Vladikavkaz Mathematical Journal
2022, Volume 24, Issue 1, P. 54–64

УДК 517.927.4
DOI 10.46698/p7919-5616-0187-g

EXISTENCE AND UNIQUENESS THEOREMS FOR A DIFFERENTIAL EQUATION WITH A DISCONTINUOUS RIGHT-HAND SIDE

M. G. Magomed-Kasumov[1,2]

[1] Southern Mathematical Institute VSC RAS,
53 Vatutin St., Vladikavkaz 362027, Russia;
[2] Daghestan Federal Research Centre of Russian Academy of Sciences,
45 M. Gadjiev St., Makhachkala 367000, Russia

E-mail: rasuldev@gmail.com

Abstract. We consider new conditions for existence and uniqueness of a Caratheodory solution for an initial value problem with a discontinuous right-hand side. The method used here is based on: 1) the representation of the solution as a Fourier series in a system of functions orthogonal in Sobolev sense and generated by a classical orthogonal system; 2) the use of a specially constructed operator A acting in l_2, the fixed point of which are the coefficients of the Fourier series of the solution. Under conditions given here the operator A is contractive. This property can be employed to construct robust, fast and easy to implement spectral numerical methods of solving an initial value problem with discontinuous right-hand side. Relationship of new conditions with classical ones (Caratheodory conditions with Lipschitz condition) is also studied. Namely, we show that if in classical conditions we replace L^1 by L^2, then they become equivalent to the conditions given in this article.

Key words: initial value problem, Cauchy problem, discontinuous right-hand side, Sobolev orthogonal system, existence and uniqueness theorem, Caratheodory solution.

AMS Subject Classification: 65L10, 34A12, 34A36, 34A37, 34B37.

For citation: *Magomed-Kasumov, M. G.* Existence and Uniqueness Theorems for a Differential Equation with a Discontinuous Right-Hand Side, *Vladikavkaz Math. J.*, 2022, vol. 24, no. 1, pp. 54–64. DOI: 10.46698/p7919-5616-0187-g.

1. Introduction

Consider an initial value problem:

$$x'(t) = f(t, x), \quad x(a) = x_0, \quad t \in [a, b], \tag{1}$$

where $f(t, x)$ can be discontinuous. Classic definition of a solution is too restrictive for differential equations with discontinuous right-hand side. There are different ways to generalize the notion of a solution in this case: Caratheodory solution, Filippov [1, 2] and Krasovskij [3, 4] solutions (based on differential inclusions), Hermes solution [5] (uses limiting transitions) and others (see [2, 6, 7, 8, 9] and references therein). In this paper we consider only Caratheodory solutions. A function $x(t)$ is called a Caratheodory solution of problem (1), if it is absolutely continuous, equality $x'(t) = f(t, x(t))$ holds for almost every $t \in [a, b]$ and $x(a) = x_0$.

We say that a function $f(t, x)$ satisfies the Caratheodory conditions in a domain D if in the domain

C1) $f(t, x)$ is continuous with respect to x for almost every t;

C2) $f(t, x)$ is measurable with respect to t for each x;

C3) there exists an integrable function $m(t)$ such that $|f(t, x)| \leqslant m(t)$.

The following results are well-known (see [2, 10]).

Theorem A. *Let $f(t,x)$ satisfy the Caratheodory conditions in $D=[a,b]\times[x_0-c,x_0+c]$. Then there exists a Caratheodory solution of problem* (1) *on $[a,a+d]$, where d is such that*

$$0<d\leqslant b-a,\quad \varphi(a+d)\leqslant c,\quad \varphi(t)=\int\limits_a^t m(s)\,ds. \tag{2}$$

We say that f satisfies L) condition in domain $D=[a,b]\times[x_0-c,x_0+c]$ if

L) there exist an integrable function $l(t)$, such that for almost every $t\in[a,b]$ and every $x,\,y\in[x_0-c,x_0+c]$

$$|f(t,x)-f(t,y)|\leqslant l(t)|x-y|. \tag{3}$$

Theorem B. *Suppose f satisfies* L) *in domain $D=[a,b]\times[x_0-c,x_0+c]$. Then if in the domain D a solution of problem* (1) *exists, it is unique.*

Thus, if $f(t,x)$ satisfies in $D=[a,b]\times[x_0-c,x_0+c]$ the Caratheodory conditions and condition L) then there exists a unique solution of problem (1) in $[a,a+d]$ where d satisfies (2).

I. Sharapudinov obtained another conditions for existence and uniqueness of a solution of problem (1). Before stating the main result obtained in [11], we give some definitions.

Let $L^p_\mu[a,b]$ be a space of functions, integrable with weight μ on the segment $[a,b]$:

$$L^p_\mu[a,b]=\left\{f:\int\limits_a^b|f(t)|^p\,\mu(t)\,dt<\infty\right\}. \tag{4}$$

By $W^r_{L^p_\mu}[a,b]$ we denote a space of $(r-1)$-times continuously differentiable functions $f=f(t)$ defined on $[a,b]$ such that $f^{(r-1)}(t)$ is absolutely continuous and $f^{(r)}\in L^p_\mu[a,b]$.

Let $\Phi=\{\varphi_k,k=0,1,\ldots\}$ be a complete orthonormal system in $L^2_\mu=L^2_\mu[0,1]$. Define a new system $\Phi_1=\{\varphi_{1,k}\}$ using formulas:

$$\varphi_{1,0}(t)=1,\quad \varphi_{1,1+k}(t)=\int\limits_a^t(t-x)\,\varphi_k(x)\,dx,\quad k\geqslant 0.$$

This system is orthogonal with respect to Sobolev-type inner product (12), where $r=1$ (see details in section 3). Suppose that the system $\Phi_1=\{\varphi_{1,k}\}$ possess the property $\kappa(\Phi_1)=\big(\sum_{k=1}^\infty\int_a^b\varphi^2_{1,k}(t)\mu(t)\,dt\big)^{1/2}<\infty$. Systems with this property exist (see [12]). The following theorem was proved in [11].

Theorem C. *If for some δ the conditions*

A) *$f(t,g(t))\in L^2_\mu[a,b]$ for any function $g(t)\in W^1_{L^2_\mu}[a,b]$;*

B) *for any $g_1(t),g_2(t)\in W^1_{L^2_\mu}[a,b]$ the following relation holds:*

$$\int\limits_a^b\left[f(t,g_1(t))-f(t,g_2(t))\right]^2\mu(t)\,dt\leqslant\delta^2\int\limits_a^b\left[g_1(t)-g_2(t)\right]^2\mu(t)\,dt;$$

C) $\delta\kappa(\Phi_1)<1$,

hold then initial value problem (1) *has a unique solution $x(t)\in W^1_{L^2_\mu}[a,b]$. This solution can* *be represented as a uniformly convergent series*

$$x(t) = x_0 + \sum_{k=1}^{\infty} c_{1,k}\varphi_{1,k}(t), \quad t \in [a,b]. \tag{5}$$

In this article, we show that in the case of unit weight $\mu(t)$, using the methods from [11–13], we can

1) remove condition C) in Theorem C;

2) replace condition B) with

B') there exists an integrable function $w(t)$, such that for any $g_1(t), g_2(t) \in W^1_{L^2_\mu}[a,b]$ the following relation holds:

$$\int_a^b [f(t,g_1(t)) - f(t,g_2(t))]^2 \, dt \leqslant \int_a^b w(t)\,[g_1(t) - g_2(t)]^2 \, dt.$$

Namely, the following theorem holds.

Theorem 1. *If f satisfies conditions* A), B'), *initial value problem* (1) *has a unique solution $x(t) \in W^1_{L^2}[a,b]$ on $[a,b]$.*

A proof of this theorem, which is given in section 5, is based on using the next theorem.

Theorem 2. *Let $\Phi = \{\varphi_k\}$ be a complete orthonormal system in $L^2[u,v]$ such that for $\Phi_1 = \{\varphi_{1,k}\}$ a condition $\delta(\Phi_1) = \left(\sup_{t\in[u,v]} \sum_{k=1}^{\infty} \varphi_{1,k}^2(t)\right)^{1/2} < \infty$ holds.*

If f satisfies A), B'), *then for any $\alpha \in [a,b)$ and any $h \leqslant b-\alpha$ that satisfies the condition*

$$h \int_{\alpha}^{\alpha+h} w(t)\,dt < \frac{v-u}{\delta^2(\Phi_1)} \tag{6}$$

an initial value problem

$$x'(t) = f(t,x), \quad x(\alpha) = x_0, \quad t \in [\alpha, \alpha+h], \tag{7}$$

has a unique solution $x(t) \in W^1_{L^2}[\alpha,\alpha+h]$ on $[\alpha,\alpha+h]$. This solution can be represented as a uniformly convergent series

$$x(t) = x_0 + \sum_{k=1}^{\infty} c_{1,k}\varphi_{1,k}(\theta(t)), \quad \theta(t) = \frac{v-u}{h}(t-\alpha) + u, \quad t \in [\alpha,\alpha+h]. \tag{8}$$

A proof of Theorem 2 is given in section 4. The proof is based on using Fourier series with respect to Φ_1-type system, orthogonal in Sobolev sense and generated by an ordinary system. Some general information about these systems we give in section 3.

Coefficients $c_{1,k}$ in (8) are Fourier coefficients with respect to Sobolev system Φ_1. To determine the coefficients $c_{1,k}$, we use a specially constructed operator A (see (26)), defined in Hilbert space l_2, consisting of sequences $C = (c_j)_{j=1}^{\infty}$ with the norm $\|C\| = \left(\sum_{j=1}^{\infty} c_j^2\right)^{1/2}$. The operator A is constructed in such a way that its fixed point is a sequence of the coefficients $c_{1,k}$. In this connection, the question of whether the operator A has a contraction property becomes important. It turns out that a positive answer to this question can be given when functions of the system Φ_1 have the following property

- under conditions of Theorem C: $\kappa(\Phi_1) < \infty$ [12–14];
- under conditions of Theorem 2: $\delta(\Phi_1) < \infty$ (see section 4).

It was shown in [12] that the properties $\kappa(\Phi_1) < \infty$ and $\delta(\Phi_1) < \infty$ hold for the system of functions $\chi_{1,k}(x)$ generated by the Haar system, and for the system of functions generated by the system of cosine functions.

It should be noted that under conditions of Theorem 2 the operator A is contractive. This property can be employed to construct robust, fast and easy to implement numerical methods of solving an initial value problem with discontinuous right-hand side.

We begin with considering a relationship between conditions C1), C2), C3), L) and A), B').

2. Relationship Between Conditions

Let's introduce modifications of conditions C3) and L) in the following way:

C3') there exists a function $m(t) \in L^2[a,b]$ such that $|f(t,x)| \leqslant m(t)$.

L') there exist a function $l(t) \in L^2[a,b]$, such that for almost every $t \in [a,b]$ and every x, $y \in [x_0 - c, x_0 + c]$

$$|f(t,x) - f(t,y)| \leqslant l(t)|x-y|. \tag{9}$$

These conditions differ from their counterparts only in that here we require functions $m(t)$ and $l(t)$ to be from $L^2[a,b]$.

Theorem 3. *A function $f(t,x)$ satisfies conditions* C1), C2), C3'), L') *if and only if it satisfies conditions* A), B').

This theorem proof is based on the following lemmas.

Lemma 1. *If f for some function $w(t)$ satisfies the condition* B') *on the segment $[a,b]$, then f will satisfy this condition with the same function $w(t)$ on any subsegment $[\alpha,\beta] \subset [a,b]$.*

$\lhd$ Let g_1, g_2 be arbitrary functions from $W^1_{L^2}[\alpha,\beta]$. Denote by $\tilde{g}_1$ the continuous extension of g_1 by the constants to the entire interval $[a,b]$:

$$\tilde{g}_1(x) = \begin{cases} g_1(\alpha), & x \in [a,\alpha), \\ g_1(x), & x \in [\alpha,\beta], \\ g_1(\beta), & x \in (\beta,b], \end{cases}$$

and by $\tilde{g}_2(x;h)$ the continuous extension of g_2 by constants $g_1(\alpha)$, $g_1(\beta)$ on the segments $[a,\alpha-h]$, $[\beta+h,b]$ and by linear functions on the segments $[\alpha-h,\alpha]$, $[\beta,\beta+h]$:

$$\tilde{g}_2(x;h) = \begin{cases} g_1(\alpha), & x \in [a,\alpha-h], \\ \frac{g_2(\alpha)-g_1(\alpha)}{h}(x-\alpha) + g_2(\alpha), & x \in (\alpha-h,\alpha), \\ g_2(x), & x \in [\alpha,\beta], \\ \frac{g_1(\beta)-g_2(\beta)}{h}(x-\beta) + g_2(\beta), & x \in (\beta,\beta+h), \\ g_1(\beta), & x \in [\beta+h,b]. \end{cases}$$

It is clear that $\tilde{g}_1(t)$ and $\tilde{g}_2(t;h)$ belong to $W^1_{L^2}[a,b]$ for any sufficiently small h. Further, for any small $h > 0$ we have

$$\int_\alpha^\beta \Big[f(t,g_1(t)) - f(t,g_2(t))\Big]^2 dt \leqslant \int_a^b \Big[f(t,\tilde{g}_1(t)) - f(t,\tilde{g}_2(t;h))\Big]^2 dt$$

$$\leqslant \int_a^b w(t)\Big[\tilde{g}_1(t) - \tilde{g}_2(t;h)\Big]^2 dt = \int_\alpha^\beta w(t)\Big[g_1(t) - g_2(t)\Big]^2 dt + I_1(h) + I_2(h), \tag{10}$$

where

$$I_1(h) = \int\limits_a^\alpha w(t)\Big[\tilde{g}_1(t) - \tilde{g}_2(t;h)\Big]^2 dt, \quad I_2(h) = \int\limits_\beta^b w(t)\Big[\tilde{g}_1(t) - \tilde{g}_2(t;h)\Big]^2 dt.$$

Consider $I_1(h)$:

$$|I_1(h)| = \left| \int\limits_{\alpha-h}^{\alpha} w(t)\left[g_1(\alpha) - \frac{g_2(\alpha)-g_1(\alpha)}{h}(t-\alpha) - g_2(\alpha)\right]^2 dt \right|$$
$$= (g_1(\alpha) - g_2(\alpha))^2 \left| \int\limits_{\alpha-h}^{\alpha} w(t)\left[1 - \frac{\alpha-t}{h}\right]^2 dt \right| \leqslant (g_1(\alpha) - g_2(\alpha))^2 \int\limits_{\alpha-h}^{\alpha} |w(t)|\, dt.$$

Last integral vanishes as $h \to 0$ (absolute continuity). Hence, $I_1(h) \to 0$, $h \to 0$. Similarly, we can show that $I_2(h) \to 0$, $h \to 0$. Then lemma's statement follows from (10). $\rhd$

Lemma 2. *If f satisfies condition* B'), *then the function $w(t)$ is nonnegative for* a. e. *$t \in [a,b]$ and the function f satisfies condition* L'), *in which $l(t) = \sqrt{w(t)} \in L^2[a,b]$.*

$\lhd$ Let x, y be arbitrary numbers. By Lemma 1, it follows that for any small $h > 0$

$$\frac{1}{h}\int\limits_u^{u+h}\Big[f(t,x) - f(t,y)\Big]^2 dt \leqslant (x-y)^2 \frac{1}{h}\int\limits_u^{u+h} w(t)\, dt, \quad u \in [a,b].$$

Hence, tending h to 0, we get that for a. e. $u \in [a,b]$ [15, th. 1.3, p. 104]

$$|f(u,x) - f(u,y)|^2 \leqslant w(u)(x-y)^2. \tag{11}$$

This implies that $w(t)$ must be nonnegative for a. e. $t \in [a,b]$. Then to obtain L') it remains to extract square roots from both sides of (11) and denote $l(u) = \sqrt{w(u)}$. $\rhd$

$\lhd$ Proof of Theorem 3. Suppose f satisfies the conditions C1), C2), C3'), L'). Arguing as in [10, Chapter VIII, § 8] (or in [16, Chapter III, § 10, Supplement II, p. 122]) one can show that $f(t,g(t)) \in L^2[a,b]$ for any measurable function $g(t)$, so condition A) holds for f. Further, it follows from L') that

$$\int\limits_a^b \Big[f(t,g_1(t)) - f(t,g_2(t))\Big]^2 dt \leqslant \int\limits_a^b w(t)\Big[g_1(t) - g_2(t)\Big]^2 dt,$$

where $w(t) = l^2(t) \in L^1[a,b]$, and condition B') also holds for f. Thus, conditions C1), C2), C3'), L') imply conditions A), B').

Now we show that the converse is also true. Condition C2) follows from A). By Lemma 2, condition B') imply L'). It follows from L') that $f(t,x)$ satisfies C1). It remains to show that $f(t,x)$ satisfies C3'). We claim that L') and A) imply C3'). Indeed, using L') we get

$$|f(t,x) - f(t,a)| \leqslant l(t)|x-a| \leqslant l(t)(b-a), \quad x \in [a,b],$$

where $l(t) \in L^2[a,b]$. This can be rewritten as $f(t,a) - l(t)(b-a) \leqslant f(t,x) \leqslant f(t,a) + l(t)(b-a)$. Hence,

$$|f(t,x)| \leqslant m(t) = \max\{|f(t,a) - l(t)(b-a)|, |f(t,a) + l(t)(b-a)|\}, \quad x \in [a,b].$$

Since $f(t,a) \in L^2[a,b]$ (due to A)), we have $f(t,a) \pm l(t)(b-a) \in L^2[a,b]$, so $m(t)$ is also from $L^2[a,b]$ and condition C3') holds. $\rhd$

3. Sobolev Orthogonal Systems

In [17–20] I. Sharapudinov considered systems of functions orthogonal with respect to Sobolev-type inner product

$$\langle f, g\rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(a)\, g^{(\nu)}(a) + \int_a^b f^{(r)}(x)\, g^{(r)}(x)\, \mu(x)\, dx. \tag{12}$$

He introduced systems $\Phi_r = \{\varphi_{r,k}\}$ defined as

$$\varphi_{r,k}(x) = \frac{(x-a)^k}{k!}, \quad k = 0, 1, \dots, r-1, \tag{13}$$

$$\varphi_{r,k}(x) = \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} \varphi_{k-r}(t)\, dt, \quad k = r, r+1, \dots, \tag{14}$$

where $\Phi = \{\varphi_k\}_{k=0}^{\infty}$ is a system, orthogonal with respect to the ordinary inner product of the form

$$\langle f, g\rangle = \int_a^b f(t) g(t)\, \mu(t)\, dt, \tag{15}$$

and showed orthogonality of these systems with respect to inner product (12). The system Φ_r is called a Sobolev orthogonal system generated by the system Φ.

A Fourier series of a function $x(t)$ in the system Φ_r has the following form [18]:

$$x(t) \sim \sum_{k=0}^{r-1} x^{(k)}(a) \frac{(t-a)^k}{k!} + \sum_{k=r}^{\infty} c_{r,k}(x) \varphi_{r,k}(t), \tag{16}$$

where

$$c_{r,k}(x) = \int_a^b x^{(r)}(t) \varphi_{k-r}(t)\, \mu(t)\, dt. \tag{17}$$

The Fourier series of form (16) turned out to be a natural and very convenient tool for solving systems of differential equations [12]. In [12–14] it was proposed an iterative method for solving an initial value problem for a nonlinear ordinary differential equation of the form

$$x'(t) = f(t, x), \quad x(a) = x_0, \quad t \in [a, b], \tag{18}$$

based on a representation of the solution of problem (18) as a Fourier series in a Φ_1-type system:

$$x(t) = x(a) + \sum_{k=1}^{\infty} c_{1,k}(x) \varphi_{1,k}(t), \tag{19}$$

where $x(a) = x_0$ is an initial value and $c_{1,k}(x)$ are unknown Fourier coefficients that should be found.

In already mentioned works [12–14] it is assumed that a function on the right-hand side of a differential equation is continuous in both variables and satisfies the Lipschitz condition with respect to y. However, it turned out that the method used there can be extended to the case of differential equations with a discontinuous right-hand side [11]. We use this method to prove Theorem 2.

4. Proof of Theorem 2

If we introduce a function $y(s) = x(\theta^{-1}(s))$, where $\theta^{-1}(s) = \frac{s-u}{v-u}h + \alpha$, then problem (7) can be written as follows

$$y'(s) = F(s, y), \quad y(u) = y_0, \quad s \in [u, v], \tag{20}$$

where $F(s, y) = \frac{h}{v-u} f(\theta^{-1}(s), y)$. It is easy to verify that $F(s, g(s)) \in L^2[u, v]$ for any $g \in W^1_{L^2}[u, v]$. Indeed, using the substitution $s = \theta(t)$, we obtain

$$\int_u^v F^2(s, g(s))\, ds = \frac{h}{v-u} \int_\alpha^{\alpha+h} f^2(t, g(\theta(t)))\, dt. \tag{21}$$

Assuming

$$\tilde{g}(t) = \begin{cases} g(\theta(\alpha)), & t \in [a, \alpha), \\ g(\theta(t)), & t \in [\alpha, \alpha+h], \\ g(\theta(\alpha+h)), & t \in (\alpha+h, b], \end{cases}$$

and noting that $\tilde{g}(t) \in W^1_{L^2}[a, b]$, from (21) and condition A) we get

$$\int_u^v F^2(s, g(s))\, ds \leqslant \frac{h}{v-u} \int_a^b f^2(t, \tilde{g}(t))\, dt < \infty.$$

Further, for problem (20) following the work [11] we introduce the operator A, the construction of which is based on the following relations:

$$y(s) = y(u) + \sum_{k=0}^{\infty} c_{1,k+1}(y)\, \varphi_{1,k+1}(s), \tag{22}$$

$$y'(s) = \sum_{k=0}^{\infty} c_{1,k+1}(y)\, \varphi_k(s), \tag{23}$$

$$q(s) = F(s, y(s)) = \sum_{k=0}^{\infty} c_k(q)\, \varphi_k(s), \tag{24}$$

where the first relation is the Fourier series in the system $\{\varphi_{1,k}\}$ of the function $y(s) \in W^1_{L^2}[u, v]$, and the second and third ones are Fourier series in system $\{\varphi_k\}$ of functions $y'(s) \in L^2[u, v]$ and $q(s) \in L^2[u, v]$ respectively. Note that in the relation (22) the Fourier series converges uniformly (see, for example, [18, Theorem 2.2]), and in (23) and (24) series converge in the metric of $L^2[u, v]$ (due to the completeness of the system φ_k in the space $L^2[u, v]$).

It follows from (7), (23) and (24) that

$$c_{1,k+1}(y) = c_k(q) = \int_u^v F\big(s, y(s)\big)\, \varphi_k(s)\, ds.$$

Combining this equality with (22) we obtain the relation

$$c_{1,k+1}(y) = \int_u^v F\Big(s, y(u) + \sum_{j=0}^{\infty} c_{1,j+1}(y)\, \varphi_{1,j+1}(s)\Big) \varphi_k(s)\, ds, \quad k \geqslant 0. \tag{25}$$

The right-hand side expression is the aforementioned operator A that takes the point $d \in l^2$ to the point $A(d) \in l^2$ according to the following rule

$$A(d) = \left(\int_u^v F\Big(s, y(u) + \sum_{j=0}^{\infty} d_j\, \varphi_{1,j+1}(s)\Big) \varphi_k(s)\, ds,\ k \geqslant 0 \right), \tag{26}$$

where

$$F(s,y) = \frac{h}{v-u} f\left(\theta^{-1}(s), y\right), \quad \theta^{-1}(s) = \frac{s-u}{v-u} h + \alpha.$$

It follows from (25) that the Fourier coefficients sequence $C(y) = \big(c_{1,k+1}(y),\ k \geqslant 0\big)$ of the solution $y(s)$ with respect to the system $\Phi_1 = \{\varphi_{1,k}(s)\}$ is a fixed point of the operator A: $A(C(y)) = C(y)$.

We now show that A is contractive provided B').

Let d^1 and d^2 be two arbitrary points in l^2. We introduce the notation:

$$g_1(s) = y(u) + \sum_{j=0}^{\infty} d_j^1\, \varphi_{1,j+1}(s), \quad g_2(s) = y(u) + \sum_{j=0}^{\infty} d_j^2\, \varphi_{1,j+1}(s). \tag{27}$$

Theorem [18, Theorem 2] implies that $g_1, g_2 \in W^1_{L^2}[u,v]$ and that their series uniformly converge on $[u,v]$. Consider the difference:

$$A(d^1) - A(d^2) = \left(\int_u^v \big[F(s, g_1(s)) - F(s, g_2(s))\big] \varphi_k(s)\, ds,\ k \geqslant 0 \right).$$

By Parseval's equality, we have:

$$\big\|A(d^1) - A(d^2)\big\|^2_{l^2} = \int_u^v \big[F(s, g_1(s)) - F(s, g_2(s))\big]^2 ds \doteq J. \tag{28}$$

Changing the variable $s = \theta(t)$ reduces the integral J to the form:

$$J = \frac{h}{v-u} \int_{\alpha}^{\alpha+h} \big[f(t, \bar{g}_1(t)) - f(t, \bar{g}_2(t))\big]^2 dt,$$

where $\bar{g}_j(t) = g_j(\theta(t))$, $j = 1, 2$. It is obvious that $\bar{g}_j(t) \in W^1_{L^2}[\alpha, \alpha+h]$, $j = 1, 2$. Since f satisfies condition B') on the segment $[a,b]$, using Lemma 1 and making the inverse change $t = \theta^{-1}(s)$ we get

$$J \leqslant \frac{h}{v-u} \int_{\alpha}^{\alpha+h} w(t)\big[\bar{g}_1(t) - \bar{g}_2(t)\big]^2 dt = \frac{h^2}{(v-u)^2} \int_u^v w(\theta^{-1}(s))\big[g_1(s) - g_2(s)\big]^2 ds. \tag{29}$$

Substituting the expressions from (27) into the last integral and applying the Cauchy–Bunyakovsky inequality we obtain:

$$\int_u^v w(\theta^{-1}(s))\big[g_1(s) - g_2(s)\big]^2 ds \leqslant \big\|d^1 - d^2\big\|^2_{l^2}\, \delta^2(\Phi_1) \frac{v-u}{h} \int_{\alpha}^{\alpha+h} w(t)\, dt.$$

This inequality with (28), (29) yields:

$$\left\|A(d^1) - A(d^2)\right\|_{l^2} \leqslant \delta(\Phi_1)\left(\frac{h}{v-u}\int\limits_{\alpha}^{\alpha+h} w(t)\,dt\right)^{1/2} \left\|d^1 - d^2\right\|_{l^2}.$$

Therefore, under the condition (6), the operator A will be contractive.

Hence, since l^2 is a complete space, the operator A will have a unique fixed point. This point as noted above (see (25)) is a Fourier coefficients sequence of the solution $y(s)$ for problem (20). So a solution exists and has a form (22), where $C(y) = \big(c_{1,k+1}(y),\ k \geqslant 0\big)$ is a fixed point of A. The solution uniqueness follows from the fact that any solution $y(s)$ of problem (20) belongs to the space $W^1_{L^2}[u,v]$ and therefore can be decomposed into uniformly convergent series (22), in which the sequence of coefficients is a fixed point of the operator A.

If $y(s)$ is a solution of problem (20), then $x(t) = y(\theta(t))$ is a solution of problem (7) on $[\alpha, \alpha+h]$.

5. Proof of Theorem 1

To obtain a solution on the segment $[a,b]$, we divide it into m subsegments $[a_i, a_{i+1}] = [ih, (i+1)h]$, $i = 0, 1, \ldots, m-1$, where $h = \frac{b-a}{m}$, $m > \delta^2(\Phi_1)\frac{b-a}{v-u}\int_a^b w(t)\,dt$ (therefore, on each segment condition (6) will hold). We will successively solve the initial value problems

$$x'(t) = f(t,x), \quad x(a_i) = x_{i,0}, \quad t \in [a_i, a_{i+1}], \tag{30}$$

on the segments $[a_i, a_{i+1}]$ with initial values defined as follows: $x_{0,0} = x_0$, $x_{i,0} = x_{i-1}(a_i)$, $i = 1, \ldots, m-1$, where $x_i(t)$ is a solution of the problem on the subsegment $[a_i, a_{i+1}]$. Then the solution on the segment $[a,b]$ will have the form:

$$x(t) = x_i(t), \quad t \in [a_i, a_{i+1}],\ i = 0, \ldots, m-1.$$

It is easy to verify that given function is a solution of problem (1). Indeed, $x(0) = x_0(0) = x_{0,0} = x_0$, so initial condition holds. It is also obvious that $x(t) \in W^1_{L^2}[u,v]$. Further, denote by $E_i \subset [a_i, a_{i+1}]$ a set with measure $\mu(E_i) = |a_{i+1} - a_i|$ such that $x_i'(t) = f(t, x_i(t))$, $t \in E_i$. If $t \in E = \cup_{i=0}^{m-1} E_i$ then t belongs to some E_i. Hence, for this t we have $x'(t) = x_i'(t) = f(t, x_i(t)) = f(t, x(t))$. Since $\mu(E) = b - a$, $x(t)$ satisfies (1) almost everywhere on $[a,b]$.

References

1. *Filippov, A. F.* Differential Equations with Discontinuous Right-Hand Side, *Matematicheskii Sbornik. Novaya Seriya*, 1960, vol. 51 (93), no. 1, pp. 99–128 (in Russian).
2. *Filippov, A. F. Differential Equations with Discontinuous Righthand Sides*, Dordrecht, The Netherlands, Kluwer Academic Publishers, 1988.
3. *Krasovskij, N. N. Game-Theoretic Problems of Capture*, Moscow, Nauka, 1970 (in Russian).
4. *Krasovskij, N. N. and Subbotin, A. I.* Positional Differential Games, Moscow, Nauka, 1974 (in Russian).
5. *Hermes, H.* Discontinuous Vector Fields and Feedback Control, *Differential Equations and Dynamical Systems*, New York, Academic Press, 1967.
6. *Cid, J. N. and Pouso, R. L.* Ordinary Differential Equations and Systems with Time-Dependent Discontinuity Sets, *Proceedings of the Royal Society of Edinburgh: Section a Mathematics*, 2004, vol. 134, no. 4, pp. 617–637. DOI: 10.1017/S0308210500003383.
7. Hajek, O. Discontinuous Differential Equations, I, *Journal of Differential Equations*, 1979, vol. 32, no. 2, pp. 149–170. DOI: 10.1016/0022-0396(79)90056-1.
8. *Cortes, J.* Discontinuous Dynamical Systems, *IEEE Control Systems Magazine*, 2008, vol. 28, no. 3, pp. 36–73. DOI: 10.1109/MCS.2008.919306.
9. *Ceragioli, F. Discontinuous Ordinary Differential Equations and Stabilization*, PhD thesis, Universita di Firenze, 2000.

10. *Sansone, G. Equazioni Differenziali Nel Campo Reale, vol. 2*, Consiglio nazionale consiglio nazionale delle ricerche. Monografie di matematic applicata, Nicola Zanichelli, 1949.
11. *Sharapudinov, I. I.* On the Existence and Uniqueness of Solutions of ODEs with Discontinuous Right-Hand Sides and Sobolev Orthogonal Systems of Functions, *Daghestan Electronic Mathematical Reports*, 2018, vol. 9, pp. 68–75 (in Russian). DOI: 10.31029/demr.9.8.
12. *Sharapudinov, I. I.* Sobolev-Orthogonal Systems of Functions and the Cauchy Problem for ODEs, *Izvestiya: Mathematics*, 2019, vol. 83, no. 2, pp. 391–412. DOI: 10.1070/im8742.
13. *Sharapudinov, I. I.* Sobolev Orthogonal Polynomials Associated with Chebyshev Polynomials of the First Kind and the Cauchy Problem for Ordinary Differential Equations, *Differential Equations*, 2018, vol. 54, no. 12, pp. 1602–1619. DOI: 10.1134/S0012266118120078.
14. *Sharapudinov, I. I.* Approximation of the Solution of the Cauchy Problem for Nonlinear ODE Systems by Means of Fourier Series in Functions Orthogonal in the Sense of Sobolev, *Daghestan Electronic Mathematical Reports*, 2017, vol. 7, pp. 66–76 (in Russian). DOI: 10.31029/demr.7.8.
15. *Stein, E. and Shakarchi, R. Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2009.
16. *Walter, W. Ordinary Differential Equations, Graduate Texts in Mathematics*, New York, Springer, 1998.
17. *Sharapudinov, I. I.* Sobolev-Orthogonal Systems of Functions and Some of their Applications, *Russian Mathematical Surveys*, 2019, vol. 74, no. 4, pp. 659–733. DOI: 10.1070/rm9846.
18. *Sharapudinov, I. I.* Sobolev-Orthogonal Systems of Functions Associated with an Orthogonal System, *Izv. Math.*, 2018, vol. 82, no. 1, pp. 212–244. DOI: 10.1070/IM8536.
19. *Sharapudinov, I. I.* Approximation Properties of Fourier Series of Sobolev Orthogonal Polynomials with Jacobi Weight and Discrete Masses, *Mathematical Notes*, 2017, vol. 101, no. 3, pp. 718–734. DOI: 10.1134/S0001434617030300.
20. *Sharapudinov, I. I.* Sobolev Orthogonal Polynomials Generated by Jacobi and Legendre Polynomials, and Special Series with the Sticking Property for their Partial Sums, *Sbornik: Mathematics*, 2018, vol. 209, no. 9, pp. 1390–1417. DOI: 10.1070/sm8910.

Received 23 мая 2021 г.

Magomedrasul G. Magomed-Kasumov
Southern Mathematical Institute VSC RAS,
53 Vatutin St., Vladikavkaz 362027, Russia,
Senior Researcher

Daghestan Federal Research Centre of Russian Academy of Sciences,
45 M. Gadjiev St., Makhachkala 367000, Russia,
Senior Researcher
E-mail: rasuldev@gmail.com

Владикавказский математический журнал
2022, Том 24, Выпуск 1, С. 54–64

ТЕОРЕМЫ СУЩЕСТВОВАНИЯ И ЕДИНСТВЕННОСТИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С РАЗРЫВНОЙ ПРАВОЙ ЧАСТЬЮ

Магомед-Касумов М. Г.[1,2]

[1] Южный математический институт ВНЦ РАН,
Россия, 362027, Владикавказ, ул. Ватутина, 53;
[2] Дагестанский федеральный исследовательский центр РАН,
Россия, 367000, Махачкала, ул. М. Гаджиева, 45,

E-mail: rasuldev@gmail.com

Аннотация. Рассмотрены новые условия существования и единственности решения Каратеодори задачи Коши для дифференциального уравнения первого порядка с разрывной правой частью. Применяемый в статье метод основан на: 1) представлении решения в виде ряда Фурье по системе функций, ортогональной относительно скалярного произведения типа Соболева и порожденной классической ортогональной системой; 2) использовании специальный образом сконструированного оператора A, действующего в пространстве l_2, неподвижной точкой которого являются коэффициенты Фурье решения. При выполнении условий, рассматриваемых в данной статье, оператор A будет сжимающим. Это свойство может быть использовано для конструирования устойчивых, быстрых и легко реализуемых спектральных численных методов решения задачи Коши с разрывной правой частью. Изучена также взаимосвязь новых условий с хорошо известными классическими условиями (условия Каратеодори вместе с условием Липшица) существования и единственности решения Каратеодори задачи Коши с разрывной правой частью. А именно, показано, что если в классических условиях заменить пространство суммируемых функций L^1 на пространство суммируемых с квадратом функций L^2, то они станут эквивалентными условиям, приведенным в данной статье.

Ключевые слова: задача Коши, разрывная правая часть, ортогональная в смысле Соболева система, теорема существования и единственности, решение Каратеодори.

AMS Subject Classification: 65L10, 34A12, 34A36, 34A37, 34B37.

Образец цитирования: *Magomed-Kasumov, M. G.* Existence and Uniqueness Theorems for a Differential Equation with a Discontinuous Right-Hand Side // Владикавк. мат. журн.—2022.—Т. 24, № 1.—С. 54–64 (in English). DOI: 10.46698/p7919-5616-0187-g.