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ON MEROMORPHIC FUNCTION WITH MAXIMAL DEFICIENCY SUM AND IT'S DIFFERENCE OPERATORS

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Abstract. The paper deals with characteristic funtion and deficiency of a meromorphic function. We mainly focused on the relation between the characteristic function of a product of difference operators with the characteristic function of a meromorphic function with maximal deficiency sum. The concept of maximal deficiency sum of a meromorphic function is employed as an effective tool for our research. In the same context, the notion of a difference polynomial of a difference operator is discussed. The paper contains the details analysis and discussion of some asymptotic behaviour of the product of difference operators, such as $\lim_{r\to\infty}\frac{T(r,\prod_{i=1}^{q}\Delta_{\eta_i}f)}{T(r,f)}$, $\lim_{r\to\infty}\frac{N(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f)}{T(r,\prod_{i=1}^{q}\Delta_{\eta_i}f)}$, $\overline{\lim}_{r\to\infty}\frac{N(r,\infty;\prod_{i=1}^{q}\Delta_{\eta_i}f)+N(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f)}{T(r,\prod_{i=1}^{q}\Delta_{\eta_i}f)}$ etc. and same resolution and discussion also developed for the difference polynomial of difference operators. Several innovative idea to establish some inequalities on the zeros and poles for $\prod_{i=1}^{q}\Delta_{\eta_i}f$ and $L(\Delta_\eta f)$ are also introduced. We broadly elaborate our results with many remarks and corollaries, and give two excellent examples for proper justification of our results. The results on product and polynomial of difference operators of our article improved and generalised the results of Z. Wu.

Key words: transcendental meromorphic function, deficiency sum, difference operators, product of difference operators.

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1. Introduction, Definations and Notations

Let f be a meromorphic function defined in $\mathbb{C}$ and $\alpha \in \mathbb{C}$. We adopt some notations from the Nevanlinna theory of meromorphic functions, such as $m(r,f)$, $N(r,\infty;f)$, $m(r,\alpha)=m\big(r,\frac{1}{f-\alpha}\big)$, $N(r,0;f)$ etc., for details see [1–3]. We defined $T(r,f)=N(r,\infty;f)+m(r,f)$ and called Nevanlinna's characteristic function, and defined the quantity $S(r,f)$ such that $S(r,f)=o(T(r,f))$ as $r\to\infty$, outside of a set of finite measure.

Definition 1.1. Deficiency of $\alpha\in\mathbb{C}$ with respect to a meromorphic function f is denoted by $\delta(\alpha,f)$ and defined as

$$\delta(\alpha,f)=\varliminf_{r\to\infty}\frac{m\big(r,\frac{1}{f-\alpha}\big)}{T(r,f)}=1-\varlimsup_{r\to\infty}\frac{N(r,\alpha;f)}{T(r,f)}.$$

By the Nevanlinna's second fundamental theorem, it can be easily shown that

$$\sum_{\alpha\in\mathbb{C}\cup\{\infty\}}\delta(\alpha,f)\leqslant 2.$$

If equality holds in the above relation, we say that f is of maximal deficiency sum.

DEFINITION 1.2. The order of a meromorphic function f is denoted by $\sigma(f)$ and define by

$$\sigma(f) = \lim_{r\to\infty} \sup \frac{\log T(r,f)}{\log r}.$$

Let $P(z,f)$ be rational function in f with small meromorphic coefficients. Then by Valiron–Mo'honko identity we have

$$T(r, P(z,f)) = \deg_f(p) T(r,f) + S(r,f). \tag{1.1}$$

Hulburd and Korhonen [4], define difference operator by $\Delta_\eta f(z) = f(z+\eta) - f(z)$, where $\eta \in \mathbb{C} \setminus \{0\}$. We take product of the difference operators by $\prod_{i=1}^{q} \Delta_{\eta_i} f$, where $\eta_i \in \mathbb{C} \setminus \{0\}$, $i = 1, 2, \dots, q (q \in \mathbb{Z}^+)$.

DEFINITION 1.3. The q-th order difference operator $\Delta_\eta^q f(z)$ is defined by $\Delta_\eta^q f(z) = \Delta_\eta^{q-1}(\Delta_\eta f(z))$, where $q\ (\geqslant 2) \in \mathbb{N}$ and $\eta \in \mathbb{C} \setminus \{0\}$, while the difference polynomial of difference operator is given by $L(\Delta_\eta f) = \sum_{i=1}^{q} a_i \Delta_\eta^i f$, where a_i $(i = 1, 2, \dots, q)$ are nonzero constants.

We can also deduce that,

$$\Delta_\eta^q f = \sum_{i=0}^{q} \binom{q}{i} f(z + (q-i)\eta).$$

A. Edrei [5] and A. Weitsman [6] proved independently the following results:

Theorem A. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum. Then,*

i) $\displaystyle\lim_{r\to\infty} \frac{T(r,f')}{T(r,f)} = 2 - \delta(\infty, f)$;

ii) $\displaystyle\lim_{r\to\infty} \frac{N(r,0;f')}{T(r,f)} = 0.$

In 2000, Fang [7] obtained the following result:

Theorem B. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum with finite order. Then*

$$K(f') = \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)},$$

where

$$K(f') = \overline{\lim_{r\to\infty}} \frac{N(r,\infty;f') + N(r,0;f')}{T(r,f')}.$$

In 2013, Z. Wu [8] proved the following results:

Theorem C. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum with the order less than one. Then,*

i) $\displaystyle\lim_{r\to\infty} \frac{T(r,\Delta_\eta f)}{T(r,f)} = 2 - \delta(\infty, f)$;

ii) $\displaystyle\lim_{r\to\infty} \frac{N(r,0;\Delta_\eta f)}{T(r,\Delta_\eta f)} = 0.$

Theorem D. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum with the order less than one. Then,*

$$K(\Delta_\eta f) \leqslant \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)},$$

where

$$K(\Delta_\eta f) = \lim_{r\to\infty} \frac{N(r,\infty;\Delta_\eta f)+N(r,0;\Delta_\eta f)}{T(r,\Delta_\eta f)}.$$

Theorem E. *Let $f(z)$ be a transcendental meromorphic function with order is less than one and $\delta(\infty,f)=1$. Then,*

$$\sum_{\alpha\in\mathbb{C}\cup\{\infty\}} \delta(\alpha,f) \leqslant \delta(0,\Delta_\eta f).$$

In the paper, we consider the product of difference operators and difference polynomial of difference operators of transcendental meromorphic functions with maximal deficiency sum and established some results generalizing the results of Z. Wu [8].

2. Main Results

In this section, we present our main results:

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum of order less than one. Then*

i) $\displaystyle \lim_{r\to\infty} \frac{T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)} = q(2-\delta(\infty,f))$;

ii) $\displaystyle \lim_{r\to\infty} \frac{N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)} = 0.$

Corollary 2.1. *The deficiency of* 0 *with respect to* $\prod_{i=1}^{q}\Delta_{\eta_i}f$ *is*

$$\delta\Big(0,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big) = 1-\lim_{r\to\infty}\frac{N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)} = 1.$$

Remark 2.1. If we put $q=1$, then Theorem 2.1 coincide with Theorem C.

Theorem 2.2. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum of order less than one. Then,*

$$K\Big(\prod_{i=1}^{q}\Delta_{\eta_i}f\Big) \leqslant \frac{2(1-\delta(\infty,f))}{(2-\delta(\infty,f))},$$

where

$$K\Big(\prod_{i=1}^{q}\Delta_{\eta_i}f\Big) = \overline{\lim_{r\to\infty}} \frac{N\Big(r,\infty;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)+N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)}.$$

Corollary 2.2. *If in the above result if we take $\delta(\infty, f) = 1$, then we get*

$$\lim_{r\to\infty} \frac{N\Big(r, \infty; \prod\limits_{i=1}^{q} \Delta_{\eta_i} f\Big)}{T\Big(r, \prod\limits_{i=1}^{q} \Delta_{\eta_i} f\Big)} = 0.$$

Moreover, the deficiency of ∞ with respect to $\prod_{i=1}^{q} \Delta_{\eta_i} f$ is

$$\delta\Big(\infty, \prod_{i=1}^{q} \Delta_{\eta_i} f\Big) = 1 - \lim_{r\to\infty} \frac{N\Big(r, \infty; \prod\limits_{i=1}^{q} \Delta_{\eta_i} f\Big)}{T\Big(r, \prod\limits_{i=1}^{q} \Delta_{\eta_i} f\Big)} = 1.$$

Remark 2.2. If we put $q = 1$, then Theorem 2.2 coincide with Theorem D.

Example 2.1. Take $f(z) = \frac{1}{e^z - \tau}$, where τ is a complex constant. Observe that

$$\prod_{i=1}^{q} \Delta_{\eta_i} f = \prod_{i=1}^{q} \frac{(\tau - e^{\eta_i})e^z}{(e^{\eta_i}e^z - \tau)(e^z - \tau)}.$$

Then $\prod_{i=1}^{q} \Delta_{\eta_i} f \not\equiv 0$, $\delta(0, f) = 1$, $\delta(-\tau, f) = 1$, $\delta(\infty, f) = 0$. Thus $f(z)$ is a meromorphic function with maximal deficiency sum. Now

$$N\Big(r, \infty; \prod_{i=1}^{q} \Delta_{\eta_i} f\Big) = \sum_{i=1}^{q} N(r, \infty; \Delta_{\eta_i} f) = \sum_{i=1}^{q} \big(N(r, \tau; e^z) + N(r, \tau e^{-\eta_i}; e^z)\big)$$

and also we have $\delta(0, e^z) = 1 = \delta(\infty, e^z)$. As $r \longrightarrow \infty$, $N(r, \tau; e^z) = N(r, \tau e^{-\eta_i}; e^z) \sim T(r, e^z)$ and with help of Valiron–Mo'honko identity (1.1), $T(r, \prod_{i=1}^{q} \Delta_{\eta_i} f) = \sum_{i=1}^{q} 2T(r, e^z)$. Therefore,

$$K\Big(\prod_{i=1}^{q} \Delta_{\eta_i} f\Big) = \frac{2(1 - \delta(\infty, f))}{2 - \delta(\infty, f)} = 1.$$

On removing the condition of maximal deficiency sum of f we obtain the follows result:

Theorem 2.3. *Let $f(z)$ be a transcendental meromorphic function of order less than one and $\delta(\infty, f) = 1$. Then*

$$\sum_{\alpha\in\mathbb{C}} \delta(\alpha, f) \leqslant \delta\Big(0, \prod_{i=1}^{q} \Delta_{\eta_i} f\Big).$$

Theorem 2.4. *Let $f(z)$ be a transcendental meromorphic function with maximal deficiency sum of order less than one. Then*

i) $(2 - \delta(\infty, f)) \leqslant \lim\limits_{r\to\infty} \dfrac{T(r, L(\Delta_\eta f))}{T(r, f)} \leqslant \Big[\dfrac{1}{2}(q + 1)(q + 2) - 1\Big](1 - \delta(\infty, f)) + \delta(\infty, f)$;

ii) $\lim\limits_{r\to\infty} \dfrac{N(r, 0; L(\Delta_\eta f))}{T(r, L(\Delta_\eta f))} \leqslant 1 - \dfrac{2 - \delta(\infty, f)}{\delta(\infty; f) + \big[\frac{1}{2}(q + 1)(q + 2) - 1\big](1 - \delta(\infty; f))}$;

iii) $\overline{\lim}_{r\to\infty} \dfrac{N(r, \infty; L(\Delta_\eta f))}{T(r, L(\Delta_\eta f))} \leqslant \dfrac{\big[\frac{1}{2}(q + 1)(q + 2) - 1\big](1 - \delta(\infty, f))}{(2 - \delta(\infty, f))}$.

Corollary 2.3. *The deficiency of* 0 *with respect to* $L(\Delta_\eta f)$ *is*

$$\delta(0, L(\Delta_\eta f)) = 1 - \lim_{r\to\infty} \frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant \frac{2-\delta(\infty,f)}{\delta(\infty;f) + \left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty;f))}.$$

Corollary 2.4. *If in the above result we take* $\delta(\infty, f) = 1$, *then we get*

$$\lim_{r\to\infty} \frac{N(r,\infty;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} = 0.$$

Moreover, the deficiency of ∞ *with respect to* $L(\Delta_\eta f)$ *is*

$$\delta(\infty, L(\Delta_\eta f)) = 1 - \lim_{r\to\infty} \frac{N(r,\infty;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} = 1.$$

Remark 2.3. If equality occurs in (i) of Theorem 2.4 whenever $q = 1$.

Remark 2.4. If $q = 1$, then we can find $\delta(0, L(\Delta_\eta f)) = 1$ and $\delta(\infty, L(\Delta_\eta f)) = 0$.

Remark 2.5. If we put $q = 1$, then Theorem 2.4 will coincide with a combined result of Theorem C and Theorem D.

Example 2.2. Take $f(z) = \frac{1}{e^z - \tau}$, where τ is a complex constant. Then,

$$L(\Delta_\eta f) = \sum_{i=1}^{q} a_i \Delta_\eta^i f = \frac{p(e^z)}{\prod_{i=0}^{q}(e^{z+i\eta} - \tau)},$$

where $p(e^z)$ is polynomial of e^z. Hence, $L(\Delta_\eta f) \not\equiv 0$. Now, $\delta(0, f) = 1$, $\delta(-\tau, f) = 1$, $\delta(\infty, f) = 0$. Thus $f(z)$ is a meromorphic function with maximal deficiency sum. Now

$$N(r,\infty;L(\Delta_\eta f)) = \sum_{i=0}^{q} N(r, \tau e^{-i\eta}; e^z)$$

and also we have $\delta(0, e^z) = 1 = \delta(\infty, e^z)$. As $r \longrightarrow \infty$, $N(r, \tau e^{-i\eta}; e^z) \sim T(r, e^z)$ and with help of Valiron–Mo'honko identity (1.1), $T(r, L(\Delta_\eta f)) = \sum_{i=0}^{q} T(r, e^z)$. Therefore,

$$\overline{\lim_{r\to\infty}} \frac{N(r,\infty;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant 1.$$

Again if we remove the condition of maximal deficiency sum of f, then we arrive at the follows result:

Theorem 2.5. *Let* $f(z)$ *be a transcendental meromorphic function of order less than one and* $\delta(\infty, f) = 1$. *Then*

$$\sum_{\alpha\in\mathbb{C}} \delta(\alpha, f) \leqslant \delta(0, L(\Delta_\eta f)).$$

3. Lemmas

In this section,we state some lemmas which will be needed in the sequel.

Lemma 3.1 [9]**.** *Let* $f(z)$ *be a meromorphic function with order* σ *(finite) and* $\eta \in \mathbb{C} \setminus \{0\}$. *Then,*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(r^{\sigma-1+\varepsilon}\right),$$

where $\varepsilon \in \mathbb{R}^+ \setminus \{0\}$.

Lemma 3.2 [8]**.** *Let $f(z)$ be a meromorphic function with order σ (<1) and $\eta \in \mathbb{C}\setminus\{0\}$. Then,*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = o(T(r,f)) = S(r,f).$$

Lemma 3.3 [8]**.** *Let $f(z)$ be a meromorphic function with order σ (<1) and $\eta \in \mathbb{C}\setminus\{0\}$. Then,*

$$N(r,\infty; f(z+\eta)) = N(r,\infty;f) + S(r,f).$$

Lemma 3.4. *Let $f(z)$ be a meromorphic function with order σ (<1) and $\eta \in \mathbb{C}\setminus\{0\}$. Then,*

$$N(r,\infty; L(\Delta_\eta f)) \leqslant \left[\frac{1}{2}(q+1)(q+2) - 1\right] N(r,\infty;f) + S(r,f).$$

◁ We have

$$L(\Delta_\eta f) = \sum_{i=1}^{q} a_i \Delta_\eta^i f = a_1\Delta_\eta^1 f + a_2 \Delta_\eta^2 f + \cdots + a_q \Delta_\eta^q f.$$

Using Lemma 3.3, we deduce that,

$$N\big(r,\infty;\Delta_\eta^1 f\big) = N(r,\infty; f(z+\eta)) + N(r,\infty;f) \leqslant 2N(r,\infty;f) + S(r,f),$$

$$\begin{aligned} N\big(r,\infty;\Delta_\eta^2 f\big) &= N(r,\infty;f(z+2\eta)) + N(r,\infty;f(z+\eta)) + N(r,\infty;f) \\ &\leqslant 3N(r,\infty;f) + S(r,f). \end{aligned}$$

Proceeding similarly we find,

$$N\big(r,\infty;\Delta_\eta^q f\big) \leqslant (q+1)N(r,\infty;f) + S(r,f),$$

and thus,

$$\begin{aligned} N(r,\infty;L(\Delta_\eta f)) &\leqslant 2N(r,\infty;f) + 3N(r,\infty;f) + \cdots + (q+1)N(r,\infty;f) + S(r,f) \\ &\leqslant \left[\frac{1}{2}(q+1)(q+2) - 1\right] N(r,\infty;f) + S(r,f). \end{aligned}$$

Hence the lemma. ▷

4. Proofs of Theorems

Proof of Theorem 2.1. With help of Lemma 3.2 and Lemma 3.3, we deduce from Nevanlinna's first fundamental theorem,

$$\begin{aligned} T\left(r, \prod_{i=1}^{q} \Delta_{\eta_i} f\right) &= m\left(r, \prod_{i=1}^{q} \Delta_{\eta_i} f\right) + N\left(r,\infty; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) \\ &= m\left(r, \frac{f^q \prod\limits_{i=1}^{q} \Delta_{\eta_i} f}{f^q}\right) + N\left(r,\infty; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) \\ &\leqslant m(r,f^q) + m\left(r, \frac{\prod\limits_{i=1}^{q} \Delta_{\eta_i} f}{f^q}\right) + \sum_{i=1}^{q} N(r,\infty;\Delta_{\eta_i} f) + O(1) \end{aligned}$$

$$\leqslant m(r, f^q) + m\left(r, \frac{\prod_{i=1}^{q} \Delta_{\eta_i} f}{f^q}\right) + \sum_{i=1}^{q} N(r, \infty; f(z+\eta_i)) + \sum_{i=1}^{q} N(r, \infty; f) + O(1)$$

$$\leqslant qm(r, f) + m\left(r, \frac{\prod_{i=1}^{q} \Delta_{\eta_i} f}{f^q}\right) + qN(r, \infty; f) + qN(r, \infty; f) + S(r, f)$$

$$\leqslant qT(r, f) + qN(r, \infty; f) + S(r, f).$$

Hence,

$$\overline{\lim_{r\to\infty}} \frac{T\left(r, \prod_{i=1}^{q} \Delta_{\eta_i} f\right)}{T(r, f)} \leqslant q + q \overline{\lim_{r\to\infty}} \frac{N(r, \infty; f)}{T(r, f)} \leqslant q + q(1 - \delta(\infty, f)) = q(2 - \delta(\infty; f)). \quad (4.1)$$

Let $\{\alpha_j : j = 1, 2, \ldots, q\}$ be the sequence of finite deficient values of $f(z)$. We construct a function $\psi(z)$ on open complex plane as

$$\psi(z) = \prod_{i=1}^{q} \sum_{j=1}^{q} \frac{1}{f - \alpha_j},$$

where $q \in \mathbb{Z}^+$. Now, $T(r, f - \alpha_j) = T(r, f) + O(1)$ and $\Delta_{\eta_i}(f - \alpha_j) = \Delta_{\eta_i} f$, which implies that $\prod_{i=1}^{q} \Delta_{\eta_i}(f - \alpha_j) = \prod_{i=1}^{q} \Delta_{\eta_i} f$. Since order of $\psi(z)$ is less than 1, from Lemma 3.2, we have

$$m\left(r, \psi(z) \prod_{i=1}^{q} \Delta_{\eta_i} f(z)\right) \leqslant \sum_{i=1}^{q} \sum_{j=1}^{q} m\left(r, \frac{\Delta_{\eta_i} f(z)}{f - \alpha_j}\right) = S(r, f).$$

Now making use of the above relation we show that,

$$m(r, \psi(z)) = m\left(r, \psi(z)\left(\prod_{i=1}^{q} \Delta_{\eta_i} f\right) \frac{1}{\prod_{i=1}^{q} \Delta_{\eta_i} f}\right) \leqslant m\left(r, \frac{1}{\prod_{i=1}^{q} \Delta_{\eta_i} f}\right) + S(r, f). \quad (4.2)$$

Since $\psi(z)$ is polynomial of degree q^2, from Valiron–Mo'honko identity (1.1), Nevanlinna's first fundamental theorem and inequality (4.2), we have,

$$q^2 T(r, f) + N\left(r, 0; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) = T(r, \psi) + N\left(r, 0; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) + O(1)$$

$$= m(r, \psi(z)) + N(r, \infty; \psi(z)) + N\left(r, 0; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) + O(1)$$

$$\leqslant m(r, \psi(z)) + N\left(r, 0; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) + \sum_{i=1}^{q} \sum_{j=1}^{q} N(r, \alpha_j; f) + O(1)$$

$$\leqslant m\left(r, \frac{1}{\prod_{i=1}^{q} \Delta_{\eta_i} f}\right) + N\left(r, 0; \prod_{i=1}^{q} \Delta_{\eta_i} f\right) + \sum_{i=1}^{q} \sum_{j=1}^{q} N(r, \alpha_j; f) + S(r, f)$$

$$\leqslant T\left(r, \prod_{i=1}^{q} \Delta_{\eta_i} f\right) + \sum_{i=1}^{q} \sum_{j=1}^{q} N(r, \alpha_j; f) + S(r, f).$$

Hence,

$$q^2T(r,f)\leqslant T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)+\sum_{i=1}^{q}\sum_{j=1}^{q}N(r,\alpha_j;f)+S(r,f),$$

$$\begin{aligned}q^2&\leqslant \frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)}+\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}N(r,\alpha_j;f)}{T(r,f)}\leqslant \varliminf_{r\to\infty}\frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)}+\varlimsup_{r\to\infty}\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}N(r,\alpha_j;f)}{T(r,f)}\\&\leqslant \varliminf_{r\to\infty}\frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)}+\sum_{i=1}^{q}\sum_{j=1}^{q}(1-\delta(\alpha_j,f)).\end{aligned}$$

Thus,

$$\varliminf_{r\to\infty}\frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)}\geqslant\sum_{i=1}^{q}\sum_{j=1}^{q}\delta(\alpha_j,f).$$

Therefore we obtain,

$$\varliminf_{r\to\infty}\frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)}\geqslant\sum_{i=1}^{q}\sum_{j=1}^{q}\delta(\alpha_j,f)=\sum_{i=1}^{q}(2-\delta(\infty,f))=q(2-\delta(\infty,f)).\tag{4.3}$$

Now combining (4.1) and (4.3), we get

$$\lim_{r\to\infty}\frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)}=q(2-\delta(\infty,f)).\tag{4.4}$$

At the same time, from Nevanlinna's first fundamental theorem and inequality (4.2), we deduce

$$\begin{aligned}&\sum_{i=1}^{q}\sum_{j=1}^{q}m(r,\alpha_j)+N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)\leqslant m(r,\psi(z))+N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)\\&\leqslant m\left(r,\frac{1}{\prod\limits_{i=1}^{q}\Delta_{\eta_i}f}\right)+N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)+S(r,f)\leqslant T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)+S(r,f).\end{aligned}\tag{4.5}$$

Hence,

$$\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}m(r,\alpha_j)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\leqslant 1+\frac{S(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}.$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{q}\sum_{j=1}^{q}\varliminf_{r\to\infty}\frac{m(r,\alpha_j)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}+\varliminf_{r\to\infty}\frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\\&\leqslant 1+\varlimsup_{r\to\infty}\frac{S(r,f)}{T(r,f)}\,\frac{T(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}=1.\end{aligned}$$

Using (4.4) we deduce

$$1 \geqslant \overline{\lim_{r\to\infty}} \frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} + \sum_{i=1}^{q}\sum_{j=1}^{q} \underline{\lim_{r\to\infty}} \frac{m(r,\alpha_j)}{T(r,f)} \underline{\lim_{r\to\infty}} \frac{T(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}$$

$$\geqslant \overline{\lim_{r\to\infty}} \frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} + \frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}\delta(\alpha_j,f)}{q(2-\delta(\infty,f))}.$$

Therefore we obtain,

$$1 \geqslant \overline{\lim_{r\to\infty}} \frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} + 1.$$

Hence,

$$\overline{\lim_{r\to\infty}} \frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} = 0.$$

Therefore,

$$\lim_{r\to\infty} \frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} = 0.$$

Proof of Theorem 2.2. Making use of Lemma 3.3 we deduce that

$$N\Big(r,\infty;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big) = \sum_{i=1}^{q} N(r,\infty;\Delta_{\eta_i}f) \leqslant \sum_{i=1}^{q} 2N(r,\infty;f) + S(r,f) \leqslant 2qN(r,\infty;f) + S(r,f).$$

In view of the above inequality, we show

$$\frac{N\Big(r,\infty;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} \frac{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T(r,f)} \leqslant 2q\frac{N(r,\infty;f)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}.$$

Now, by Theorem 2.1(1), we have

$$q(2-\delta(\infty,f)) \overline{\lim_{r\to\infty}} \frac{N\Big(r,\infty;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} \leqslant 2q(1-\delta(\infty,f)).$$

Hence,

$$\overline{\lim_{r\to\infty}} \frac{N\Big(r,\infty;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} \leqslant \frac{2(1-\delta(\infty,f))}{(2-\delta(\infty,f))}.$$

Finally, making use of Theorem 2.1(2), we get

$$K\Big(\prod_{i=1}^{q}\Delta_{\eta_i}f\Big) \leqslant \frac{2(1-\delta(\infty,f))}{(2-\delta(\infty,f))}.$$

Proof of Theorem 2.3. To prove the result, consider two cases.

Case I. Under the assumption $\sum_{\alpha\in\mathbb{C}}\delta(\alpha,f)=0$ the theorem is obviously true.

Case II. We assume $\sum_{\alpha\in\mathbb{C}}\delta(\alpha,f)>0$. Let $\{\alpha_j : j=1,2,\ldots,q\}$ is sequence of deficient values of $f(z)$ where $\alpha_j\in\mathbb{C}$ and q is any arbitrary positive integer. Now from inequality (4.5), we have

$$\sum_{i=1}^{q}\sum_{j=1}^{q}m(r,\alpha_j)+N\Big(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f\Big) \leqslant T\Big(r,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)+S(r,f).$$

Then for $r\to\infty$, we have

$$\frac{T(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\left(\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}m(r,\alpha_j)}{T(r,f)}-o(1)\right)+\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}m(r,\alpha_j)+N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} \leqslant 1.$$

Now, taking into account the inequality (4.1), we deduce that

$$\overline{\lim_{r\to\infty}}\left[\frac{T(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\left(\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}m(r,\alpha_j)}{T(r,f)}-o(1)\right)+\frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\right] \leqslant 1$$

$$\Longrightarrow \underline{\lim_{r\to\infty}}\left[\frac{T(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\left(\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}m(r,\alpha_j)}{T(r,f)}-o(1)\right)\right]+\overline{\lim_{r\to\infty}}\frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} \leqslant 1$$

$$\Longrightarrow \underline{\lim_{r\to\infty}}\frac{T(r,f)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}\ \underline{\lim_{r\to\infty}}\frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}m(r,\alpha_j)}{T(r,f)}+\overline{\lim_{r\to\infty}}\frac{N\Big(r,0;\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)}{T\Big(r,\prod\limits_{i=1}^{q}\Delta_{\eta_i}f\Big)} \leqslant 1$$

$$\Longrightarrow \frac{\sum\limits_{i=1}^{q}\sum\limits_{j=1}^{q}\delta(\alpha_j,f)}{q(2-\delta(\infty,f))}+\left(1-\delta\Big(0,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big)\right) \leqslant 1.$$

According to our hypotheses that $\delta(\infty,f)=1$ and q is arbitrary, we have

$$\sum_{\alpha\in\mathbb{C}}\delta(\alpha,f) \leqslant \delta\Big(0,\prod_{i=1}^{q}\Delta_{\eta_i}f\Big).$$

Proof of Theorem 2.4. With the help of Lemma 3.2, Lemma 3.3, and Lemma 3.4, we deduce from Nevanlinna's first fundamental theorem,

$$\begin{aligned}
T(r, L(\Delta_\eta f)) = m(r, L(\Delta_\eta f)) + N(r,\infty; L(\Delta_\eta f)) &= m\left(r, \frac{fL(\Delta_\eta f)}{f}\right) + N(r,\infty; L(\Delta_\eta f)) \\
&\leqslant m(r,f) + m\left(r, \frac{L(\Delta_\eta f)}{f}\right) + N(r,\infty; L(\Delta_\eta f)) + O(1) \\
&\leqslant m(r,f) + m\left(r, \frac{L(\Delta_\eta f)}{f}\right) + N(r,\infty; L(\Delta_\eta f)) + O(1) \\
&\leqslant m(r,f) + m\left(r, \frac{L(\Delta_\eta f)}{f}\right) + \left[\frac{1}{2}(q+1)(q+2) - 1\right] N(r,\infty; f) + S(r,f) \\
&\leqslant m(r,f) + \left[\frac{1}{2}(q+1)(q+2) - 1\right] N(r,\infty; f) + S(r,f).
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{\lim_{r\to\infty}} \frac{T(r, L(\Delta_\eta f))}{T(r,f)} &\leqslant \overline{\lim_{r\to\infty}} \frac{m(r,f)}{T(r,f)} + \left[\frac{1}{2}(q+1)(q+2) - 1\right] \overline{\lim_{r\to\infty}} \frac{N(r,\infty;f)}{T(r,f)} \\
&= \delta(\infty; f) + \left[\frac{1}{2}(q+1)(q+2) - 1\right](1 - \delta(\infty; f)).
\end{aligned} \tag{4.6}$$

Let $\{\alpha_j : j = 1, 2, \ldots, q\}$ be the sequence of finite deficient values of $f(z)$. We construct a function $\psi(z)$ on open complex plane as

$$\psi(z) = \sum_{j=1}^{q} \frac{1}{f - \alpha_j},$$

where $q \in \mathbb{Z}^+$. Now, $T(r, f - \alpha_j) = T(r,f) + O(1)$ and $\Delta_\eta^j (f - \alpha_j) = \Delta_\eta^j f$, which implies that $\Delta_\eta^j (f - \alpha_j) = L(\Delta_\eta f)$. Since order of $\psi(z)$ is less than 1, it follows from Lemma 3.2 that

$$m\big(r, \psi(z) L(\Delta_\eta f(z))\big) \leqslant \sum_{j=1}^{q} m\left(r, \frac{\Delta_\eta^j f(z)}{f - \alpha_j}\right) = S(r,f).$$

Now with the help of above relation we show that,

$$m(r, \psi(z)) = m\left(r, \psi(z)(L(\Delta_\eta f)) \frac{1}{(L(\Delta_\eta f))}\right) \leqslant m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + S(r,f). \tag{4.7}$$

Since $\psi(z)$ is polynomial of degree q, from Valiron–Mo'honko identity (1.1), Nevanlinna's first fundamental theorem and inequality (4.7), we have

$$\begin{aligned}
qT(r,f) + N(r, 0; L(\Delta_\eta f)) &= T(r,\psi) + N(r, 0; L(\Delta_\eta f)) + O(1) \\
&= m(r, \psi(z)) + N(r,\infty; \psi(z)) + N(r, 0; L(\Delta_\eta f)) + O(1) \\
&\leqslant m(r, \psi(z)) + N(r, 0; L(\Delta_\eta f)) + \sum_{j=1}^{q} N(r, \alpha_j; f) + O(1) \\
&\leqslant m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + N(r, 0; L(\Delta_\eta f)) + \sum_{j=1}^{q} N(r, \alpha_j; f) + S(r,f) \\
&\leqslant T(r, L(\Delta_\eta f)) + \sum_{j=1}^{q} N(r, \alpha_j; f) + S(r,f).
\end{aligned}$$

Hence, $qT(r,f) \leqslant T(r, L(\Delta_\eta f)) + \sum_{j=1}^{q} N(r, \alpha_j; f) + S(r,f)$,

$$q \leqslant \frac{T(r, L(\Delta_\eta f))}{T(r,f)} + \frac{\sum\limits_{j=1}^{q} N(r, \alpha_j; f)}{T(r,f)} \leqslant \varliminf_{r\to\infty} \frac{T(r, L(\Delta_\eta f))}{T(r,f)} + \varlimsup_{r\to\infty} \frac{\sum\limits_{j=1}^{q} N(r, \alpha_j; f)}{T(r,f)}$$
$$\leqslant \varliminf_{r\to\infty} \frac{T(r, L(\Delta_\eta f))}{T(r,f)} + \sum_{j=1}^{q}(1 - \delta(\alpha_j, f)).$$

Thus,

$$\varliminf_{r\to\infty} \frac{T(r, L(\Delta_\eta f))}{T(r,f)} \geqslant \sum_{j=1}^{q} \delta(\alpha_j, f).$$

Therefore we obtain,

$$\varliminf_{r\to\infty} \frac{T(r, L(\Delta_\eta f))}{T(r,f)} \geqslant \sum_{j=1}^{q} \delta(\alpha_j, f) = (2 - \delta(\infty, f)) = (2 - \delta(\infty, f)). \tag{4.8}$$

Now combining (4.6) and (4.8), we have

$$(2 - \delta(\infty, f)) \leqslant \varliminf_{r\to\infty} \frac{T(r, L(\Delta_\eta f))}{T(r,f)} \leqslant \delta(\infty; f) + \left[\frac{1}{2}(q+1)(q+2) - 1\right](1 - \delta(\infty; f)). \tag{4.9}$$

At the same time, from Nevanlinna's first fundamental theorem and inequality (4.7), we have

$$\begin{aligned} &\sum_{j=1}^{q} m(r, \alpha_j) + N(r, 0; L(\Delta_\eta f)) \leqslant m(r, \psi(z)) + N(r, 0; L(\Delta_\eta f)) \\ &\leqslant m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + N(r, 0; L(\Delta_\eta f)) + S(r,f) \leqslant T(r, L(\Delta_\eta f)) + S(r,f). \end{aligned} \tag{4.10}$$

Hence,

$$\frac{\sum\limits_{j=1}^{q} m(r, \alpha_j)}{T(r, L(\Delta_\eta f))} + \frac{N(r, 0; L(\Delta_\eta f))}{T(r, L(\Delta_\eta f))} \leqslant 1 + \frac{S(r,f)}{T(r, L(\Delta_\eta f))}.$$

It follows that

$$\sum_{j=1}^{q} \varliminf_{r\to\infty} \frac{m(r, \alpha_j)}{T(r, L(\Delta_\eta f))} + \varlimsup_{r\to\infty} \frac{N(r, 0; L(\Delta_\eta f))}{T(r, L(\Delta_\eta f))} \leqslant 1 + \varlimsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} \frac{T(r,f)}{T(r, L(\Delta_\eta f))} = 1.$$

Taking into account (4.9) we estimate

$$1 \geqslant \varlimsup_{r\to\infty} \frac{N(r, 0; L(\Delta_\eta f))}{T(r, L(\Delta_\eta f))} + \sum_{j=1}^{q} \varliminf_{r\to\infty} \frac{m(r, \alpha_j)}{T(r,f)} \varliminf_{r\to\infty} \frac{T(r,f)}{T(r, L(\Delta_\eta f))}$$
$$\geqslant \varlimsup_{r\to\infty} \frac{N(r, 0; L(\Delta_\eta f))}{T(r, L(\Delta_\eta f))} + \frac{\sum\limits_{j=1}^{q} \delta(\alpha_j, f)}{\delta(\infty; f) + \left[\frac{1}{2}(q+1)(q+2) - 1\right](1 - \delta(\infty; f))}.$$

Therefore we obtain,

$$1 \geqslant \overline{\lim_{r\to\infty}} \frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} + \frac{2-\delta(\infty,f)}{\delta(\infty;f)+\left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty;f))}.$$

Hence,

$$\overline{\lim_{r\to\infty}} \frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant 1 - \frac{2-\delta(\infty,f)}{\delta(\infty;f)+\left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty;f))}.$$

Therefore,

$$\lim_{r\to\infty} \frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant 1 - \frac{2-\delta(\infty,f)}{\delta(\infty;f)+\left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty;f))}.$$

Again, making use of Lemma 3.3 and Lemma 3.4, we deduce

$$N(r,\infty;L(\Delta_\eta f)) \leqslant \left[\frac{1}{2}(q+1)(q+2)-1\right]N(r,\infty;f)+S(r,f).$$

Using the above inequality, we show

$$\frac{N(r,\infty;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))}\frac{T(r,L(\Delta_\eta f))}{T(r,f)} \leqslant \left[\frac{1}{2}(q+1)(q+2)-1\right]\frac{N(r,\infty;f)}{T(r,f)}+\frac{S(r,f)}{T(r,f)}.$$

Now using the result (i) of Theorem 2.4, we have

$$(2-\delta(\infty,f))\ \overline{\lim_{r\to\infty}} \frac{N(r,\infty;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant \left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty,f)).$$

Hence,

$$\overline{\lim_{r\to\infty}} \frac{N(r,\infty;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant \frac{\left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty,f))}{(2-\delta(\infty,f))}.$$

Proof of Theorem 2.5. To prove the result, consider two cases.

Case I. We assume $\sum_{\alpha\in\mathbb{C}}\delta(\alpha,f)=0$, then the theorem is trivially true.

Case II. We assume $\sum_{\alpha\in\mathbb{C}}\delta(\alpha,f)>0$. Let $\{\alpha_j : j=1,2,\ldots,q\}$ be sequence of the deficient values of $f(z)$ where $\alpha_j\in\mathbb{C}$ and q arbitrary positive integer. Then from (4.10) we have

$$\sum_{j=1}^{q} m(r,\alpha_j)+N(r,0;L(\Delta_\eta f)) \leqslant T(r,L(\Delta_\eta f))+S(r,f).$$

For sufficiently large r, we can estimate

$$\frac{T(r,f)}{T(r,L(\Delta_\eta f))}\left(\frac{\sum\limits_{j=1}^{q} m(r,\alpha_j)}{T(r,f)}-o(1)\right)+\frac{\sum\limits_{j=1}^{q} m(r,\alpha_j)+N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant 1.$$

Now with the help of inequality (4.6), we deduce that

$$\overline{\lim_{r\to\infty}}\left[\frac{T(r,f)}{T(r,L(\Delta_\eta f))}\left(\frac{\sum\limits_{j=1}^{q} m(r,\alpha_j)}{T(r,f)}-o(1)\right)+\frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))}\right] \leqslant 1$$

$$\implies \lim_{r\to\infty}\left[\frac{T(r,f)}{T(r,L(\Delta_\eta f))}\left(\frac{\sum\limits_{j=1}^{q} m(r,\alpha_j)}{T(r,f)}-o(1)\right)\right]+\overline{\lim_{r\to\infty}}\frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant 1$$

$$\Longrightarrow \varliminf_{r\to\infty} \frac{T(r,f)}{T(r,L(\Delta_\eta f))} \varliminf_{r\to\infty} \frac{\sum\limits_{j=1}^{q} m(r,\alpha_j)}{T(r,f)} + \varlimsup_{r\to\infty} \frac{N(r,0;L(\Delta_\eta f))}{T(r,L(\Delta_\eta f))} \leqslant 1$$

$$\Longrightarrow \frac{\sum\limits_{j=1}^{q} \delta(\alpha_j,f)}{\delta(\infty;f) + \left[\frac{1}{2}(q+1)(q+2)-1\right](1-\delta(\infty;f))} + \big(1-\delta(0,L(\Delta_\eta f))\big) \leqslant 1.$$

Now, according to our hypotheses that $\delta(\infty,f)=1$ and $q\in\mathbb{Z}^+$, we have

$$\sum_{\alpha\in\mathbb{C}} \delta(\alpha,f) \leqslant \delta(0,L(\Delta_\eta f)).$$

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О МЕРОМОРФНЫХ ФУНКЦИЯХ С МАКСИМАЛЬНОЙ СУММОЙ ДЕФЕКТОВ И СООТВЕТСТВУЮЩИЕ РАЗНОСТНЫЕ ОПЕРАТОРЫ

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Аннотация. В этой статье мы имеем дело в основном с характеристической функцией и дефектом мероморфной функции. В качестве эффективного инструмента исследования пользуемся понятием максимальной дефектной суммы (= максимальной суммы величин дефектов) мероморфной функции. Основное внимание уделяется связи между характеристической функцией произведения разностных операторов и характеристической функцией мероморфной функции с максимальной дефектной суммой. В этом же контексте рассматривается разностный полином от разностного оператора. Статья содержит также детальный анализ и обсуждение асимптотического поведения произведения разностных операторов, таких, например, как $\lim_{r\to\infty}\frac{T(r,\prod_{i=1}^{q}\Delta_{\eta_i}f)}{T(r,f)}$, $\lim_{r\to\infty}\frac{N(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f)}{T(r,\prod_{i=1}^{q}\Delta_{\eta_i}f)}$, $\overline{\lim}_{r\to\infty}\frac{N(r,\infty;\prod_{i=1}^{q}\Delta_{\eta_i}f)+N(r,0;\prod_{i=1}^{q}\Delta_{\eta_i}f)}{T(r,\prod_{i=1}^{q}\Delta_{\eta_i}f)}$ и др.; аналогичным образом рассмотрен также разностный полином разностных операторов. Представлены также некоторые неравенства для нулей и полюсов для $\prod_{i=1}^{q}\Delta_{\eta_i}f$ и $L(\Delta_\eta f)$. По ходу изложения представлены несколько замечаний и следствий, а также даны два примера для надлежащего обоснования наших результатов. Эти результаты усиливают или обобщают результаты З. Ву.

Ключевые слова: трансцендентная мероморфная функция, сумма дефектов, разностный оператор, произведение разностных операторов.

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