If $(t^*,x^*)\in\widetilde{J}$, then $y(t,x)=\|\phi\|_C$ and the previous inequality holds.

If $(t,x)\in J$, Lemma 2.1 implies that there exists $\delta=\delta(r_1,r_2)$ such that we have

$$y(t,x)\leqslant\left[\|z(t,x)\|+\frac{a^{r_1}b^{r_2}(\|p\|_\infty+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]\left[1+\frac{\delta(\|q\|_\infty+k)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\,d\tau ds\right]$$

$$\leqslant\left[\|z(t,x)\|+\frac{a^{r_1}b^{r_2}(\|p\|_\infty+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]\left[1+\frac{\delta a^{r_1}b^{r_2}(\|q\|_\infty+k)}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]:=M.$$

Since for every $(t,x)\in J$, $\|u_{(t,x)}\|_C\leqslant y(t,x)$, we have $\|u\|_\infty\leqslant\max(\|\phi\|_C,M):=M^*$. This shows that the set $\mathscr{E}$ is bounded. As a consequence of Theorem 2.1 we deduce that $F+G$ has a fixed point u which is a solution of problem (1)–(3). $\triangleright$

4. Existence Results for the Infinite Delay Case

4.1. The phase space $\mathscr{B}$. The notation of the phase space $\mathscr{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [22]). For further applications see for instance [23, 24] and their references.

For any $(t,x)\in J$ denote $E_{(t,x)} := [0,t]\times\{0\}\cup\{0\}\times[0,x]$, furthermore in case $t=a$, $x=b$ we write simply E. Consider the space $(\mathscr{B}, \|(\cdot,\cdot)\|_{\mathscr{B}})$ is a seminormed linear space of functions mapping $(-\infty,0]\times(-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $y:(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ continuous on J and $y_{(t,x)}\in\mathscr{B}$, for all $(t,x)\in E$, then there are constants $H,K,M>0$ such that for any $(t,x)\in J$ the following conditions hold:

(i) $y_{(t,x)}$ is in $\mathscr{B}$;

(ii) $\|y(t,x)\|\leqslant H\|y_{(t,x)}\|_{\mathscr{B}}$;

(iii) $\|y_{(t,x)}\|_{\mathscr{B}}\leqslant K\sup_{(s,\tau)\in[0,t]\times[0,x]}\|y(s,\tau)\|+M\sup_{(s,\tau)\in E_{(t,x)}}\|y_{(s,\tau)}\|_{\mathscr{B}}$.

(A_2) For the function $y(\cdot,\cdot)$ in (A_1), $y_{(t,x)}$ is a $\mathscr{B}$-valued continuous function on J.

(A_3) The space $\mathscr{B}$ is complete.

Now, we present some examples of phase spaces [25, 26].

Example 4.1. Let $\mathscr{B}$ be the set of all functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0]$, $\alpha,\beta\geqslant 0$, with the seminorm $\|\phi\|_{\mathscr{B}}=\sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,\tau)\|$. Then we have $H=K=M=1$. The quotient space $\widehat{\mathscr{B}}=\mathscr{B}/\|\cdot\|_{\mathscr{B}}$ is isometric to the space $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ of all continuous functions from $[-\alpha,0]\times[-\beta,0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 4.2. Let $\gamma\in\mathbb{R}$ and let C_γ be the set of all continuous functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ for which a limit $\lim_{\|(s,\tau)\|\to\infty}e^{\gamma(s+\tau)}\phi(s,\tau)$ exists, with the norm $\|\phi\|_{C_\gamma}=\sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]}e^{\gamma(s+\tau)}\|\phi(s,\tau)\|$. Then we have $H=1$ and $K=M=\max\{e^{-\gamma(a+b)},1\}$.

Example 4.3. Let $\alpha,\beta,\gamma\geqslant 0$ and let

$$\|\phi\|_{CL_\gamma}=\sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,\tau)\|+\int_{-\infty}^{0}\int_{-\infty}^{0}e^{\gamma(s+\tau)}\|\phi(s,\tau)\|\,d\tau ds$$

be the seminorm for the space CL_γ of all functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0]$ measurable on $(-\infty,-\alpha]\times(-\infty,0]\cup(-\infty,0]\times(-\infty,-\beta]$, and such that $\|\phi\|_{CL_\gamma}<\infty$. Then $H=1$, $K=\int_{-\alpha}^{0}\int_{-\beta}^{0}e^{\gamma(s+\tau)}d\tau ds$, $M=2$.

4.2. Main Results. Let us start in this section by defining what we mean by a solution of the problem (4)–(6). Let the space $\Omega:=\{u:(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n : u_{(t,x)}\in\mathscr{B}$ for $(t,x)\in E$ and $u|_J\in C(J,\mathbb{R}^n)\}$.

Definition 4.1. A function $u\in\Omega$ is said to be a solution of (4)–(6) if u satisfies equations (4) and (6) on J and the condition (5) on $\widetilde{J}'$.

Set $\mathscr{R}':=\mathscr{R}'_{(\rho_1^-,\rho_2^-)}=\{(\rho_1(s,\tau,u),\ \rho_2(s,\tau,u)) : (s,\tau,u)\in J\times\mathscr{B},\ \rho_i(s,\tau,u)\leqslant 0;\ i=1,2\}$. We always assume that $\rho_1:J\times\mathscr{B}\to(-\infty,a]$, $\rho_2:J\times\mathscr{B}\to(-\infty,b]$ are continuous and the function $(s,\tau)\mapsto u_{(s,\tau)}$ is continuous from $\mathscr{R}'$ into $\mathscr{B}$.

Our main result in this section is based upon the fixed point theorem due to Burton and Kirk.

We will need to introduce the following hypothesis:

(H_ϕ) There exists a continuous bounded function $L : \mathscr{R}'_{(\rho_1^-,\rho_2^-)} \to (0,\infty)$ such that $\|\phi_{(s,\tau)}\|_{\mathscr{B}} \leqslant L(s,\tau)\|\phi\|_{\mathscr{B}}$, for any $(s,\tau) \in \mathscr{R}'$.

In the sequel we will make use of the following generalization of a consequence of the phase space axioms [27, Lemma 2.1].

Lemma 4.1. *If $u \in \Omega$, then*

$$\|u_{(s,\tau)}\|_{\mathscr{B}} = (M + L')\,\|\phi\|_{\mathscr{B}} + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,\tau\}]} \|u(\theta,\eta)\|,$$

where $L' = \sup_{(s,\tau)\in\mathscr{R}'} L(s,\tau)$.

Theorem 4.1. *Assume (H_ϕ) and that the following hypothesis holds:*

$(H1)$ The functions $f, g : J \times \mathscr{B} \to \mathbb{R}^n$ are continuous.

$(H2)$ There exists $\ell > 0$ such that $\|g(t,x,u) - g(t,x,v)\| \leqslant \ell\|u - v\|_{\mathscr{B}}$, for any $u, v \in \mathscr{B}$ and $(t,x) \in J$.

$(H3)$ There exist $p, q \in C(J,\mathbb{R}_+)$ such that $\|f(t,x,u)\| \leqslant p(t,x) + q(t,x)\|u\|_{\mathscr{B}}$, for $(t,x) \in J$ and each $u \in \mathscr{B}$.

If

$$\frac{\ell\, K a^{r_1} b^{r_2}}{\Gamma(r_1+1)\,\Gamma(r_2+1)} < 1, \tag{1}$$

then the IVP (4)–(6) has at least one solution on $(-\infty,a] \times (-\infty,b]$.

$\lhd$ Transform the problem (4)–(6) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$(Nu)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \widetilde{J}, \\ z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \quad \times f\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\,d\tau ds & \\ \quad\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \quad\quad \times g\big(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\,d\tau ds, & (t,x) \in J. \end{cases} \tag{2}$$

Let $v(\cdot,\cdot) : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n$ be a function defined by,

$$v(t,x) = \begin{cases} z(t,x), & (t,x) \in J, \\ \phi(t,x), & (t,x) \in \widetilde{J}. \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$.

For each $w \in C(J,\mathbb{R}^n)$ with $w(t,x) = 0$ for each $(t,x) \in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} w(t,x) & (t,x) \in J, \\ 0, & (t,x) \in \widetilde{J}. \end{cases}$$

If $u(\cdot,\cdot)$ satisfies the integral equation

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times f\big(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\big)\,d\tau ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g\big(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\big)\,d\tau ds,$$

we can decompose $u(\cdot,\cdot)$ as $u(t,x) = \overline{w}(t,x) + v(t,x)$; $(t,x) \in J$, which implies $u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}$, $(t,x) \in J$, and the function $w(\cdot,\cdot)$ satisfies

$$w(t,x) = \frac{1}{\Gamma(r_1)\,\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}\,(x-\tau)^{r_2-1}$$

$$\times f\big(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\big)\,d\tau ds + \frac{1}{\Gamma(r_1)\,\Gamma(r_2)}$$

$$\times \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\, g\big(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\big)\,d\tau ds.$$

Set $C_0 = \{w \in C(J,\mathbb{R}^n) : w(t,x) = 0 \text{ for } (t,x) \in E\}$, and let $\|\cdot\|_{(a,b)}$ be the seminorm in C_0 defined by $\|w\|_{(a,b)} = \sup_{(t,x)\in E} \|w_{(t,x)}\|_{\mathscr{B}} + \sup_{(t,x)\in J} \|w(t,x)\| = \sup_{(t,x)\in J} \|w(t,x)\|$, $w \in C_0$. C_0 is a Banach space with norm $\|\cdot\|_{(a,b)}$. Let the operators $A, B : C_0 \to C_0$ defined by

$$(Aw)(t,x) = \frac{1}{\Gamma(r_1)\,\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}\,(x-\tau)^{r_2-1}$$

$$\times\ f\big(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\big)\,d\tau ds$$

and

$$(Bw)(t,x) = \frac{1}{\Gamma(r_1)\,\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}\,(x-\tau)^{r_2-1}$$

$$\times\ g\big(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\big)\,d\tau ds.$$

Then the operator N has a fixed point is equivalent to finding the fixed point of the operator equation $(Aw)(t,x) + (Bw)(t,x) = w(t,x)$, $(t,x) \in J$. We shall show that the operators A and B satisfies all the conditions of Theorem 2.1.

For better readability, we break the proof into a sequence of steps.

Step 1. F is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0. Then

$$\big\|(Aw_n)(t,x) - (Aw)(t,x)\big\| \leqslant \frac{1}{\Gamma(r_1)\,\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}\,(x-\tau)^{r_2-1}$$

$$\times \big\|f\big(s,\tau,\overline{w}_{n(\rho_1(s,\tau,u_{n(s,\tau)}),\rho_2(s,\tau,u_{n(s,\tau)}))} + v_{n(\rho_1(s,\tau,u_{n(s,\tau)}),\rho_2(s,\tau,u_{n(s,\tau)}))}\big)$$

$$- f\big(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\big\|\,d\tau ds$$

$$\leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\,\big\|f(s,\tau,\overline{w}_{n(s,\tau)}+v_{n(s,\tau)}) - f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})\big\|\,d\tau ds.$$

Since f is a continuous function, we have

$$\big\|(Aw_n) - (Aw)\big\|_\infty \leqslant \frac{t^{r_1}x^{r_2}\|f(\cdot,\cdot,\overline{w_n}_{(\cdot,\cdot)} + v_{n(\cdot,\cdot)}) - f(\cdot,\cdot,\overline{w}_{(\cdot,\cdot)} + v_{(\cdot,\cdot)})\|_\infty}{\Gamma(r_1+1)\,\Gamma(r_2+1)}$$

$$\leqslant \frac{a^{r_1}b^{r_2}\|f(\cdot,\cdot,\overline{w_n}_{(\cdot,\cdot)} + v_{n(\cdot,\cdot)}) - f(\cdot,\cdot,\overline{w}_{(\cdot,\cdot)} + v_{(\cdot,\cdot)})\|_\infty}{\Gamma(r_1+1)\,\Gamma(r_2+1)} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus A is continuous.

STEP 2. A maps bounded sets into bounded sets in C_0.

Indeed, it is enough show that, for any $\eta > 0$. there exists a positive constant $\tilde{\ell}$ such that, for each $w \in B_\eta = \{w \in C_0 : \|w\|_{(a,b)} \leqslant \eta\}$, we have $\|A(w)\|_\infty \leqslant \tilde{\ell}$.

Lemma 4.1 implies that

$$\left\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\right\|_{\mathscr{B}} \leqslant \left\|\overline{w}_{(s,\tau)}\right\|_{\mathscr{B}} + \left\|v_{(s,\tau)}\right\|_{\mathscr{B}} \leqslant K\eta + K\left\|\phi(0,0)\right\| + (M+L')\|\phi\|_{\mathscr{B}}.$$

Set $\eta^* := K\eta + K\|\phi(0,0)\| + (M+L')\|\phi\|_{\mathscr{B}}$. Let $w \in B_\eta$. By $(H3)$ we have for each $(t,x) \in J$,

$$\begin{aligned}
\left\|(Aw)(t,x)\right\| &\leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} \\
&\quad \times \left\|f\left(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right)\right\| d\tau ds \\
&\leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} p(s,\tau)\, d\tau ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} \\
&\quad \times (x-\tau)^{r_2-1} q(s,\tau) \left\|\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right\|_{\mathscr{B}} d\tau ds \\
&\leqslant \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds + \frac{\|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&\leqslant \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} t^{r_1} x^{r_2} \leqslant \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} a^{r_1} b^{r_2} := \ell^*.
\end{aligned}$$

Hence $\|A(w)\|_\infty \leqslant \ell^*$.

STEP 3. A maps bounded sets into equicontinuous sets in C_0.

Let $(t_1,x_1),(t_2,x_2) \in (0,a] \times (0,b]$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set as in Step 2, and let $w \in B_\eta$. Then

$$\begin{aligned}
\left\|(Aw)(t_2,x_2) - (Aw)(t_1,x_1)\right\| &\leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} \left[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}\right] \\
&\quad \times \left\|f\left(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right)\right\| d\tau ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2-s)^{r_1-1} (x_2-\tau)^{r_2-1} \\
&\quad \left\|f\left(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right)\right\| d\tau ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2-s)^{r_1-1} (x_2-\tau)^{r_2-1} \\
&\quad \times \left\|f\left(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right)\right\| d\tau ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2-s)^{r_1-1} (x_2-\tau)^{r_2-1} \\
&\quad \times \left\|f\left(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right)\right\| d\tau ds
\end{aligned}$$

$$\leqslant \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^{t_1}\int\limits_0^{x_1} \left[(t_1 - s)^{r_1-1}(x_1-\tau)^{r_2-1} - (t_2-s)^{r_1-1}(x_2-t)^{r_2-1}\right] d\tau ds$$

$$+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int\limits_{t_1}^{t_2}\int\limits_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\, d\tau ds + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int\limits_0^{t_1}\int\limits_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\, d\tau ds + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int\limits_{t_1}^{t_2}\int\limits_0^{x_1} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\, d\tau ds$$

$$\leqslant \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \left[x_2^{r_2}(t_2-t_1)^{r_1} + t_2^{r_1}(x_2-x_1)^{r_2} - (t_2-t_1)^{r_1}(x_2-x_1)^{r_2} + t_1^{r_1}x_1^{r_2} - t_2^{r_1}x_2^{r_2}\right]$$

$$+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} (t_2-t_1)^{r_1}(x_2-x_1)^{r_2} + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \left[t_2^{r_1} - (t_2-t_1)^{r_1}\right](x_2-x_1)^{r_2}$$

$$+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} (t_2-t_1)^{r_1}\left[x_2^{r_2} - (x_2-x_1)^{r_2}\right] \leqslant \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$\times \left[2x_2^{r_2}(t_2-t_1)^{r_1} + 2t_2^{r_1}(x_2-x_1)^{r_2} + t_1^{r_1}x_1^{r_2} - t_2^{r_1}x_2^{r_2} - 2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}\right].$$

As $t_1 \to t_2$, $x_1 \to x_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \leqslant 0 \leqslant t_2$, $x_1 \leqslant 0 \leqslant x_2$ is obvious. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we can conclude that $A : C_0 \to C_0$ is continuous and completely continuous.

STEP 4. B is a contraction.

Let $w, w^* \in C_0$. Then we have for each $(t,x) \in J$

$$\left\|(Bw)(t,x) - (Bw^*)(t,x)\right\| \leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^{t}\int\limits_0^{x} (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times \left\|g\big(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\right.$$

$$\left. - g\big(s,\tau,\overline{w^*}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\big)\right\| d\tau ds$$

$$\leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^{t}\int\limits_0^{x} (t-s)^{r_1-1}(x-\tau)^{r_2-1}\, \ell \left\|\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right.$$

$$\left. - \overline{w^*}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right\|_{\mathscr{B}} \leqslant \frac{\ell}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^{t}\int\limits_0^{x} (t-s)^{r_1-1}(x-\tau)^{r_2-1}\left\|\overline{w}_{(s,\tau)} - \overline{w^*}_{(s,\tau)}\right\|_{\mathscr{B}}$$

$$\leqslant \frac{\ell K}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^{t}\int\limits_0^{x} (t-s)^{r_1-1}(x-\tau)^{r_2-1} \sup_{(s,\tau)\in[0,T]\times[0,X]} \left\|\overline{w}(s,\tau) - \overline{w^*}(s,\tau)\right\| d\tau ds$$

$$\leqslant \frac{\ell K}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^{t}\int\limits_0^{x} (t-s)^{r_1-1}(x-\tau)^{r_2-1}\, d\tau ds \left\|\overline{w} - \overline{w^*}\right\|_{(a,b)} \leqslant \frac{\ell K t^{r_1} x^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \left\|\overline{w} - \overline{w^*}\right\|_{(a,b)}.$$

Therefore $\|(Bw) - (Bw^*)\|_{(a,b)} \leqslant \frac{\ell K a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|\overline{w} - \overline{w^*}\|_{(a,b)}$. Since by (1), B is a contraction.

STEP 5 (A priori bounds). Now it remains to show that the set $\mathscr{E} = \{w \in C(J,\mathbb{R}) : w = \lambda A(w) + \lambda B(\frac{w}{\lambda})\}$ for some $\lambda \in (0,1)$ is bounded.

Let $w \in \mathscr{E}$, then and $w = \lambda A(w) + \lambda B\left(\frac{w}{\lambda}\right)$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J$, we have

$$w(t,x) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times f\left(s,\tau,\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\right) d\tau ds + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g\left(s,\tau,\frac{\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}+v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}}{\lambda}\right) d\tau ds.$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x) \in J$, we have

$$\|w(t,x)\| \leqslant \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times \left[p(s,\tau) + q(s,\tau)\left\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\right\|_{\mathscr{B}}\right] d\tau ds + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$

$$\times \left| g\left(s,\tau,\frac{\overline{w}_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} + v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}}{\lambda}\right) - g(s,\tau,0)\right| d\tau ds$$

$$+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\left|g(s,\tau,0)\right| d\tau ds \leqslant \frac{a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$+ \frac{a^{r_1}b^{r_2}g^*}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\left\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\right\|_{\mathscr{B}} d\tau ds$$

$$+ \frac{\ell}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\left\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\right\|_{\mathscr{B}} d\tau ds \leqslant \frac{a^{r_1}b^{r_2}(\|p\|_\infty + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$+ \frac{(\|q\|_\infty + \ell)}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\left\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\right\|_{\mathscr{B}} d\tau ds,$$

where $g^* = \sup_{(s,\tau)\in J}|g(s,\tau,0)|$ and

$$\begin{aligned} \left\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\right\|_{\mathscr{B}} &\leqslant \left\|\overline{w}_{(s,\tau)}\right\|_{\mathscr{B}} + \|v_{(s,\tau)}\|_{\mathscr{B}} \\ &\leqslant K \sup\left\{w(\tilde{s},\tilde{\tau}) : (\tilde{s},\tilde{\tau}) \in [0,s]\times[0,\tau]\right\} + (M+L')\|\phi\|_{\mathscr{B}} + K\|\phi(0,0)\|. \end{aligned} \tag{3}$$

If we name $y(s,\tau)$ the right hand side of (3), then we have $\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathscr{B}} \leqslant y(t,x)$, and therefore, for each $(t,x) \in J$ we obtain

$$\|w(t,x)\| \leqslant \frac{a^{r_1}b^{r_2}(\|p\|_\infty + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{\|q\|_\infty + \ell}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,\tau)\, d\tau ds. \tag{4}$$

Using the above inequality and the definition of y for each $(t,x)\in J$ we have

$$y(t,x)\leqslant (M+L')\,\|\phi\|_{\mathscr{B}}+K\,\|\phi(0,0)\|+\frac{Ka^{r_1}b^{r_2}(\|p\|_\infty+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}$$
$$+\frac{K(\|q\|_\infty+\ell)}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x(t-s)^{r_1-1}\,(x-\tau)^{r_2-1}\,y(s,t)\,d\tau ds.$$

Then by Lemma 2.1, there exists $\delta=\delta(r_1,r_2)$ such that we have

$$\|y(t,x)\|\leqslant R+\delta\,\frac{K(\|q\|_\infty+\ell)}{\Gamma(r_1)\Gamma(r_2)}\int\limits_0^t\int\limits_0^x(t-s)^{r_1-1}\,(x-\tau)^{r_2-1}\,R\,d\tau ds,$$

where

$$R=(M+L')\,\|\phi\|_{\mathscr{B}}+K\,\|\phi(0,0)\|+\frac{Ka^{r_1}b^{r_2}(\|p\|_\infty+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|y\|_\infty\leqslant R+\frac{R\,\delta Ka^{r_1}b^{r_2}(\|q\|_\infty+\ell)}{\Gamma(r_1+1)\Gamma(r_2+1)}:=\widetilde{R}.$$

Then, (4) implies that

$$\|w\|_\infty\leqslant\frac{a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\left[\|p\|_\infty+g^*+\widetilde{R}(\|q\|_\infty+\ell)\right]:=R^*.$$

This shows that the set $\mathscr{E}$ is bounded. As a consequence of Theorem 2.1 we deduce that $A+B$ has a fixed point w which is a solution of problem (4)–(6). ▷

5. Examples

Example 5.1. As an application of our results we consider the following fractional order perturbed hyperbolic partial functional differential equations with finite delay of the form

$$({}^cD_0^r u)(t,x)=\frac{|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|+2}{10e^{t+x+4}(1+|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|)},$$
$$\text{if}\quad (t,x)\in J:=[0,1]\times[0,1],\quad (1)$$

$$u(t,0)=t,\quad u(0,x)=x^2,\quad (t,x)\in J,\qquad (2)$$

$$u(t,x)=t+x^2,\quad (t,x)\in\tilde{J}:=[-1,1]\times[-2,1]\backslash[0,1]\times[0,1],\qquad (3)$$

where $\sigma_1\in C(\mathbb{R},[0,1])$, $\sigma_2\in C(\mathbb{R},[0,2])$.

$$\rho_1(t,x,\varphi)=t-\sigma_1(\varphi(0,0)),\quad (t,x,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R}),$$

$$\rho_2(t,x,\varphi)=x-\sigma_2(\varphi(0,0)),\quad (t,x,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R}),$$

$$f(t,x,\varphi)=\frac{|\varphi|}{(10e^{t+x+4})(1+|\varphi|)},\quad (t,x)\in J,\ \varphi\in C([-1,0]\times[-2,0],\mathbb{R}),$$

and

$$g(t,x,\varphi)=\frac{2}{(10e^{t+x+4})(1+|\varphi|)},\quad (t,x)\in J,\ \varphi\in C([-1,0]\times[-2,0],\mathbb{R}).$$

For each $\varphi, \overline{\varphi} \in C([-1,0]\times[-2,0],\mathbb{R})$ and $(t,x)\in J$ we have

$$\left|g(t,x,\varphi)-g(t,x,\overline{\varphi})\right| \leqslant \frac{1}{5e^4}\left\|\varphi-\overline{\varphi}\right\|_C.$$

Hence condition $(H2)$ is satisfied with $k=\frac{1}{5e^4}$. We shall show that condition (3) holds with $a=b=1$. Indeed

$$\frac{ka^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{5e^4\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

which is satisfied for each $(r_1,r_2)\in(0,1]\times(0,1]$. Also, the function f is continuous on $[0,1]\times[0,1]\times[0,\infty)$ and $|f(t,x,\varphi)|\leqslant|\varphi|$, for each $(t,x,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R})$. Thus conditions $(H1)$ and $(H3)$ hold. Consequently Theorem 3.1 implies that problem (1)–(3) has at least one solution defined on $[-1,1]\times[-2,1]$.

Example 5.2. We consider now the following fractional order perturbed hyperbolic partial functional differential equations with infinite delay of the form

$$({}^cD_0^r u)(t,x) = \frac{3+|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|}{9e^{t+x+5}(1+|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|)},$$
$$\text{if} \quad (t,x)\in J:=[0,1]\times[0,1], \quad (4)$$

$$u(t,0)=t, \quad u(0,x)=x^2, \quad (t,x)\in J, \quad (5)$$

$$u(t,x)=t+x^2, \quad (t,x)\in\widetilde{J}, \quad (6)$$

where $\widetilde{J}:=(-\infty,1]\times(-\infty,1]\backslash[0,1]\times(0,1]$, $\sigma_1\in C(\mathbb{R},[0,1])$, $\sigma_2\in C(\mathbb{R},[0,2])$.

$$\mathscr{B}_\gamma=\left\{u\in C((-\infty,0]\times(-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) \text{ exists } \in\mathbb{R}\right\}.$$

The norm of $\mathscr{B}_\gamma$ is given by $\|u\|_\gamma=\sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta,\eta)|$.

Let $E:=[0,1]\times\{0\}\cup\{0\}\times[0,1]$, and $u:(-\infty,1]\times(-\infty,1]\to\mathbb{R}$ such that $u_{(t,x)}\in\mathscr{B}_\gamma$ for $(t,x)\in E$, then

$$\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(t,x)}(\theta,\eta)=\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-t+\eta-x)}u(\theta,\eta)=e^{-\gamma(t+x)}\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta)<\infty.$$

Hence $u_{(t,x)}\in\mathscr{B}_\gamma$. Finally we prove that

$$\|u_{(t,x)}\|_\gamma = K\sup\left\{|u(s,\tau)| : (s,\tau)\in[0,t]\times[0,x]\right\} + M\sup\left\{\|u_{(s,\tau)}\|_\gamma : (s,\tau)\in E_{(t,x)}\right\},$$

where $K=M=1$ and $H=1$.

If $t+\theta\leqslant 0$, $x+\eta\leqslant 0$ we get $\|u_{(t,x)}\|_\gamma=\sup\{|u(s,\tau)| : (s,\tau)\in(-\infty,0]\times(-\infty,0]\}$, and if $t+\theta\geqslant 0$, $x+\eta\geqslant 0$, then we have $\|u_{(t,x)}\|_\gamma=\sup\{|u(s,\tau)| : (s,\tau)\in[0,t]\times[0,x]\}$. Thus, for all $(t+\theta,x+\eta)\in[0,1]\times[0,1]$, we get

$$\|u_{(t,x)}\|_\gamma = \sup\left\{|u(s,\tau)| : (s,\tau)\in(-\infty,0]\times(-\infty,0]\right\} + \sup\left\{|u(s,\tau)| : (s,\tau)\in[0,t]\times[0,x]\right\}.$$

Then $\|u_{(t,x)}\|_\gamma=\sup\{\|u_{(s,\tau)}\|_\gamma : (s,\tau)\in E\}+\sup\{|u(s,\tau) : (s,\tau)\in[0,t]\times[0,x]|\}$. $(\mathscr{B}_\gamma,\|\cdot\|_\gamma)$ is a Banach space. We conclude that $\mathscr{B}_\gamma$ is a phase space.

$$\rho_1(t,x,\varphi)=t-\sigma_1(\varphi(0,0)), \quad (t,x,\varphi)\in J\times\mathscr{B}_\gamma,$$

$$\rho_2(t,x,\varphi) = x - \sigma_2(\varphi(0,0)), \quad (t,x,\varphi) \in J \times \mathscr{B}_\gamma,$$

$$f(t,x,\varphi) = \frac{|\varphi|}{(9e^{t+x+5})(1+|\varphi|)}, \quad (t,x) \in J,\ \varphi \in \mathscr{B}_\gamma,$$

and

$$g(t,x,\varphi) = \frac{3}{(9e^{t+x+5})(1+|\varphi|)}, \quad (t,x) \in J,\ \varphi \in \mathscr{B}_\gamma.$$

For each φ, $\overline{\varphi} \in \mathscr{B}_\gamma$ and $(t,x) \in J$ we have

$$\left|g(t,x,\varphi) - g(t,x,\overline{\varphi})\right| \leqslant \frac{1}{3e^5} \left\|\varphi - \overline{\varphi}\right\|_{\mathscr{B}_\gamma}.$$

Hence condition $(H2)$ is satisfied with $\ell = \frac{1}{3e^5}$. We shall show that condition (1) holds with $a = b = K = 1$ we get

$$\frac{\ell a^{r_1} b^{r_2} K}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3e^5\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. Also, the function f is continuous on $[0,1] \times [0,1] \times [0,\infty)$ and $|f(t,x,\varphi)| \leqslant 3 + |\varphi|$, for each $(t,x,\varphi) \in [0,1] \times [0,1] \times \mathscr{B}_\gamma$. Thus conditions $(H1)$ and $(H3)$ hold. Consequently Theorem 4.1 implies that problem (4)–(6) has at least one solution defined on $(-\infty, 1] \times (-\infty, 1]$.

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References

1. *Diethelm, K. and Freed, A. D.* On the Solution of Nonlinear Fractional Order Differential Equations Used in the Modeling of Viscoplasticity, *Scientific Computing in Chemical Engineering II, Computational Fluid Dynamics, Reaction Engineering and Molecular Properties*, F. Keil, W. Mackens, H. Voss, and J. Werther eds., Heidelberg, Springer-Verlag, 1999, pp. 217–224,
2. *Miller, K. S. and Ross, B. An Introduction to the Fractional Calculus and Differential Equations*, New York, John Wiley, 1993.
3. *Podlubny, I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of their Applications*, San Diego, Academic Press, 1999.
4. *Abbas, S. and Benchohra, M.* Partial Hyperbolic Differential Equations with Finite Delay Involving the Caputo Fractional Derivative, *Communications in Mathematical Analysis*, 2009, vol. 7, no. 2, pp. 62–72.
5. *Abbas, S. and Benchohra, M.* Darboux Problem for Perturbed Partial Differential Equations of Fractional Order with Finite Delay, *Elsevier Nonlinear Analysis: Hybrid Systems*, 2009, vol. 3, no. 4, pp. 597–604. DOI: 10.1016/j.nahs.2009.05.001.
6. *Benchohra, M., Henderson, J., Ntouyas, S. K. and Ouahab A.* Existence Results for Functional Differential Equations of Fractional Order, *Journal of Mathematical Analysis and Applications*, 2008, vol. 338, pp. 1340–1350. DOI: 10.1016/j.jmaa.2007.06.021.
7. *Hale, J. K. and Verduyn Lunel, S. M. Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, New York, Springer-Verlag, 1993.
8. *Lakshmikantham, V., Wen, L. and Zhang, B. Theory of Differential Equations with Unbounded Delay*, Mathematics and its Applications, vol. 298, Dordrecht, Kluwer Academic Publishers, 1994.
9. *Corduneanu, C. and Lakshmikantham, V.* Equations with Unbounded Delay, *Nonlinear Analysis: Theory, Methods and Applications*, 1980, vol. 4, no. 5, pp. 831–877. DOI: 10.1016/0362-546X(80)90001-2.
10. *Willé, D. R. and Baker, C. T. H.* Stepsize Control and Continuity Consistency for State-Dependent Delay-Differential Equations, *Journal of Computational and Applied Mathematics* 1994, vol. 53, no. 2, pp. 163–170. DOI: 10.1016/0377-0427(94)90043-4.
11. *Anguraj A., Arjunan, M. M. and Hernàndez, E. M.* Existence Results for an Impulsive Neutral Functional Differential Equation with State-Dependent Delay, *Applicable Analysis*, 2007, vol. 86, no. 7, pp. 861–872. DOI: 10.1080/00036810701354995.

12. *Hartung, F.* Linearized Stability for a Class of Neutral Functional Differential Equations with State-Dependent Delays, *Nonlinear Analysis: Theory, Methods and Applications*, 2008, vol. 69, no. 5–6, pp. 1629–1643. DOI: 0.1016/j.na.2007.07.004.
13. *Hernandez, E., Sakthivel, R. and Tanaka Aki, S.* Existence Results for Impulsive Evolution Differential Equations with State-Dependent Delay, *Electronic Journal of Differential Equations*, 2008, no. 28, pp. 1–11.
14. *Darwish, M. A. and Ntouyas, S. K.* Semilinear Functional Differential Equations of Fractional Order with State-Dependent Delay, *Electronic Journal of Differential Equations*, 2009, no. 38, pp. 1–10.
15. *Benchohra, M. and Hellal, M.* Perturbed Partial Functional Fractional Order Differential Equations with Infinite Delay, *Journal of Advanced Research in Dynamical and Control System*, 2013, vol. 5, no. 2, pp. 1–15.
16. *Benchohra, M. and Hellal, M.* Perturbed Partial Fractional Order Functional Differential Equations with Infinite Delay in Fréchet Spaces, *Nonlinear Dynamics and System Theory*, 2014, vol. 14, no. 3, pp. 244–257.
17. *Helal, M.* Perturbed Partial Functional Differential Equations on Unbounded Domains with Finite Delay, *Applied Mathematics E-Notes*, 2020, vol. 20, pp. 82–92.
18. *Helal, M.* On Fractional Differential Perturbed Equations with State-Dependent Delay in Fréchet Spaces, *Nonlinear Studies*, 2021, vol. 28, no. 4, pp. 1085–1106.
19. *Vityuk, A. N. and Golushkov, A. V.* Existence of Solutions of Systems of Partial Differential Equations of Fractional Order, *Nonlinear Oscillations*, 2004, vol. 7, no. 3, pp. 318–325. DOI: 10.1007/s11072-005-0015-9.
20. *Henry, D. Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Lecture Notes in Mathematics, vol. 840, Berlin–New York, Springer-Verlag, 1981.
21. *Burton, T. A. and Kirk, C.* A Fixed Point Theorem of Krasnoselskii–Schaefer Type, *Mathematische Nachrichten*, 1998, vol. 189, pp. 23–31. DOI: 10.1002/MANA.19981890103.
22. *Hale, J. and Kato, J.* Phase Space for Retarded Equations with Infinite Delay, *Funkcialaj Ekvacioj*, 1978, vol. 21, pp. 11–41.
23. *Hartung, F.* Differentiability of Solutions with Respect to Parameters in Neutral Differential Equations with State-Dependent Delays, *Journal of Mathematical Analysis and Applications*, 2006, vol. 324, no. 1, pp. 504–524. DOI: 10.1016/j.jmaa.2005.12.025.
24. *Hino, Y., Murakami, S. and Naito, T. Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, vol. 1473, Berlin, Springer-Verlag, 1991.
25. *Czlapinski, T.* On the Darboux Problem for Partial Differential-Functional Equations with Infinite Delay at Derivatives, *Nonlinear Analysis: Theory, Methods and Applications*, 2001, vol. 44, no. 3, pp. 389–398. DOI: 10.1016/S0362-546X(99)00275-8.
26. *Czlapinski, T.* Existence of Solutions of the Darboux Problem for Partial Differential-Functional Equations with Infinite Delay in a Banach Space, *Commentationes Mathematicae*, 1995, vol. 35, pp. 111–122.
27. *Hernández, E., Prokopczyk, A. and Ladeira, L.* A Note on Partial Functional Differential Equations with State-Dependent Delay, *Nonlinear Analysis: Real World Applications*, 2006, vol. 7, no. 4, pp. 510–519. DOI: 10.1016/j.nonrwa.2005.03.014.

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РЕЗУЛЬТАТЫ СУЩЕСТВОВАНИЯ ДЛЯ ФУНКЦИОНАЛЬНО ВОЗМУЩЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ДРОБНОГО ПОРЯДКА С ЗАПАЗДЫВАНИЕМ В БАНАХОВЫХ ПРОСТРАНСТВАХ

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Аннотация. В данной работе мы приводим достаточные условия существования решений начальной задачи для функционально возмущенных гиперболических дифференциальных уравнений в частных производных дробного порядок с участием дробной производной Капуто с запаздыванием, зависящим от состояния, сводя исследование к поиску существования и единственности неподвижных точек соответствующих операторов. Наш основной результат для этой задачи основан на нелинейной альтернативной теореме о неподвижной точке Бёртона и Кирка для суммы вполне непрерывного оператора и сжатия в банаховых пространствах и дробной версии неравенства Гронуолла. Чтобы получить результаты существования необходимо принимать во внимание как структуру пространства, так и свойства возникающих операторов. Насколько нам известно, очень мало работ, посвященных уравнениям дробных производных с конечным и/или бесконечным постоянным запаздыванием на ограниченных областях. В этом направлении возникает множество проблемных вопросов относительно существования решений в весовых пространствах непрерывных функций, единственности решения, строения множества решений, а также того, являются ли оптимальными условия, которым подчинены рассматриваемые операторы. Данную статью можно рассматривать как вклад в указанную проблематику. Приведены также иллюстрирующие примеры.

Ключевые слова: уравнение в частных производных, дробный порядок, решение, левосторонний смешанный интеграл Римана — Лиувилля, дробная производная Капуто, зависящая от состояния запаздывание, неподвижная точка.

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