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ON POLETSKY-TYPE MODULUS INEQUALITIES FOR SOME CLASSES OF MAPPINGS#

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Abstract. It is well-known that the theory of mappings with bounded distortion was laid by Yu. G. Reshetnyak in 60-th of the last century [1]. In papers [2, 3], there was introduced the two-index scale of mappings with weighted bounded (q, p)-distortion. This scale of mappings includes, in particular, mappings with bounded distortion mentioned above (under $q = p = n$ and the trivial weight function). In paper [4], for the two-index scale of mappings with weighted bounded (q, p)-distortion, the Poletsky-type modulus inequality was proved under minimal regularity; many examples of mappings were given to which the results of [4] can be applied. In this paper we show how to apply results of [4] to one such class. Another goal of this paper is to exhibit a new class of mappings in which Poletsky-type modulus inequalities is valid. To this end, for $n = 2$, we extend the validity of the assertions in [4] to the limiting exponents of summability: $1 < q \leqslant p \leqslant \infty$. This generalization contains, as a special case, the results of recently published papers. As a consequence of our results, we also obtain estimates for the change in capacity of condensers.

Key words: quasiconformal analysis, Sobolev space, modulus of a family of curves, modulus estimate.

AMS Subject Classification: 30C65 (26B35, 31B15, 46E35).

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1. Introduction

The goal of this work is to show the application of results of [4] for output of Poletsky-type modulus inequalities for some classes of mappings. For doing this we formulate first the main result of [4], and then we provide how it can be applied for some concrete classes of mappings.

The main classes of mappings studied in [4] were defined in [2, 3].

Definition 1. Let $\omega\colon \mathbb{R}^n \to [0,\infty]$ be a measurable function, called a *weight*, with $0 < \omega < \infty$ holding $\mathscr{H}^n$-almost everywhere, and $\Omega \subset \mathbb{R}^n$ is a domain in $\mathbb{R}^n$. A mapping $f : \Omega \to \mathbb{R}^n$ with $n \geqslant 2$ is called a *mapping with* (*inner*) *bounded* ω*-weighted* (q, p)*-codistortion*, or briefly, $f \in \mathscr{ID}(\Omega; q, p; \omega, 1)$, where $n - 1 \leqslant q \leqslant p < \infty$, whenever

(1) f is continuous, open and discrete;

(2) f belongs to the Sobolev class $W^1_{n-1,\mathrm{loc}}(\Omega)$;

(3) the Jacobian determinant satisfies $\det Df(x) \geqslant 0$ for almost all $x \in \Omega$;

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(4) the mapping f has *bounded codistortion*: $\operatorname{adj} Df(x) = 0$ a. e. on the set $Z = \{x \in \Omega : \det Df(x) = 0\}$;

(5) the *local ω-weighted (q,p)-codistortion function*

$$\Omega \ni x \mapsto \mathscr{K}^{\omega,1}_{q,p}(x,f) = \begin{cases} \frac{\omega^{\frac{n-1}{q}}(x)|\operatorname{adj} Df(x)|}{\det Df(x)^{\frac{n-1}{p}}} & \text{if } \det Df(x) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

belongs to $L_\varrho(\Omega)$, where ϱ satisfies $\frac{1}{\varrho} = \frac{n-1}{q} - \frac{n-1}{p}$, while $\varrho = \infty$ for $q = p$.

Put $\mathscr{K}^{\omega,1}_{q,p}(f;\Omega) = \|\mathscr{K}^{\omega,1}_{q,p}(\cdot,f) \mid L_\varrho(\Omega)\|$.

Definition 2. Let $\omega\colon \mathbb{R}^n \to [0,\infty]$ be a measurable function, called a *weight*, with $0 < \omega < \infty$ holding $\mathscr{H}^n$-almost everywhere, and $\Omega \subset \mathbb{R}^n$ is a domain in $\mathbb{R}^n$. A mapping $f : \Omega \to \mathbb{R}^n$ with $n \geqslant 2$ is called a *mapping with* (*outer*) *bounded ω-weighted (q,p)-distortion*, or briefly $f \in \mathscr{OD}(\Omega; q, p; \omega, 1)$, with $n - 1 \leqslant q \leqslant p < \infty$, whenever:

(1) f is continuous, open and discrete;

(2) f belongs to the Sobolev class $W^1_{n-1,\mathrm{loc}}(\Omega)$;

(3) the Jacobian determinant satisfies $\det Df(x) \geqslant 0$ for a. e. $x \in \Omega$;

(4) the mapping f has *bounded distortion*: $Df(x) = 0$ a. e. on the set $Z = \{x \in \Omega : \det Df(x) = 0\}$;

(5) the *local ω-weighted (q,p)-distortion function*

$$\Omega \ni x \mapsto K^{\omega,1}_{q,p}(x,f) = \begin{cases} \frac{\omega^{\frac{1}{q}}(x)|Df(x)|}{\det Df(x)^{\frac{1}{p}}} & \text{if } \det Df(x) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

belongs to $L_\varkappa(\Omega)$, where $\varkappa$ satisfies $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$, while $\varkappa = \infty$ for $q = p$.

Put $K^{\omega,1}_{q,p}(f;\Omega) = \|K^{\omega,1}_{q,p}(\cdot,f) \mid L_\varkappa(\Omega)\|$.

Remark 1. It is established in [3] that

$$\mathscr{OD}(\Omega; q, p; \omega, 1) \subset \mathscr{ID}(\Omega; q, p; \omega, 1) \tag{3}$$

in case of $n - 1 < q \leqslant p < \infty$.

For justifying (3) we refer to [3, Theorem 8] where it is proved that every mapping $f : \Omega \to \Omega'$ of $\mathscr{OD}(\Omega; q, p; \omega, 1)$, $n-1 < q \leqslant p < \infty$, belongs also to the class $\mathscr{ID}(\Omega; q, p; \omega, 1)$, and the estimate

$$\|\mathscr{K}^{\omega,1}_{q,p}(\cdot,f) \mid L_\varrho(\Omega)\| \leqslant \|K^{\omega,1}_{q,p}(\cdot,f) \mid L_\varkappa(\Omega)\|^{n-1} \tag{4}$$

holds. (Here ϱ and $\varkappa$ are defined after formulas (1) and (2) respectively).

In [4] it was proved the following result.

Theorem 1 [4, Theorem 4.1]**.** *Let $n - 1 < q \leqslant p < \infty$. Suppose that $f : \Omega \to \mathbb{R}^n$ is a mapping with with inner bounded ω-weighted (q,p)-codistortion ($f \in \mathscr{ID}(\Omega; q, p; \omega, 1)$), while the weight function $\theta(x) = \omega^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable. If Γ is a family of curves in the domain Ω then we have the inequality*

$$\big(\operatorname{mod}_s f(\Gamma)\big)^{1/s} \leqslant \mathscr{K}^{\omega,1}_{q,p}(f;\Omega)\big(\operatorname{mod}^\theta_r \Gamma\big)^{1/r}, \tag{5}$$

with $s = \frac{p}{p-(n-1)}$ and $r = \frac{q}{q-(n-1)}$.

Below we recall the concept of the modulus of a family of curves (see [4] for more details).

A *curve* in $\mathbb{R}^n$ is a continuous mapping $\alpha\colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$, that is, a set of the form $\langle a, b\rangle$, where each angular parenthesis can be either round or square, $a, b \in \mathbb{R}$ with $a \leqslant b$. We also allow infinite intervals. A curve α is called closed (open) if the interval I is compact (open). Put $|\alpha| = \alpha(I)$. The expression $\gamma' \subset \gamma$ will mean that the curve γ' is a restriction of the curve γ to a subinterval or a point.

If $\alpha\colon I = [a, b] \to \mathbb{R}^n$ is a closed curve then its *length* is

$$\ell(\alpha) = \sup \sum_{i=1}^{l} \bigl|\alpha(t_i) - \alpha(t_{i+1})\bigr|,$$

where the supremum is taken over all finite partitions $a = t_1 \leqslant t_2 \leqslant \ldots \leqslant t_l \leqslant t_{l+1} = b$. If a curve α is not closed then put its length equal to $\ell(\alpha) = \sup \ell(\alpha|_J)$, where the supremum is taken over all closed subintervals J of I.

A curve $\alpha\colon I \to \mathbb{R}^n$ is called *rectifiable* whenever $\ell(\alpha) < \infty$. A curve is called *locally rectifiable* if each closed subcurve of it is rectifiable.

Consider a closed curve $\alpha\colon [a, b] \to \mathbb{R}^n$ and suppose that it is rectifiable. Define a function $s_\alpha\colon [a, b] \to \mathbb{R}$ by the equality $s_\alpha(t) = \ell\bigl(\alpha|_{[a,t]}\bigr)$. For the rectifiable curve α there exists a unique curve $\alpha^0\colon [0, \ell(\alpha)] \to \mathbb{R}^n$ obtained from α by a monotonely increasing change of parameter such that $s_{\alpha^0}(t) = t$ and $\alpha = \alpha^0 \circ s_\alpha$ [5, Section 2.4]. The curve α^0 is called the *positive natural parametrization* of α.

Take a Borel set $A \subset \mathbb{R}^n$ and a Borel function $\rho\colon A \to [0, \infty]$. The integral of ρ along a rectifiable curve $\alpha\colon [a, b] \to \mathbb{R}^n$ is defined as

$$\int\limits_{\alpha} \rho\, ds = \int\limits_{0}^{\ell(\alpha)} \rho\bigl(\alpha^0(\tau)\bigr)\, d\mathscr{H}^1(\tau)$$

with an usual Lebesgue integral in the right-hand side. If α is absolutely continuous then so is the function $s_\alpha(t) = [a, b] \to [0, \ell(\alpha)]$. Putting $\tau = s_\alpha(t)$ in the last integral, using the change-of-variables theorem for Lebesgue integrals, and accounting for $\dot\alpha(t) = \frac{d}{d\tau}\alpha^0(s_\alpha(t))\dot s_\alpha(t)$ and $\frac{d}{d\tau}\alpha^0(\tau) = 1$, we infer that

$$\int\limits_{\alpha} \rho\, ds = \int\limits_{a}^{b} \rho(\alpha(t))|\dot\alpha(t)|\, d\mathscr{H}^1(t). \tag{6}$$

Observe that by the change of variable formula we can express this as

$$\int\limits_{\alpha} \rho\, ds = \int\limits_{a}^{b} \rho(\alpha(t))|\dot\alpha(t)|\, d\mathscr{H}^1(t) = \int\limits_{|\alpha|} \rho(y)\mathscr{N}\bigl(y, \alpha, [a, b]\bigr)\, d\mathscr{H}^1(y), \tag{7}$$

where $\mathscr{N}(y, \alpha, [a, b]) = \#\{[a, b] \cap \alpha^{-1}(y)\}$ is the Banach indicatrix.

For a locally rectifiable curve $\alpha\colon I \to \mathbb{R}^n$, put

$$\int\limits_{\alpha} \rho\, ds = \sup_{\beta} \int\limits_{\beta} \rho\, ds, \tag{8}$$

where the supremum is taken over all closed subcurves β of α.

Consider a family Γ of curves in $\mathbb{R}^n$, where $n \geqslant 2$. A Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ whenever

$$\int\limits_{\gamma} \rho\, ds \geqslant 1 \tag{9}$$

for each locally rectifiable curve $\gamma \in \Gamma$. Denote the collection of all admissible functions by $\operatorname{adm}\Gamma$. Given a weight function $\theta\colon \mathbb{R}^n \to (0,\infty)$ and a number $p \in [1,\infty)$, define the *θ-weighted p-modulus* of Γ as

$$\operatorname{mod}_p^\theta \Gamma = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p \theta \, d\mathscr{H}^n.$$

Properties of the weight function will be prescribed separately; at least, we assume that it is locally summable and $0 < \theta < \infty$ holds $\mathscr{H}^n$-almost everywhere. For $\theta \equiv 1$ we obtain the usual definition of p-modulus, and instead of $\operatorname{mod}_p^1 \Gamma$ we write $\operatorname{mod}_p \Gamma$. If $\operatorname{adm}\Gamma = \varnothing$ then we put $\operatorname{mod}_p^\theta \Gamma = \infty$; this case is realized only if Γ contains at least one curve determining a constant mapping.

Remark 2. The definition of modulus implies that every family of curves which are not locally rectifiable has zero modulus. Moreover, if Γ is a family of curves and $\Gamma_1 = \{\gamma \in \Gamma :$ γ is locally rectifiable$\}$ then $\operatorname{mod}_p^\theta(\Gamma) = \operatorname{mod}_p^\theta(\Gamma_1)$.

Suppose that α is a rectifiable closed curve in $\mathbb{R}^n$. A mapping $g\colon |\alpha| \to \mathbb{R}^n$ is called *absolutely continuous on* α if the composition $g \circ \alpha^0$ is absolutely continuous on $[0, \ell(\alpha)]$.

Theorem 2 [5, Fuglede's Theorem; 6]**.** *Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a mapping of class $W_p^1(\Omega)$ with $1 \leqslant p < \infty$, and Γ is a family of locally rectifiable curves in Ω such that each curve has a closed subcurve on which f is not absolutely continuous. Then* $\operatorname{mod}_p \Gamma = 0$.

2. Modification of Theorem 1 in the case of $n = 2$ and $p = \infty$

In this case parameters q, p may be taken within $(1,\infty]$: $1 < q \leqslant p \leqslant \infty$. The case $1 < q \leqslant p < \infty$ is taken into consideration in Theorem 1.

Theorem 3. *Let $1 < q < p = \infty$. Suppose that $\Omega \subset \mathbb{R}^2$ is a domain, and $f\colon \Omega \to \mathbb{R}^2$ is a mapping with inner bounded ω-weighted (q,∞)-codistortion ($f \in \mathscr{ID}(\Omega; q, \infty; \omega, 1)$[1]), while the weight function $\theta(x) = \omega^{-\frac{1}{q-1}}(x)$ is locally summable. If Γ is a family of curves in the domain Ω then we have the inequality*

$$(\operatorname{mod}_1 f(\Gamma)) \leqslant \mathscr{K}_{q,\infty}^{\omega,1}(f;\Omega)(\operatorname{mod}_r^\theta \Gamma)^{1/r} \tag{10}$$

with $r = \frac{q}{q-1}$.

In this theorem $\mathscr{K}_{q,\infty}^{\omega,1}(f;\Omega) = \big\|\mathscr{K}_{q,\infty}^{\omega,1}(\cdot, f) \mid L_r(\Omega)\big\|$.

Theorem 4. *Suppose that $\Omega \subset \mathbb{R}^2$ is a domain, and $f\colon \Omega \to \mathbb{R}^2$ is a mapping belonging to the Sobolev class $W_{1,\mathrm{loc}}^1(\Omega)$ with the nonnegative Jacobian determinant:* $\det Df(x) \geqslant 0$ *for almost all $x \in \Omega$. Assume that*

1) *f is continuous, open and discrete;*

(2) *the mapping f has bounded codistortion:* $\operatorname{adj} Df(x) = 0$ *a. e. on the set* $Z = \{x \in \Omega :$ $\det Df(x) = 0\}$.

[1] In the case $p = \infty$ we have to replace $\det Df(x)^{\frac{1}{p}}$ in (1) by 1.

Let, for a weight $\omega\colon \mathbb{R}^n \to [0,\infty]$, (∞,∞)*-codistortion function*

$$\Omega \ni x \mapsto \mathscr{K}^{\omega,1}_{\infty,\infty}(x,f) = \begin{cases} \omega(x)|\mathrm{adj}\, Df(x)| & \text{if } \det Df(x) > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

belongs to $L_\infty(\Omega)$ *(in another words* $f \in \mathscr{ID}(\Omega;\infty,\infty;\omega,1)$*). If the weight function* $\theta(x) = \omega^{-1}(x)$ *is locally summable then, for any family of curves* Γ *in the domain* Ω*, we have the inequality*

$$\operatorname{mod}_1 f(\Gamma) \leqslant \mathscr{K}^{\omega,1}_{\infty,\infty}(f;\Omega) \operatorname{mod}^\theta_1 \Gamma. \tag{12}$$

In this theorem $\mathscr{K}^{\omega,1}_{\infty,\infty}(f;\Omega) = \big\|\mathscr{K}^{\omega,1}_{\infty,\infty}(\cdot,f) \mid L_\infty(\Omega)\big\|$.

Theorems 3 and 4 will be proved in Section 6.

3. Application

In paper [7, Example 32] the following class of mappings is considered. Suppose that $n-1 < p < \infty$, and consider a continuous, open and discrete mapping $f : D' \to \mathbb{R}^n$ of an open connected domain $D' \subset \mathbb{R}^n$, where $n \geqslant 2$, such that

(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;

(2) $\det Df(y) \geqslant 0$ and f has finite codistortion; i. e., $\mathrm{adj}\, Df(y) = 0$ $\mathscr{H}^n$-almost everywhere on $Z = \{y \in D' : \det Df(y) = 0\}$;

(3) the inner operator distortion function

$$D' \ni y \mapsto \mathscr{K}^{1,1}_{n-1,s}(y,f) = \begin{cases} \frac{|\mathrm{adj} Df(y)|}{\det Df(y)^{\frac{n-1}{s}}} & \text{if } \det Df(y) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

belongs to $L_{p,\mathrm{loc}}(D')$, where $\frac{1}{p} = \frac{n-1}{n-1} - \frac{n-1}{s}$ holds with $s = \frac{(n-1)p}{p-1} > n-1$;

(4) the weight function σ defined as

$$\sigma(y) = \begin{cases} \frac{|\,\mathrm{adj}\, Df(y)|^p}{\det Df(y)^{p-1}} & \text{if } y \in D' \setminus Z', \\ 1 & \text{otherwise}, \end{cases} \tag{14}$$

is in $\in L_{1,\mathrm{loc}}(D')$, here $Z' = \{y \in D' : Df(y) = 0\}$.

Taking into acount saying above we see that $f : D' \to D$ meets the assumptions of Theorem 1 with D' instead of Ω:

(2a) $f \in W^1_{n-1,\mathrm{loc}}(D')$;

(2b) $\det Df(y) \geqslant 0$ and f has finite codistortion;

(2c) $f\colon D' \to D$ is a mapping of bounded ω-weighted (s,s)-codistortion with $\omega(y) = \sigma^{-\frac{1}{p-1}}(y)$, that is, the ω-weighted (s,s)-codistortion function

$$D' \ni y \mapsto \mathscr{K}^{\omega,1}_{s,s}(y,f) = \begin{cases} \dfrac{\omega^{\frac{n-1}{s}}(y)|\,\mathrm{adj}\, Df(y)|}{\det Df(y)^{\frac{n-1}{s}}} & \text{if } J(y,f) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $L_\infty(D')$ and

$$\big\|\mathscr{K}^{\omega,1}_{s,s}(\cdot,f) \mid L_\infty(D')\big\| = 1 \tag{15}$$

(the last equality is proved in [7, Theorem 3] under more general assumption).

Taking into account saying above, by Theorem 1, we come to the following statement.

Proposition 1. *Suppose that a continuous, open and discrete mapping $f : D' \to \mathbb{R}^n$ of an open connected domain $D' \subset \mathbb{R}^n$, where $n \geqslant 2$, has the following properties*:

(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;

(2) $\det Df(y) \geqslant 0$ *and f has finite codistortion* ($\operatorname{adj} Df(y) = 0$ $\mathscr{H}^n$*-almost everywhere on* $Z = \{y \in D' : \det Df(y) = 0\}$);

(3) *the inner operator distortion function*

$$D' \ni y \mapsto \mathscr{K}^{1,1}_{n-1,s}(y, f) = \begin{cases} \dfrac{|\operatorname{adj} Df(y)|}{\det Df(y)^{\frac{n-1}{s}}} & \text{if } \det Df(y) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

belongs to $L_{p,\mathrm{loc}}(D')$ with some $p > n-1$, where $\frac{1}{p} = \frac{n-1}{n-1} - \frac{n-1}{s}$ holds with $s = \frac{(n-1)p}{p-1} > n-1$.

If Γ is a family of curves in the domain D' then we have the inequality

$$\operatorname{mod}_p f(\Gamma) \leqslant \operatorname{mod}^{\sigma}_p \Gamma \tag{17}$$

where the weight function σ is defined in (7).

$\lhd$ When deriving inequality (17) the properties (2a)–(2c) formulated above, should be taken into account. Really, we see that $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \omega, 1)$ with $q = p = s$ and $\omega(y) = \sigma^{-\frac{1}{p-1}}(y)$. Therefore, by Theorem 1, we get the inequality

$$\big(\operatorname{mod}_{s'} f(\Gamma)\big)^{1/s'} \leqslant \mathscr{K}^{\omega,1}_{s,s}\big(f; D'\big)\big(\operatorname{mod}^{\theta}_{s'} \Gamma\big)^{1/s'}$$

with $s' = \frac{s}{s-(n-1)}$ (here $\mathscr{K}^{\omega,1}_{s,s}(f; D') = \|\mathscr{K}^{\omega,1}_{s,s}(\cdot, f) \mid L_\infty(D')\|$). Because of (15), $s' = p$ and $\theta(y) = \omega^{-\frac{n-1}{s-(n-1)}}(y) = \sigma(y)$ inequality (17) holds. $\rhd$

Taking into account [2, Theorem 34] or [4, Theorem 5.2] and its proof we come to

Proposition 2. *Suppose that for a continuous, open and discrete mapping $f : D' \to \mathbb{R}^n$ of an open connected domain $D' \subset \mathbb{R}^n$, where $n \geqslant 2$, conditions of Proposition* 1 *hold. If $E = (A, C)$ is a condenser in Ω, then the estimate holds:* $\operatorname{cap}_p f(E) \leqslant \operatorname{cap}^{\sigma}_p E$.

4. The special case of the mappings under consideration: $n = 2$

In the case $n = 2$ we have the following modification of the results of the previous section.

We have $1 < p < \infty$ and a continuous, open and discrete mapping $f : D' \to \mathbb{R}^2$ of on open connected domains $D' \subset \mathbb{R}^2$ such that

(1) $f \in W^1_{1,\mathrm{loc}}(D')$;

(2) $\det Df(y) \geqslant 0$ and f has finite codistortion; i. e., $\operatorname{adj} Df(y) = 0$ $\mathscr{H}^2$-almost everywhere on $Z = \{y \in D' : \det Df(y) = 0\}$;

(3) the inner operator distortion function

$$D' \ni y \mapsto \mathscr{K}^{1,1}_{1,\frac{p}{p-1}}(y, f) = \begin{cases} \dfrac{|\operatorname{adj} Df(y)|}{\det Df(y)^{\frac{p-1}{p}}} & \text{if } \det Df(y) \neq 0, \\ 0 & \text{if } \det Df(y) = 0, \end{cases}$$

belongs to $L_{p,\mathrm{loc}}(D')$.

(4) the weight function σ defined as

$$\sigma(y)=\begin{cases}\frac{|\operatorname{adj}Df(y)|^p}{\det Df(y)^{p-1}} & \text{if } y\in D'\setminus Z',\\ 1 & \text{otherwise},\end{cases} \tag{18}$$

is in $\in L_{1,\mathrm{loc}}(D')$, here $Z'=\{y\in D' : Df(y)=0\}$.

It is not hard to see that the continuous, open and discrete mapping $f: D'\to\mathbb{R}^2$ meets the assumptions of Proposition 1 under $n=2$:

(3a) $f\in W^1_{1,\mathrm{loc}}(D')$;

(3b) f has finite distortion;

(3c) $f\colon D'\to D$ is a mapping with bounded ω-weighted (p',p')-distortion where $p'=\frac{p}{p-1}$ and $\omega(y)=\sigma^{-\frac{1}{p-1}}(y)$, that is the ω-weighted (p',p')-distortion function

$$D'\ni y\mapsto K^{\omega,1}_{p',p'}(y,f)=\begin{cases}\dfrac{\omega^{\frac{1}{p'}}(y)|Df(y)|}{\det Df(y)^{\frac{1}{p'}}} & \text{if } \det Df(y)\neq 0,\\ 0 & \text{otherwise},\end{cases}$$

belongs to $L_\infty(D')$, and

$$\big\|K^{\omega,1}_{p',p'}(\cdot,f)\mid L_\infty(D')\big\|=1. \tag{19}$$

Taking into account saying above, by Proposition 1, we come to the following statement.

Corollary 1. *Suppose that a continuous, open and discrete mapping $f: D'\to\mathbb{R}^2$ of an open connected domain $D'\subset\mathbb{R}^2$ has the following properties*:

(1) $f\in W^1_{1,\mathrm{loc}}(D')$;

(2) *f has finite codistortion* ($\operatorname{adj}Df(y)=0$ *$\mathscr{H}^2$-almost everywhere on* $Z=\{y\in D' : \det Df(y)=0\}$);

(3) *the inner operator distortion function*

$$D'\ni y\mapsto \mathscr{K}^{1,1}_{1,p'}(y,f)=\begin{cases}\frac{|\operatorname{adj}Df(y)|}{\det Df(y)^{\frac{1}{p'}}} & \textit{if } \det Df(y)\neq 0,\\ 0 & \textit{otherwise},\end{cases} \tag{20}$$

belongs to $L_{p,\mathrm{loc}}(D')$ with some $p>1$, where $\frac{1}{p}+\frac{1}{p'}=1$.

If Γ is a family of curves in the domain D' then we have the inequality

$$\operatorname{mod}_p f(\Gamma)\leqslant \operatorname{mod}^\sigma_p\Gamma \tag{21}$$

holds where the weight function σ is defined in (18).

5. One more special case of the mappings under consideration: $n=2$ and $p=1$

In this section we prove that Corollary 1 is valid also in the case $p=1$. To show this we have to modify some arguments of the previous section. A counterpart of Corollary 1 is formulated in the following statement.

Proposition 3. *Suppose that a continuous, open and discrete mapping $f: D'\to\mathbb{R}^2$ of an open connected domain $D'\subset\mathbb{R}^2$ has the following properties*:

(1) $f\in W^1_{1,\mathrm{loc}}(D')$;

(2) $\det Df(y) \geqslant 0$ *and* f *has finite codistortion* ($\operatorname{adj} Df(y) = 0$ $\mathscr{H}^2$*-almost everywhere on* $Z = \{y \in D' \mid \det Df(y) = 0\}$);

(3) *the inner operator codistortion function*

$$D' \ni y \mapsto \mathscr{K}^{1,1}_{1,\infty}(y, f) = \begin{cases} |\operatorname{adj} Df(y)| & \text{if } \det Df(y) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{22}$$

belongs to $L_{1,\mathrm{loc}}(D')$.

If Γ *is a family of curves in* D' *then we have*

$$\operatorname{mod}_1 f(\Gamma) \leqslant \operatorname{mod}_1^{\sigma} \Gamma \tag{23}$$

with σ *defined in* (25).

$\lhd$ We show that the proof of Proposition 3 can be reduced to Theorem 3. For doing this formulate first additional properties of f and $\varphi = f^{-1}$.

Properties of $\varphi = f^{-1}$. If $f : D' \to D$ is a homeomorphism then the inverse homeomorphism $\varphi = f^{-1} : D \to D'$ enjoys the following properties:

(4) by [9, Theorem 4] or [7, Theorem 27] we have $\varphi \in W^1_{1,\mathrm{loc}}(D)$ (see also [10, Theorem 3.2]);

(5) φ has finite distortion by [7, Theorem 27] (see also [10, Theorem 3.3]);

(6) φ is differentiable a. e. in D by [7, Theorem 27];

while $f : D' \to D$

(6) φ belongs to $\mathscr{Q}_{1,1}(D, D'; \sigma)$ (see [4]), that is the distortion function

$$D \ni x \mapsto K^{1,\sigma}_{1,1}(x, \varphi) = \begin{cases} \frac{|D\varphi(x)|}{\sigma(\varphi(x)) \det D\varphi(x)} & \text{if } \det D\varphi(x) \neq 0, \\ 0 & \text{if } \det D\varphi(x) = 0, \end{cases} \tag{24}$$

of the inverse mapping $\varphi = f^{-1}$ with the weight function $\sigma \in L_{1,\mathrm{loc}}(D')$ defined as

$$\sigma(y) = \begin{cases} |\operatorname{adj} Df(y)| & \text{if } y \in D' \setminus Z', \\ 1 & \text{otherwise}, \end{cases} \quad \text{where } Z' = \{y \in D' : Df(y) = 0\}, \tag{25}$$

is in $L_\infty(D)$ and $K^{1,\sigma}_{1,1}(\varphi; D) = \|K^{1,\sigma}_{1,1}(\cdot, \varphi) \mid L_\infty(D)\| = 1$ (see [4, Theorem 25, formulas (30) and (37); 8]).

Properties of f. Taking into account saying above, we see that$f : D' \to D$ meets some additional properties:

(7) $f \in W^1_{1,\mathrm{loc}}(D')$ and f is differentiable a. e. in D' by [7, Theorem 27];

(8) $\det Df(y) \geqslant 0$ and f has finite distortion by [7, Theorem 27] (see also [10, Theorem 3.3]);

(9) $f\colon D' \to D$ is a mapping with bounded ω-weighted (∞, ∞)-codistortion with the weight function $\omega = \sigma^{-1}$, that is the ω-weighted (∞, ∞)-codistortion function

$$D' \ni y \mapsto \mathscr{K}^{\omega,1}_{\infty,\infty}(y, f) = \begin{cases} \omega(y) |\operatorname{adj} Df(y)| & \text{if } \det Df(y) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $L_\infty(D')$, and

$$\|\mathscr{K}^{\omega,1}_{\infty,\infty}(\cdot, f) \mid L_\infty(D')\| = \|K^{1,\sigma}_{1,1}(\cdot, \varphi) \mid L_\infty(D)\| = 1. \tag{26}$$

Now it is evident that f enjoys the conditions of Theorem 3, and therefore (23) holds for f. $\rhd$

6. Proof of Theorems 3 and 4

$\lhd$ We verify that the proof of Theorem 1 given in [4, Theorem 4.1] for mappings with bounded θ-weighted (q,p)-codistortion, where $n-1<q\leqslant p<\infty$, works also in the case $1<q\leqslant p=\infty$ at $n=2$. To do this we need properties of Poletsky function and Poletsky's Lemma in this case. We formulate and prove them below. $\rhd$

1. Properties of Poletsky function. Take a continuous mapping $f:\Omega\to\mathbb{R}^2$ and a domain D compactly embedded into Ω, meaning that D is bounded and $\bar{D}\subset\Omega$, written briefly as $D\Subset\Omega$, and take $y\notin f(\partial D)$. Denote by $\mu(y,f,D)$ the degree of f at y with respect to D. Say that f is sense-preserving whenever $\mu(y,f,D)>0$ for all domains $D\Subset\Omega$ and all points $y\in f(D)\backslash f(\partial D)$. For $A\subset\Omega$ refer as the multiplicity function to $\mathbb{R}^2\ni y\mapsto N(y,f,A)=\#\left\{f^{-1}(y)\cap A\right\}$. Moreover, put $N(f,A)=\sup_{y\in\mathbb{R}^2}N(y,f,A)$.

Suppose that $f:\Omega\to\mathbb{R}^2$ is a continuous, open, and discrete mapping. A domain $D\Subset\Omega$ is called normal whenever $f(\partial D)=\partial f(D)$. A normal neighborhood of $x\in\Omega$ is a normal domain $U\subset\Omega$ such that $\overline{U}\cap f^{-1}(f(x))=\{x\}$. The quantity $i(x,f)=\mu(f(x),f,U)$ is independent of the choice of a normal neighborhood U of x (see [11, Chapter II, § 2] for instance) and is called the local index of f at x. A point $x\in\Omega$ is called a branch point of f whenever f is not a homeomorphism of any neighborhood of x. Denote the collection of all branch points of f by B_f. If D is a normal domain for a mapping f then $\mu(y,f,D)$ is independent of $y\in f(D)$. We will call this constant by $\mu(f,D)$.

In the following two lemmas we state propositions of interest in their own right. Both of them are applied in the proof of the main result of this section.

Lemma 1 [3, Lemma 10]**.** *Assume that $f:\Omega\to\mathbb{R}^2$ is a continuous, open and discrete mapping in $W^1_{1,\mathrm{loc}}(\Omega)$ with finite distortion. Then for every open connected set $U\subset\Omega$ the set $\{x\in U\backslash B_f:\ J(x,f)\neq 0\}$ has positive measure.*

$\lhd$ If, on the contrary, $J(x,f)=0$ a. e. on a connected set $U\subset\Omega\backslash B_f$ on which f is a homeomorphism then $Df(x)=0$ a. e. on U because f has finite distortion. Then f is constant on U, and consequently, f cannot be open. $\rhd$

Proposition 4. *If $f:\Omega\to\mathbb{R}^2$ is a continuous, open and discrete mapping in $W^1_{1,\mathrm{loc}}(\Omega)$ with finite distortion, then f is differentiable a. e. on $\Omega\backslash B_f$ and sense-preserving.*

$\lhd$ For a connected open set $U\subset\Omega\backslash B_f$ on which f is a homeomorphism, it is enough to apply the statement [9, Theorem 4] or [7, Theorem 27] twice. For the restriction $f|_U:U\to f(U)$ it provides that the inverse homeomorphism $(f|_U)^{-1}:f(U)\to U$ is in $W^1_1(f(U))$, is of finite distortion, and is differentiable a. e. on $f(U)$. Then applying [7, Theorem 27] to $(f|_U)^{-1}:f(U)\to U$ we get similar properties to the given mapping $f|_U:U\to f(U)$. By Lemma 1, $\det Df(x)\geqslant 0$ and properties of degree we conclude that f is sense-preserving. $\rhd$

Definition 3. For a sense-preserving, continuous, open and discrete mapping $f:\Omega\to\mathbb{R}^2$ and a normal domain $D\Subset\Omega$, define the Poletsky function $g_D:V\to\mathbb{R}^2$ on $V=f(D)$ [12] by putting

$$V\ni y\mapsto g_D(y)=\Lambda\sum_{x\in f^{-1}(y)\cap D}i(x,f)x, \tag{27}$$

where $\Lambda=\mu(f,D)$.

The function of the form (27) was introduced by Poletsky in [12] for mappings with bounded distortion ($p=q=n$, $\omega\equiv 1$). The next statement presents the properties of the Poletsky function for the classes of mappings under consideration.

Proposition 5 [2, 3]**.** *Suppose that* $f : \Omega \to \mathbb{R}^2$ *belongs to* $\mathscr{OD}(\Omega;\infty,\infty;\omega,1)$ (*properties* (4*a*)–(4*c*) *hold*). *Then*

(1) *the function* g_D *defined in* (27) *is continuous and belongs to* $\mathrm{ACL}(V)$;

(2) $Dg_D(y) = 0$ *a. e. on* $Z' \cup \Sigma'$;

(3) *Poletsky function* g_D *defined in* (27) *is in* $W^1_1(V)$*; furthermore,*

$$\|Dg_D \mid L_1(V)\| \leqslant \Lambda \|K^{\omega,1}_{\infty,\infty}(\cdot; f) \mid L_\infty(D)\| \int\limits_D \sigma(x)\, dx.$$

We emphasize that the formulated statement is proved in [2, Theorem 18] for mappings $f \in \mathscr{ID}(\Omega; p, p; \omega, 1)$, $p \in (1,\infty)$. The same proof works also in the case $p = \infty$ at $n = 2$.

2. Poletsky's Lemma. Consider a continuous, open and discrete mapping $f : \Omega \to \mathbb{R}^2$. Take a closed rectifiable curve $\beta : I_0 \to \mathbb{R}^n$ and a curve $\alpha : I \to \Omega$ with $f\circ\alpha \subset \beta$. In particular, we have $I \subset I_0$. If the function $s_\beta : I_0 \to [0, \ell(\beta)]$ is constant on some interval $J \subset I$, then the mapping β is constant on J. In turn, since f is discrete, α is also constant on J. Consequently, there exists a unique mapping $\alpha^* : s_\beta(I) \to \Omega$ satisfying $\alpha = \alpha^* \circ s_\beta|_I$. We can prove that α^* is continuous and $f \circ \alpha^* \subset \beta^0$. The curve α^* is called an f-representative of α (with respect to β) whenever $\beta = f \circ \alpha$. Suppose now that $\beta = f \circ \alpha$. The above arguments show that

$$f \circ \alpha^* = (f \circ \alpha)^0.$$

Therefore, the curve $f\circ\alpha^*$ admits a positive natural parametrization, and hence it is Lipschitz. Thus we can integrate along this curve using (6) where $\left|\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha^*)(t)\right| = 1$ for $\mathscr{H}^1$-almost all $t \in I$.

The mapping f is called *absolutely precontinuous* on α provided that α^* is absolutely continuous.

Lemma 2. *Suppose that* $f : \Omega \to \mathbb{R}^2$ *is a mapping of class* $\mathscr{ID}(\Omega;\infty,\infty;\omega,1)$*. Consider a family* Γ *of curves in* Ω *such that for every* $\gamma \in \Gamma$ *the following holds: the curve* $f \circ \gamma$ *is locally rectifiable and* γ *has a closed subcurve* α *on which* f *is not absolutely precontinuous. Then* $\mathrm{mod}_1 f(\Gamma) = 0$.

The formulated Lemma is proved in [4, Lemma 3.3] for mappings $f \in \mathscr{ID}(\Omega; p, p; \omega, 1)$, $p \in (1,\infty)$. The same proof works also in the case $p = \infty$ at $n = 2$.

In the proof of Lemma 2 we also need the following statement.

Lemma 3. *Consider a homeomorphism* $\varphi : \Omega \to \Omega'$ *of class* $\mathscr{ID}(\Omega; q,\infty; \theta, 1)$*, where* $\Omega, \Omega' \subset \mathbb{R}^2$ *and* $1 < q \leqslant \infty$*.*
Then

(1) *the inverse homeomorphism is* $\varphi^{-1} \in W^1_{1,\mathrm{loc}}(\Omega')$;

(2) φ^{-1} *has finite distortion:* $D\varphi^{-1}(y) = 0$ *almost everywhere on* Z';

(3) $K^{1,\omega}_{1,r}(\cdot,\varphi^{-1}) \in L_\varrho(\Omega')$*, where*

$$r = \begin{cases} \frac{q}{q-n+1} & \text{if } q < \infty,\\ 1 & \text{if } q = \infty, \end{cases} \qquad \omega = \begin{cases} \theta^{-\frac{1}{q-1}} & \text{if } q < \infty,\\ \theta^{-1} & \text{if } q = \infty; \end{cases}$$

(4) *if the weight function* ω *is locally summable then the inverse homeomorphism induces, by the change-of-variable rule, the bounded operator*

$$\varphi^{-1*} : L^1_r(\Omega;\omega) \cap W^1_{\infty,\mathrm{loc}} \to L^1_1(\Omega').$$

We have the relations

$$\|K^{1,\omega}_{1,r}(\cdot,\varphi^{-1}) \mid L_\varrho(\Omega')\| = \|\mathscr{K}^{\theta,1}_{q,\infty}(\cdot,\varphi) \mid L_\varrho(\Omega)\|$$

and

$$\beta_{q,\infty}\big\|K_{1,r}^{1,\omega}(\cdot,\varphi^{-1})\mid L_{\varrho}(\Omega')\big\| \leqslant \|\varphi^{-1*}\| \leqslant \big\|K_{1,r}^{1,\omega}(\cdot,\varphi^{-1})\mid L_{\varrho}(\Omega')\big\|,$$

where $\beta_{q,\infty}$ *is some constant.*

$\lhd$ Properties (1) and (2) of $\varphi = f^{-1}$ were proved just after Proposition 3. Taking into account (1) and (2) Properties (3) and (4) can be proved by analogy with Theorem 9 of [2]. $\rhd$

Remark 3. By means of Theorems 3 and 4 for homeomorphisms $\varphi : \Omega \to \Omega'$ of class $\mathscr{I}\mathscr{D}(\Omega; q, \infty; \theta, 1)$, where $\Omega, \Omega' \subset \mathbb{R}^2$ and $1 < q \leqslant \infty$, we can prove some more inequalities such that Väisälä inequality and the capacity inequality (see proofs in [4, Theorem 22] and [4, Theorem 28] respectively).

Remark 4. It is not hard to see that assumptions of Theorem 4 are weaker comparing with those in paper [13]. For instance, Theorem 1.3 of [13] is formulated under addition condition that the given mapping is closed. Therefore Theorem 4 with weaker assumptions contains the main result of paper [13].

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О МОДУЛЬНЫХ НЕРАВЕНСТВАХ ТИПА ПОЛЕЦКОГО ДЛЯ НЕКОТОРЫХ КЛАССОВ ОТОБРАЖЕНИЙ

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Аннотация. Хорошо известно, что теория отображений с ограниченным искажение была заложена Ю. Г. Решетняком в 60-е годы прошлого века [1]. В работах [2, 3] была введена двухиндексная шкала отображений с весовым ограниченным (q, p)-искажением. Эта шкала отображений включает в себя, в частности, отображения с ограниченным искажением, упомянутые выше (при $q = p = n$ и тривиальной весовой функции). В работе [4] для двухиндексной шкалы отображений с весовым ограниченным (q, p)-искажения доказано модульное неравенство типа Полецкого при минимальной регулярности; приведено много примеров отображений, к которым можно применить результаты [4]. В этой статье мы приведем одно такое применение. Другая цель этой статьи — показать новый класс отображений, в которых выполняются модульные неравенства типа Полецкого. Для этого мы расширяем при $n = 2$ справедливость утверждений работы [4] на предельные показатели: $1 < q \leqslant p \leqslant \infty$. Это обобщение содержит в качестве частного случая результаты недавно опубликованных работ. Как следствие результатов этой статьи мы получаем также оценки изменения емкости конденсаторов.

Ключевые слова: квазиконформный анализ, пространство Соболева, модуль семейства кривых, оценка модуля.

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