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# ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR A PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH DEGENERATE KERNEL

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**Abstract.** In this article the problems of the unique classical solvability and the construction of the solution of a nonlinear boundary value problem for a fifth order partial integro-differential equations with degenerate kernel are studied. Dirichlet boundary conditions are specified with respect to the spatial variable. So, the Fourier series method, based on the separation of variables is used. A countable system of the second order ordinary integro-differential equations with degenerate kernel is obtained. The method of degenerate kernel is applied to this countable system of ordinary integro-differential equations. A system of countable systems of algebraic equations is derived. Then the countable system of nonlinear Fredholm integral equations is obtained. Iteration process of solving this integral equation is constructed. Sufficient coefficient conditions of the unique solvability of the countable system of nonlinear integral equations are established for the regular values of parameter. In proof of unique solvability of the obtained countable system of nonlinear integral equations the method of successive approximations in combination with the contraction mapping method is used. In the proof of the convergence of Fourier series the Cauchy–Schwarz and Bessel inequalities are applied. The smoothness of solution of the boundary value problem is also proved.

**Key words:** nonlinear boundary value problem, integro-differential equation, degenerate kernel, Fourier series, classical solvability.

**AMS Subject Classification:** 35A02, 35M10, 35S05.

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## 1. Formulation of the Problem

The theory of boundary value problems is currently one of the most important directions of the theory of higher order partial differential equations and nonlinear mathematical physics. So, a large number of research works are devoted to the study these theories of higher order partial differential equations and nonlinear mathematical physics (see, in particular [1–11]). Studies of the many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by the aid of the high-order partial differential equations. Partial differential equations of Boussinesq and Benney-Luke types have differential applications in different branches of sciences (see, for example, [12–14]). When the boundary of the flow domain of a physical process is unavailable for measurements, nonlocal (often nonlinear) conditions

in integral form can serve as an information sufficient for the unique solvability of the problem [15]. Therefore, in recent years, research on the study of nonlocal boundary value problems for differential and integro-differential equations with integral conditions has been intensified (see, for example, [16–30]).

In this paper, we study the regular solvability of a fifth order partial integro-differential equations with some nonlinear integral conditions and parameter. This paper is further development of the works [31, 32].

In the domain $\Omega = \{0 < t < T,\, 0 < x < l\}$ the following form of the partial integro-differential equations is considered

$$U_{t\,t}(t,x) + \omega_1(t)\, U_{t\,t\,x\,x}(t,x) - \omega_2(t) \int\limits_0^T U_{x\,x\,x\,x}(\theta,\, x)\, d\,\theta = \nu \int\limits_0^T K(t,s)\, U_{t\,x\,x\,x\,x}(s,x)\, ds, \quad (1)$$

where $T$ and $l$ are given positive real numbers, $\nu$ is a real non-zero parameter $0 \leqslant \omega_i(t) \in C[0;T]$, $i = 1, 2$, $0 \neq K(t,s) = \sum_{i=1}^{k} a_i(t)\, b_i(s)$, $a_i(t), b_i(s) \in C\,[0;T]$. It is supposed that the system of functions $a_i(t)$, $i = 1, \ldots, k$, and the system of functions $b_i(s)$, $i = 1, \ldots, k$, are linear independent.

Problem: Find in the domain $\Omega$ a function from the class

$$U(t,x) \in C(\bar{\Omega}) \cap C^{2,4}_{t,x}(\Omega) \cap C^{2+2}_{t,x}(\Omega) \cap C^{1+4}_{t,x}(\Omega), \quad (2)$$

satisfying the integro-differential equation (1) and the following boundary conditions

$$U(0,x) = \varphi\left(x, \int\limits_0^T \int\limits_0^l R_1(\theta, z, U(\theta,z))\, d\theta\, dz\right), \quad 0 \leqslant x \leqslant l, \quad (3)$$

$$U_t(T,x) = \psi\left(x, \int\limits_0^T \int\limits_0^l R_2(\theta, z, U(\theta,z))\, d\theta\, dz\right), \quad 0 \leqslant x \leqslant l, \quad (4)$$

$$U(t,0) = U(t,l) = U_{x\,x}(t,0) = U_{x\,x}(t,l) = 0, \quad (5)$$

where $\psi(x,\cdot)$, $\varphi(x,\cdot)$ are enough smooth functions in the domain $\Omega_l \times (-\infty;\infty)$, $0 < \int_0^T \int_0^l |\, R_i(\theta, z, U(\theta,z))\,|\, d\theta\, dz < \infty$, $i = 1, 2$, $C^{r,s}_{t,x}(\Omega)$ is the class of functions $U(t,x)$, possessing the continuous derivatives $\frac{\partial^r U}{\partial t^r}$, $\frac{\partial^s U}{\partial x^s}$ in the domain $\Omega$, $C^{r+s}_{t,x}(\Omega)$ is the class of functions $U(t,x)$, possessing the continuous derivatives $\frac{\partial^{r+s} U}{\partial t^r\, \partial x^s}$ in the domain $\Omega$, $r$, $s$ are arbitrary natural numbers, $\bar{\Omega} = \{0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant l\}$.

## 2. Expansion of the Solution of the Problem into Fourier Series

Taking into account the Dirichlet conditions (5) the nontrivial solutions of the problem (1)–(5) are sought as a following sine Fourier series

$$U(t,x) = \sum_{n=1}^{\infty} u_n(t)\, \vartheta_n(x), \quad (6)$$

where

$$u_n(t) = \int\limits_{\Omega_l} U(t,x)\, \vartheta_n(x)\, dx, \quad (7)$$

$$\vartheta_n(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi\, n}{l} x, \quad \Omega_l = [0; l],\ n = 1, 2, \ldots$$

We also suppose that the following functions can be expanded into Fourier series

$$\varphi\,(x, \cdot) = \sum_{n=1}^{\infty} \varphi_n(\cdot)\, \vartheta_n(x), \quad \psi(x, \cdot) = \sum_{n=1}^{\infty} \psi_n(\cdot) \vartheta_n(x), \tag{8}$$

where

$$\begin{aligned} \varphi_n(\cdot) &= \int_0^l \varphi \left( x, \int_0^T \int_0^l R_1(\theta, z, U(\theta, z))\, d\theta\, dz \right) \vartheta_n(x)\, dx, \\ \psi_n(\cdot) &= \int_0^l \psi \left( x, \int_0^T \int_0^l R_2(\theta, z, U(\theta, z))\, d\theta\, dz \right) \vartheta_n(x)\, dx. \end{aligned} \tag{9}$$

Substituting Fourier series (6) into partial integro-differential equation (1), we obtain the countable system of ordinary integro-differential equations of second order

$$u_n''(t) - \lambda_n^2(t) \int_0^T u_n(\theta)\, d\theta = \nu\, \lambda_n^2(t) \int_0^T \sum_{i=1}^{k} a_i(t)\, b_i(s)\, u_n'(s)\, ds, \tag{10}$$

where $\lambda_n^2(t) = \frac{\omega_2(t)\, \mu_n^4}{1+\omega_1(t)\, \mu_n^2}$, $\mu_n = \frac{\pi n}{l}$. By the aid of the designations

$$\tau_{i,n} = \int_0^T b_i(s)\, u_n'(s)\, ds \tag{11}$$

the countable system (10) cab be rewritten as

$$u_n''(t) = \nu\, \lambda_n^2(t) \sum_{i=1}^{k} a_i(t)\, \tau_{i,n} + \lambda_n^2(t) \int_0^T u_n(\theta)\, d\theta. \tag{12}$$

The second order countable system of integro-differential equations (12) is solved by the variation method of arbitrary constants

$$u_n(t) = A_{1,n}\, t + A_{2,n} + \eta_n(t), \tag{13}$$

where we are used the following denotations

$$\eta_n(t) = \nu \sum_{i=1}^{k} \tau_{i,n} h_{i,n}(t) + \delta_n(t) \int_0^T u_n(\theta)\, d\theta,$$

$$h_{i,n}(t) = \int_0^t (t - s)\, \lambda_n(s)\, a_i(s)\, ds, \quad i = 1, \ldots, k, \quad \delta_n(t) = \int_0^t (t - s)\, \lambda_n(s)\, ds.$$

By virtue of Fourier coefficients (7) and (9), the conditions (3) and (4) take the forms

$$u_n(0) = \int_{\Omega_l} U(0, x)\, \vartheta_n(x)\, dx = \int_{\Omega_l} \varphi(x, \cdot)\, \vartheta_n(x)\, dx = \varphi_n(\cdot), \tag{14}$$

$$u_n'(T) = \int\limits_{\Omega_l} U_t(T,x)\,\vartheta_n(x)\,dx = \int\limits_{\Omega_l} \psi(x,\cdot)\,\vartheta_n(x)\,dx = \psi_n(\cdot). \tag{15}$$

To find the unknown Fourier coefficients $A_{1,n}$ and $A_{2,n}$ in (13), we use the boundary value conditions (14) and (15). Applying (14) to the representation (13), we find

$$A_{2,n} = \varphi_n(\cdot). \tag{16}$$

After differentiation (13) once, by the aid of the condition (15) we have

$$u_n'(t) = A_{1,n} + \nu \sum_{i=1}^{k} \tau_{i,n}\, h_{i,n}'(t) + \delta_n'(t) \int\limits_{-T}^{T} u_n(\theta)\,d\theta, \tag{17}$$

where

$$h_{i,n}'(t) = \int\limits_0^t \lambda_n(s)\,a_i(s)\,ds, \quad \delta_n'(t) = \int\limits_0^t \lambda_n(s)\,ds.$$

By virtue of the condition (15), from (17) we obtain

$$A_{1,n} = \psi_n(\cdot) - \nu \sum_{i=1}^{k} \tau_{i,n}\, h_{i,n}'(T) - \delta_n'(T) \int\limits_0^T u_n(\theta)\,d\theta. \tag{18}$$

Substituting determined Fourier coefficients (16) and (18) into presentation (13), we find

$$u_n(t) = \varphi_n(\cdot) + \psi_n(\cdot)\,t + \nu \sum_{i=1}^{k} \tau_{i,n}\, M_{i,n}(t) + N_n(t) \int\limits_0^T u_n(\theta)\,d\theta, \tag{19}$$

where

$$M_{i,n}(t) = h_{i,n}(t) - t \cdot h_{i,n}'(T), \quad N_n(t) = \delta_n(t) - t \cdot \delta_{i,n}'(T),$$

$$h_{i,n}(t) = \int\limits_0^t (t-s)\,\lambda_n(s)\,a_i(s)\,ds, \quad i = 1,\dots,k, \quad \delta_n(t) = \int\limits_0^t (t-s)\,\lambda_n(s)\,ds.$$

After differentiation (19) once, we have

$$u_n'(t) = \psi_n(\cdot) + \nu \sum_{i=1}^{k} \tau_{i,n}\, M_{i,n}'(t) + N_n'(t) \int\limits_0^T u_n(\theta)\,d\theta, \tag{20}$$

where

$$M_{i,n}'(t) = h_{i,n}'(t) - h_{i,n}'(T), \quad N_n'(t) = \delta_n'(t) - \delta_{i,n}'(T),$$

$$h_{i,n}'(t) = \int\limits_0^t \lambda_n(s)\,a_i(s)\,ds, \quad \delta_n'(t) = \int\limits_0^t \lambda_n(s)\,ds.$$

Substituting derivative (20) into designation (11), we obtain the system of algebraic equations (SAE)

$$\tau_{i,n} + \nu \sum_{j=1}^{k} \tau_{i,n} H_{i,j,n} = \psi_n(\cdot)\,\Phi_{1,i,n} + \Phi_{2,i,n} \int\limits_0^T u_n(\theta)\,d\theta, \quad i = 1,\dots,k, \tag{21}$$

where

$$H_{i,j,n} = \int\limits_{-T}^{T} b_i(s)\, M'_{j,n}(s)\, ds, \quad \Phi_{1,i,n} = \int\limits_{0}^{T} b_i(s)\, ds, \quad \Phi_{2,i,n} = \int\limits_{0}^{T} b_i(s)\, N'_n(s)\, ds.$$

We recall that the systems of functions $a_i(t)$, $i = 1, \ldots, k$, and $b_i(s)$, $i = 1, \ldots, k$ are independent. So, here follows that $H_{i,j,n} \neq 0$. The SAE (21) is unique solvable for all bounded right-hand side of SAE, if the following Fredholm condition is fulfilled

$$\Delta(\nu) = \begin{vmatrix} 1+\nu H_{1\,1} & \nu H_{1\,2} & \ldots & \nu H_{1\,k} \\ \nu H_{2\,1} & 1+\nu H_{2\,2} & \ldots & \nu H_{2\,k} \\ \ldots & \ldots & \ldots & \ldots \\ \nu H_{k\,1} & \nu H_{k\,2} & \ldots & 1+\nu H_{k\,k} \end{vmatrix} \neq 0. \tag{22}$$

The values of the parameter $\nu$, at which the condition (22) is satisfied are called regular. On the set of regular values of the parameter $\nu$, the solutions of the SAE (21) can be written as

$$\tau_{i,n} = \psi_n \frac{\Delta_{1,i}(\nu)}{\Delta(\nu)} + \frac{\Delta_{2,i}(\nu)}{\Delta(\nu)} \int\limits_{0}^{T} u_n(\theta)\, d\theta, \tag{23}$$

where

$$\Delta_{m,i}(\nu) = \begin{vmatrix} 1+\nu H_{1\,1} & \ldots & \nu H_{1\,(i-1)} & \Phi_{m\,1} & \nu H_{1\,(i+1)} & \ldots & \nu H_{1\,k} \\ \nu H_{2\,1} & \ldots & \nu H_{2\,(i-1)} & \Phi_{m\,2} & \nu H_{2\,(i+1)} & \ldots & \nu H_{2\,k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \nu H_{k\,1} & \ldots & \nu H_{k\,(i-1)} & \Phi_{m\,k} & \nu H_{k\,(i+1)} & \ldots & 1+\nu H_{k\,k} \end{vmatrix},$$

$i = 1, \ldots, k$, $m = 1, 2$.

Substituting solutions (23) into representation (19), we obtain the solution of the countable system of integral equations (CSIE)

$$u_n(t) = \varphi_n(\cdot) + \psi_n(\cdot)\, V_n(t) + W_n(t) \int\limits_{0}^{T} u_n(\theta)\, d\theta.$$

Hence, taking into account (9), the CSIE we rewrite as

$$\begin{aligned} u_n(t) = I(t; u_n) \equiv \int\limits_{0}^{l} \varphi \left( x, \int\limits_{0}^{T}\int\limits_{0}^{l} R_1 \left( \theta, z, \sum_{m=1}^{\infty} u_m(\theta)\, \vartheta_m(z) \right) d\theta\, dz \right) \vartheta_n(x)\, dx \\ + V_n(t) \int\limits_{0}^{l} \psi \left( x, \int\limits_{0}^{T}\int\limits_{0}^{l} R_2 \left( \theta, z, \sum_{m=1}^{\infty} u_m(\theta)\, \vartheta_m(z) \right) d\theta\, dz \right) \vartheta_n(x)\, dx + W_n(t) \int\limits_{0}^{T} u_n(\theta)\, d\theta, \end{aligned} \tag{24}$$

where

$$V_n(t) = t + \nu \sum_{i=1}^{k} \frac{\Delta_{1,i}(\nu)}{\Delta_1(\nu)}\, M_{i,n}(t), \quad W_n(t) = N(t) + \nu \sum_{i=1}^{k} \frac{\Delta_{2,i}(\nu)}{\Delta(\nu)}\, M_{i,n}(t),$$

$\varphi_n(\cdot)$ and $\psi_n(\cdot)$ are Fourier coefficients in series (8) and are determined from (9), respectively.

Now, to obtain an expansion of the formal solution to the problem (1)–(5), the representation (24) we substitute into the Fourier series (6)

$$U(t,x)=\sum_{n=1}^{\infty}\vartheta_n(x)\left[\varphi_n(\cdot)+\psi_n(\cdot)\,V_n(t)+W_n(t)\int\limits_0^T u_n(\theta)\,d\theta\right]. \tag{25}$$

## 3. One Value Solvability CSIE (24)

We consider the concepts of the following well-known Banach spaces: the space $B_2(T)$ of sequences of continuous functions $\{u_i(t)\}_{i=1}^{\infty}$ on the segment $[0;T]$ with norm

$$\|\,u\,(t)\,\|_{B_2(T)}=\sqrt{\sum_{i=1}^{\infty}\left(\max_{t\in\,[0;T]}|\,u_i(t)\,|\right)^2}<\infty;$$

the Hilbert coordinate space $\ell_2$ of number sequences $\{\varphi_i\}_{i=1}^{\infty}$ with norm

$$\|\,\varphi\,\|_{\ell_2}=\sqrt{\sum_{i=1}^{\infty}|\,\varphi_i\,|^2}<\infty;$$

the space $L_2\,(\Omega_l)$ of square-summable functions on the domain $\Omega_l=[0;l]$ with norm

$$\|\,\vartheta(x)\,\|_{L_2(\Omega_l)}=\sqrt{\int\limits_0^l|\,\vartheta(y)\,|^2\,dy}<\infty.$$

Assume that for the smooth functions $V_n(t),\ W_n(t)$ from (24) we obtain that the following conditions are fulfilled:

$$C_1=\max_n\left\{\max_{t\in[0;\,T]}|\,V_n(t)\,|\,;\ \max_{t\in[0;\,T]}\left|\,V_n'(t)\,\right|;\ \max_{t\in[0;\,T]}\left|\,V_n''(t)\,\right|\right\}<\infty, \tag{26}$$

$$\sqrt{\sum_{n=1}^{\infty}\left(n^4C_{2\,n}\right)^2}<\infty, \tag{27}$$

where

$$C_{2\,n}=\max\left\{\max_{t\in[0;\,T]}|\,W_n(t)\,|\,;\ \max_{t\in[0;\,T]}\left|\,W_n'(t)\,\right|;\ \max_{t\in[0;\,T]}\left|\,W_n''(t)\,\right|\right\}.$$

**Conditions of smoothness.** Let the functions $\varphi(x),\psi(x)\ \in\ C^4(\Omega_l)$ in the given domain $\Omega_l$ have piecewise continuous derivatives up to the fifth order and

$$\varphi(0,\cdot)=\varphi(l,\cdot)=\varphi_{x\,x}(x,\cdot)=\varphi_{x\,x}(x,\cdot)=\varphi_{x\,x\,x\,x}(0,\cdot)=\varphi_{x\,x\,x\,x}(l,\cdot)=0. \tag{28}$$

Conditions similar to (28) also hold for the function $\psi(x)$:

$$\psi(0,\cdot)=\psi(l,\cdot)=\psi_{x\,x}(x,\cdot)=\psi_{x\,x}(x,\cdot)=\psi_{x\,x\,x\,x}(0,\cdot)=\psi_{x\,x\,x\,x}(l,\cdot)=0.$$

Then, after integration the integrals (9):

$$\varphi_n(\cdot) = \int\limits_{\Omega_l} \varphi(x,\cdot)\,\vartheta_n(x)\,dx, \quad \psi_n(\cdot) = \int\limits_{\Omega_l} \psi(x,\cdot)\,\vartheta_n(x)\,dx$$

by parts five times over the variable $x$, we derived

$$\varphi_n(\cdot) = \left(\frac{l}{\pi}\right)^5 \frac{\varphi_n^{(5)}(\cdot)}{n^5}, \quad \psi_n = \left(\frac{l}{\pi}\right)^5 \frac{\psi_n^{(5)}(\cdot)}{n^5}, \tag{29}$$

where

$$\varphi_n^{(5)}(\cdot) = \int\limits_{\Omega_l} \frac{\partial^5 \varphi(x,\cdot)}{\partial\, x^5}\,\vartheta_n(x)\,dx, \quad \psi_n^{(5)}(\cdot) = \int\limits_{\Omega_l} \frac{\partial^5 \psi(x,\cdot)}{\partial\, x^5}\,\vartheta_n(x)\,dx. \tag{30}$$

Then for the functions in (30) the Bessel inequalities are valid

$$\sum_{n=1}^{\infty} \left[\varphi_n^{(5)}(\cdot)\right]^2 \leqslant \frac{2}{l} \int\limits_{\Omega_l} \left[\frac{\partial^5 \varphi(x,\cdot)}{\partial\, x^5}\right]^2 dx, \tag{31}$$

$$\sum_{n=1}^{\infty} \left[\psi_n^{(5)}(\cdot)\right]^2 \leqslant \frac{2}{l} \int\limits_{\Omega_l} \left[\frac{\partial^5 \psi(x,\cdot)}{\partial\, x^5}\right]^2 dx. \tag{32}$$

Then we prove the unique solvability of a countable system of nonlinear Fredholm integral equations (24). We define the iterative Picard process for countable system (24) as follows

$$\begin{cases} u_n^0(t) = 0, \\ u_n^k(t) = I(t; u_n^{k-1}),\ k = 1, 2, \ldots \end{cases} \tag{33}$$

**Theorem 1.** *Let conditions* (22), (26), (27), *smoothness be fulfilled. If the following conditions be fulfilled:*

1) $|\,\varphi(x,p_1) - \varphi(x,p_2)\,| \leqslant P_1(x)\ |p_1 - p_2|\,;$
2) $|\,\psi(x,p_1) - \psi(x,p_2)\,| \leqslant P_2(x)\ |p_1 - p_2|\,;$
3) $|\,R_i(t,x,p_1) - R_i(t,x,p_2)\,| \leqslant Q_i(t,x)\ |p_1 - p_2|$, $i = 1, 2$;
4) $\rho = \sum_{i=1}^{2} \|P_i(x)\|_{L_2(\Omega_l)} \int_0^T \|Q_i(\theta,x)\|_{L_2(\Omega_l)}\,d\theta + T\ \|C_2\|_{\ell_2} < 1.$

*Then the countable system of nonlinear integral equations* (24) *is uniquely solvable in the segment* $[0;T]$. *The solution of the countable system of nonlinear integral equations* (24) *can be find by the Picard iteration process* (33).

◁ Taking into account the formulas (26), (27), (29), (32) and applying the Cauchy–Schwarz and Bessel inequalities for the first approximation, from the iteration (33) we obtain the

estimate

$$\begin{aligned}
&\sum_{n=1}^{\infty}\max_{t\in[0;\,T]}\left|u_n^1(t)-u_n^0(t)\right|\leqslant\sum_{n=1}^{\infty}|\,\varphi_n(\cdot)\,|+\sum_{n=1}^{\infty}\max_{t\in[0;\,T]}|V_n(t)|\cdot|\,\psi_n(\cdot)\,|\\
&\leqslant\sum_{n=1}^{\infty}|\,\varphi_n(\cdot)\,|+C_1\sum_{n=1}^{\infty}|\,\psi_n(\cdot)\,|\leqslant\sum_{n=1}^{\infty}\frac{1}{n^5}\left|\varphi_n^{(5)}(\cdot)\right|+C_1\sum_{n=1}^{\infty}\frac{1}{n^5}\left|\psi_n^{(5)}(\cdot)\right|\\
&\leqslant\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^{10}}}\left[\sqrt{\sum_{n=1}^{\infty}\left[\varphi_n^{(5)}(\cdot)\right]^2}+C_1\sqrt{\sum_{n=1}^{\infty}\left[\psi_n^{(5)}(\cdot)\right]^2}\right]\\
&\leqslant\sqrt{\frac{2}{l}}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^{10}}}\left[\sqrt{\int\limits_{\Omega_l}\left[\frac{\partial^5\varphi(x,\,\cdot)}{\partial\,x^5}\right]^2dx}+C_1\sqrt{\int\limits_{\Omega_l}\left[\frac{\partial^5\psi(x,\,\cdot)}{\partial\,x^5}\right]^2dx}\right]<\infty.
\end{aligned}\tag{34}$$

Taking into account (27) and applying the Cauchy–Schwarz inequality for an arbitrary difference of the approximation (33), we obtain the estimate

$$\begin{aligned}
&\sum_{n=1}^{\infty}\max_{t\in[0;T]}\left|u_n^{k+1}(t)-u_n^k(t)\right|\\
&\leqslant\sum_{i=1}^{2}\sum_{n=1}^{\infty}\left|\int\limits_0^l P_i(x)\vartheta_n(x)\,dx\int\limits_0^T\int\limits_0^l Q_i(\theta,z)\sum_{m=1}^{\infty}\left|u_m^k(\theta)-u_m^{k-1}(\theta)\right|\,\vartheta_m(z)\,d\theta\,dz\right|\\
&\qquad+\sum_{n=1}^{\infty}\max_{t\in[0;T]}|\,W_n(t)\,|\int\limits_0^T\left|\,u_n^{k-1}(t)-u_n^{k-2}(t)\,\right|dt\\
&\leqslant\sum_{i=1}^{2}\|P_i(x)\|_{L_2(\Omega_l)}\left|\int\limits_0^T\int\limits_0^l Q_i(\theta,z)\sum_{m=1}^{\infty}\left|u_m^k(\theta)-u_m^{k-1}(\theta)\right|\,\vartheta_m(z)\,d\theta\,dz\right|\\
&\qquad+\int\limits_0^T\sum_{n=1}^{\infty}C_{2\,n}\left|\,u_n^{k-1}(t)-u_n^{k-2}(t)\,\right|dt\\
&\leqslant\sum_{i=1}^{2}\|P_i(x)\|_{L_2(\Omega_l)}\int\limits_0^T\|Q_i(\theta,x)\|_{L_2(\Omega_l)}\left\|u_m^k(\theta)-u_m^{k-1}(\theta)\right\|_{B_2(T)}d\theta\\
&+T\,\|C_2\|_{\ell_2}\left\|\,u^k(t)-u^{k-1}(t)\,\right\|_{B_2(T)}\leqslant\rho\left\|\,u^k(t)-u^{k-1}(t)\,\right\|_{B_2(T)},\quad k=2,3,\ldots
\end{aligned}\tag{35}$$

According to the last condition of the theorem, it follows from (35) that the operator on the right-hand side of (24) is contracting. It follows from (34) and (35) that there is a unique fixed point $u(t)\in B_2(T)$ on the segment $[0;T]$. This implies the unique solvability of the countable system of integral equations (24) on the interval $[0;T]$. Consequently, the iterative Picard process (33) converges absolutely and uniformly to the function $u(t)\in B_2(T)$, if the conditions of the theorem are fulfilled. The Theorem 1 is proved. ▷

### 4. Convergence of Fourier Series

**Theorem 2.** *Let the conditions of the theorem 1 are fulfilled. Then the boundary value problem* (1)–(5) *is uniquely solvable in the domain* $\Omega$. *The solution is determined by the series* (25). *The function* (25) *is differentiable with respect to all variables, i. e. the derivatives of the solution* (25) *included in equation* (1) *exist and are continuous.*

$\lhd$ If these conditions of the theorem are fulfilled, then we have

$$|\,U\,(t,x)\,| \leqslant \sum_{n=1}^{\infty} |\,\vartheta_n(x)\,| \left| \varphi_n(\cdot) + \psi_n(\cdot)\,V_n(t) + W_n(t) \int_0^T u_n(\theta)\,d\theta \right|$$

$$\leqslant \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} |\,\varphi_n(\cdot)\,| + C_1 \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} |\psi_n(\cdot)| + \sqrt{\frac{2}{l}} \int_0^T \sum_{n=1}^{\infty} C_{2\,n}\ |u_n(t)|\,dt$$

$$\leqslant \gamma \left[ \left\| \frac{\partial^5 \varphi(x,\cdot)}{\partial\, x^5} \right\|_{L_2(\Omega_l)} + \left\| \frac{\partial^5 \psi(x,\cdot)}{\partial\, x^5} \right\|_{L_2(\Omega_l)} \right] + T \sqrt{\frac{2}{l}}\ \|C_2\|_{\ell_2}\, \|u(t)\|_{B_2(T)} < \infty,$$

where $\gamma = \left(\frac{2}{l}\right)^{\frac{3}{2}} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}}$.

The function (25) is formally differentiated the required number of times with respect to the required arguments

$$U_{t\,t}(t,x) = \sum_{n=1}^{\infty} \vartheta_n(x) \left[ \varphi_n(\cdot) + \psi_n(\cdot)\,V_n''(t) + W_n''(t) \int_0^T u_n(\theta)\,d\theta \right], \tag{36}$$

$$U_{x\,x\,x\,x}(t,x) = \sum_{n=1}^{\infty} \left(\frac{\pi\, n}{l}\right)^4 \vartheta_n(x) \left[ \varphi_n(\cdot) + \psi_n(\cdot)\,V_n(t) + W_n(t) \int_0^T u_n(\theta)\,d\theta \right], \tag{37}$$

$$U_{t\,t\,x\,x\,x\,x}\,(t,x) = \sum_{n=1}^{\infty} \left(\frac{\pi\, n}{l}\right)^4 \vartheta_n(x) \left[ \varphi_n(\cdot) + \psi_n(\cdot)\,V_n''(t) + W_n''(t) \int_0^T u_n(\theta)\,d\theta \right]. \tag{38}$$

For the function (25), it is easy to verify that the continuity of all derivatives entering into equation (1). The proof of the convergence of functions (36) exactly coincides with the proof of the convergence of function (25). For the function (37) we present a proof of the convergence

$$|\,U_{x\,x\,x\,x}(t,x)\,| \leqslant \frac{\pi^4}{l^4} \sum_{n=1}^{\infty} n^4\ |u_n(t)| \cdot |\vartheta_n(x)|$$

$$\leqslant \sqrt{\frac{2}{l}}\,\frac{\pi^4}{l^4} \sum_{n=1}^{\infty} n^4\ |\varphi_n| + C_1 \sqrt{\frac{2}{l}}\,\frac{\pi^4}{l^4} \sum_{n=1}^{\infty} n^4\,|\psi_n| + \sqrt{\frac{2}{l}}\,\frac{\pi^4}{l^4} \int_0^T \sum_{n=1}^{\infty} n^4 C_{2\,n}\,|u_n(t)|\ dt$$

$$\leqslant \gamma \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \left[ \left\| \frac{\partial^5 \varphi(x,\,\cdot)}{\partial\, x^5} \right\|_{L_2(\Omega_l)} + C_1 \left\| \frac{\partial^5 \psi(x,\,\cdot)}{\partial\, x^5} \right\|_{L_2(\Omega_l)} \right]$$

$$+ \sqrt{\frac{2}{l}}\,\frac{\pi^4}{l^4} \sqrt{\sum_{n=1}^{\infty} \left(n^4\, C_{2n}\right)^2}\, \|u(t)\|_{B_2(T)} < \infty.$$

The convergence of the function (38) is proved in exactly the same way as above.

Consequently, in the domain $\Omega$ the function $U(t,x)$, defined by series (25), satisfies the conditions (2) of the problem (1)–(5) for all possible $n$. The Theorem 2 is proved. $\triangleright$

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# О НЕЛОКАЛЬНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ С ВЫРОЖДЕННЫМ ЯДРОМ

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**Аннотация.** В статье исследуются вопросы однозначной классической разрешимости и построения решения нелинейной краевой задачи для интегро-дифференциального уравнения в частных производных пятого порядка с вырожденным ядром. Для этой задачи заданы граничные условия Дирихле

по про странственной переменной. Итак, используется метод рядов Фурье, основанный на разделение переменных. Получена счетная система обыкновенных интегро-дифференциальных уравнений второго порядка с вырожденным ядром. К этой счетной системе обыкновенных интегро-дифференциальных уравнений применяется метод вырожденного ядра. Выводится система счетных систем алгебраических уравнений. Далее, получена счетная система нелинейных интегральных уравнений Фредгольма. Построен итерационный процесс решения этого интегрального уравнения. Установлены достаточные коэффициентные условия однозначной разрешимости счетной системы нелинейных интегральных уравнений при регулярных значениях параметра. Для доказательства однозначной разрешимости полученной счетной системы нелинейных интегральных уравнений используется метод последовательных приближений в сочетании его с методом сжимающего отображения. При доказательстве сходимости рядов Фурье используются неравенства Коши — Шварца и Бесселя. Доказана гладкость решения краевой задачи.

**Ключевые слова:** нелинейная краевая задача, интегро-дифференциальное уравнение, вырожденное ядро, ряд Фурье, классическая разрешимость.

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