Vladikavkaz Mathematical Journal
2020, Volume 22, Issue 3, P. 18–29

УДК 517.518
DOI 10.46698/h8083-6917-3687-w

A BERNSTEIN–NIKOL'SKII INEQUALITY FOR WEIGHTED LEBESGUE SPACES#

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Dedicated to the first author's Teacher
Professor Yurii Fedorovich Korobeinik
on the occasion of his 90th birthday

Abstract. In this paper, we give some results concerning Bernstein–Nikol'skii inequality for weighted Lebesgue spaces. The main result is as follows: Let $1 < u, p < \infty$, $0 < q + 1/p < v + 1/u < 1$, $v - q \geqslant 0$, $\kappa > 0$, $f \in L^u_v(\mathbb{R})$ and $\operatorname{supp} \widehat{f} \subset [-\kappa, \kappa]$. Then $D^m f \in L^p_q(\mathbb{R})$, $\operatorname{supp} \widehat{D^m f} = \operatorname{supp} \widehat{f}$ and there exists a constant C independent of f, m, κ such that $\|D^m f\|_{L^p_q} \leqslant C m^{-\varrho} \kappa^{m+\varrho} \|f\|_{L^u_v}$, for all $m = 1, 2, \dots$, where $\varrho = v + \frac{1}{u} - \frac{1}{p} - q > 0$, and the weighted Lebesgue space L^p_q consists of all measurable functions such that $\|f\|_{L^p_q} = \big(\int_{\mathbb{R}} |f(x)|^p |x|^{pq} dx\big)^{1/p} < \infty$. Moreover, $\lim_{m\to\infty} \|D^m f\|^{1/m}_{L^p_q} = \sup\big\{|x| : x \in \operatorname{supp}\widehat{f}\big\}$. The advantage of our result is that $m^{-\varrho}$ appears on the right hand side of the inequality ($\varrho > 0$), which has never appeared in related articles by other authors. The corresponding result for the n-dimensional case is also obtained.

Key words: weighted Lebesgue spaces, Bernstein inequality, Nikol'skii inequality.

Mathematical Subject Classification (2010): 26D10, 46E30.

For citation: *Bang, H. H. and Huy, V. N.* A Bernstein–Nikol'skii Inequality for Weighted Lebesgue Spaces, *Vladikavkaz Math. J.*, 2020, vol. 22, no. 3, pp. 18–29. DOI: 10.46698/h8083-6917-3687-w.

1. Introduction

In 1912, S. N. Bernstein proved in [1] the following inequality: Let f be any trigonometric polynomial f of degree κ. Then

$$\|D^m f\|_\infty \leqslant \kappa^m \|f\|_\infty \quad (\forall\, m = 1, 2, \dots),$$

which provides the behavior of the norm of derivatives of f with respect to differential order and its spectrum. The constants κ^m are best possible. This inequality is also true for L^p-norm, $1 \leqslant p \leqslant \infty$ (see [2]), and for entire functions of exponential type $\kappa > 0$ with respect to $L^p(\mathbb{R})$-norm, $1 \leqslant p \leqslant \infty$ (see [3]).

This work was supported by Vietnamese Academy of Science and Technology, grant number NVCC01.05/19-19.

In 1951, S. M. Nilkol'skii gave the following inequality

$$\|f\|_p \leqslant C_{p,q}\kappa^{1/q-1/p}\|f\|_q, \quad 1 \leqslant q \leqslant p \leqslant \infty,$$

for any entire function f of exponential type κ belonging to $L^q(\mathbb{R})$ [4]. Bernstein inequality was studied also in [5–11]and Nikol'skii inequality was studied in [3, 4, 12, 13]. Note that the inequalities of Bernstein and Nikol'skii play an important role in the Approximation Theory [2, 3, 15, 16]. Combining the above inequalities, we have the following Bernstein-Nikol'skii inequality

$$\|D^m f\|_p \leqslant C_{p,q}\kappa^{m+1/q-1/p}\|f\|_q \tag{1}$$

for $1 \leqslant q \leqslant p \leqslant \infty$, $\operatorname{supp}\widehat{f} \subset [-\kappa, \kappa]$ and $f \in L^q(\mathbb{R})$.

The main purpose of this paper is to derive a new Bernstein-Nikol'skii inequality for weighted Lebesgue spaces, which is a generalization of the corresponding result in [17]. Note that the obtained inequality in [17] is better than (1). We also extend the result in [18] to weighted spaces.

2. Main Results

Given a function $f : \mathbb{R} \to \mathbb{C}$ in $L^1(\mathbb{R})$, its Fourier transform is defined by

$$\widehat{f}(x) = (2\pi)^{-1/2}\int\limits_{\mathbb{R}} e^{-ixz} f(z)\, dz.$$

The Fourier transform of a tempered generalized function f can be defined via the formula

$$\langle \mathscr{F}f, \varphi\rangle = \langle f, \mathscr{F}\varphi\rangle, \quad \varphi \in \mathscr{S}(\mathbb{R}),$$

where $\mathscr{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions.

Let $1 \leqslant p < \infty$, $q \in \mathbb{R}$. The weighted Lebesgue space $L^p_q := L^p_q(\mathbb{R})$ consists of all measurable functions such that

$$\|f\|_{L^p_q} = \left(\int\limits_{\mathbb{R}} |f(x)|^p |x|^{pq}\, dx\right)^{1/p} < \infty.$$

Then $L^p_q(\mathbb{R})$ is a Banach space.

The following Bernstein–Nikol'skii inequality for weighted Lebesgue spaces is our main result:

Theorem 2.1. *Let $1 < u, p < \infty$, $0 < q + (1/p) < v + (1/u) < 1$, $v - q \geqslant 0$, $\kappa > 0$, and $f \in L^u_v(\mathbb{R})$ and $\operatorname{supp}\widehat{f} \subset [-\kappa, \kappa]$. Then $D^m f \in L^p_q(\mathbb{R})$, $\operatorname{supp}\widehat{D^m f} = \operatorname{supp}\widehat{f}$ and there exists a constant C independent of f, m, κ such that*

$$\|D^m f\|_{L^p_q} \leqslant C m^{-\varrho}\kappa^{m+\varrho}\|f\|_{L^u_v} \tag{2}$$

for all $m = 1, 2, \dots$, where $\varrho = v + \frac{1}{u} - \frac{1}{p} - q > 0$. Moreover,

$$\lim_{m\to\infty} \|D^m f\|^{1/m}_{L^p_q} = \sup\{|x| : x \in \operatorname{supp}\widehat{f}\}. \tag{3}$$

Note that equality (3) was proved in [18] for the case $q = 0$ and the Bernstein–Nikol'skii inequality for usual Lebesgue spaces was studied in [7, 8, 15–17] by other techniques.

To prove Theorem 1, we need the following lemmas.

Lemma 2.2 (Young's Inequality for the weighted Lebesgue spaces [19]). *Let* $1 < u, p, r < \infty$, $1/p \leqslant 1/u + 1/r$, $1/p = 1/u + 1/r + v + q + \gamma - 1$, $v < 1 - 1/u$, $q < 1/p$, $\gamma < 1 - 1/r$, $\gamma + q \geqslant 0$, $\gamma + v \geqslant 0$, $q + v \geqslant 0$ *and* $f \in L^u_v(\mathbb{R})$, $g \in L^r_\gamma(\mathbb{R})$. *Then* $f * g \in L^p_{-q}(\mathbb{R})$ *and there exists a constant* C *independent of* f, g *such that*

$$\|f * g\|_{L^p_{-q}} \leqslant C\|f\|_{L^u_v}\|g\|_{L^r_\gamma},$$

where

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy.$$

Lemma 2.3 [20]. *If the support of a generalized function* $f \in \mathscr{S}'(\mathbb{R})$ *consists of a single point* $x = 0$, *then it is uniquely representable in the form*

$$f(x) = \sum_{j=0}^{N} c_j D^j \delta(x)$$

where N *is the order of* f, *and* c_j *are certain constants.*

Clearly, we have the following

Lemma 2.4. *Let* $1 < p < \infty, q \in \mathbb{R}, \kappa > 0$ *and* $f \in L^p_q(\mathbb{R})$. *Then* ${}_\kappa f \in L^p_q(\mathbb{R})$ *and*

$$\|{}_\kappa f\|_{L^p_q} = \kappa^{q+(1/p)}\|f\|_{L^p_q}\ ,$$

where ${}_\kappa f(x) = f(x/\kappa)$.

Lemma 2.5. *Let* $1 < u, p < \infty$, $0 < q + (1/p) < v + (1/u) < 1$, $v - q \geqslant 0$, $f \in L^u_v(\mathbb{R})$ *and* $\operatorname{supp}\widehat{f} \subset [-1, 1]$. *Then there exists a constant* C *independent of* f, m *such that*

$$\|D^m f\|_{L^p_q} \leqslant Cm^{-\varrho}\|f\|_{L^u_v} \tag{4}$$

for all $m = 1, 2, \ldots$, *where*

$$\varrho = v + \frac{1}{u} - \frac{1}{p} - q > 0.$$

$\lhd$ We denote $\Omega := [-1, 1]$ and $\Omega_\epsilon := [-(1+\epsilon), 1+\epsilon]$ for each $\epsilon > 0$. The function $\mathscr{G}(z)$ is defined as follows

$$\mathscr{G}(z) = \begin{cases} C_1 e^{1/(z^2-1)}, & |z| < 1; \\ 0, & |z| \geqslant 1, \end{cases}$$

where C_1 is chosen such that $\int_{\mathbb{R}} \mathscr{G}(z)\,dz = 1$. We define the sequence of functions $(\phi_m(z))_{m\geqslant 1}$ via the formula

$$\phi_m(z) = (1_{\Omega_{3/(4m)}} * \mathscr{H}_m)(z),$$

where

$$\mathscr{H}_m(z) = 4m\mathscr{G}(4mz).$$

Then $\mathscr{H}_m(z) = 0$ for all $z \notin [-1/(4m), 1/(4m)]$, $\int_{\mathbb{R}} \mathscr{H}_m(z)\,dz = 1$. Hence, for any $m \geqslant 1$ we have $\phi_m(z) \in C_0^\infty(\mathbb{R})$, and $\phi_m(z) = 1$ for all $z \in \Omega_{1/(2m)}$, $\phi_m(z) = 0$ for all $z \notin \Omega_{1/m}$. So, $\widehat{f} = \phi_m(-z)\widehat{f}$ follows from $\operatorname{supp}\widehat{f} \subset \Omega$. Therefore, since

$$\widehat{D^m f} = (iz)^m \widehat{f},$$

$$\widehat{D^m f} = \phi_m(-z)(iz)^m \widehat{f}.$$

Hence,

$$D^m f = (2\pi)^{-1/2} f * \mathscr{F}^{-1}(\phi_m(-z)(iz)^m) = (2\pi)^{-1/2} f * \mathscr{F}(\phi_m(z)(-iz)^m). \tag{5}$$

We consider two numbers r, γ satisfying $1 < r < \infty$, $q + \frac{1}{p} - v - \frac{1}{u} = \frac{1}{r} + \gamma - 1$, $\gamma + v \geqslant 0$, $\gamma - q \geqslant 0$, $v - q + \gamma \leqslant 1$. From the hypothesis, we have $\frac{1}{p} \leqslant \frac{1}{u} + \frac{1}{r}$, $\gamma < 1 - \frac{1}{r}$, $v < 1 - 1/u$ and $-q < 1/p$. Therefore, due to (5) and Lemma 2.2, there exists a constant C_2 independent of f, m such that

$$\|D^m f\|_{L^p_q} = (2\pi)^{-1/2}\|f * \mathscr{F}(\phi_m(z)z^m)\|_{L^p_q} \leqslant C_2\|f\|_{L^u_v}\|\mathscr{F}(\phi_m(z)z^m)\|_{L^r_\gamma}. \tag{6}$$

Define

$$\eta_m := 1 + \frac{1}{m}, \quad \varphi_m(z) = \phi_m\left(\frac{z}{\eta_m}\right), \quad \Phi_m(z) = \phi_m(z) - \varphi_m(z).$$

Then

$$\big(\mathscr{F}(\varphi_m(z)z^m)\big)(x) = (\eta_m)^m\left(\mathscr{F}\left(\phi_m\left(\frac{z}{\eta_m}\right)\left(\frac{z}{\eta_m}\right)^m\right)\right)(x) = (\eta_m)^{m+1}\big(\mathscr{F}\big(\phi_m(z)z^m\big)\big)(\eta_m x).$$

So, by Lemma 2.4, one gets

$$\Big\|\mathscr{F}(\varphi_m(z)z^m)\Big\|_{L^r_\gamma} = (\eta_m)^{m+1-\gamma-\frac{1}{r}}\Big\|\mathscr{F}(\phi_m(z)z^m)\Big\|_{L^r_\gamma}.$$

Then it follows from $(\eta_m)^{m+1-\gamma-\frac{1}{r}} \geqslant (\eta_m)^m = \left(1 + \frac{1}{m}\right)^m \geqslant 2$ that

$$\Big\|\mathscr{F}\big(\varphi_m(z)z^m\big)\Big\|_{L^r_\gamma} \geqslant 2\Big\|\mathscr{F}\big(\phi_m(z)z^m\big)\Big\|_{L^r_\gamma}.$$

Therefore, since $\Phi_m(z) = \phi_m(z) - \varphi_m(z)$,

$$\Big\|\mathscr{F}\big(\Phi_m(z)z^m\big)\Big\|_{L^r_\gamma} \geqslant \Big\|\mathscr{F}\big(\varphi_m(z)z^m\big)\Big\|_{L^r_\gamma} - \Big\|\mathscr{F}\big(\phi_m(z)z^m\big)\Big\|_{L^r_\gamma} \geqslant \Big\|\mathscr{F}\big(\phi_m(z)z^m\big)\Big\|_{L^r_\gamma}. \tag{7}$$

From (6)–(7) we obtain

$$\big\|D^m f\big\|_{L^p_q} \leqslant C_2\|f\|_{L^u_v}\big\|\mathscr{F}(\Phi_m(z)z^m)\big\|_{L^r_\gamma}. \tag{8}$$

Next, we estimate $\|\mathscr{F}(\Phi_m(z)z^m)\|_{L^r_\gamma}$. To do that, we put $C_3 = \max\big\{\big\|\mathscr{G}^{(j)}\big\|_{L^1} : j \leqslant 3\big\}$. Since $\mathscr{H}_m(x) = 4m\mathscr{G}(4mx)$, $\mathscr{H}_m^{(j)}(x) = (4m)^{j+1}\mathscr{G}^{(j)}(4mx)$ and then we obtain

$$\big\|\mathscr{H}_m^{(j)}\big\|_{L^1} = (4m)^j\big\|\mathscr{G}^{(j)}\big\|_{L^1} \leqslant C_3(4m)^j \quad (\forall\, j \leqslant 3).$$

Therefore,

$$\big\|\phi_m^{(j)}\big\|_{L^\infty} = \big\|(1_{\Omega_{3/(4m)}}\mathscr{H}_m^{(j)})\big\|_{L^\infty} \leqslant \big\|\mathscr{H}_m^{(j)}\big\|_{L^1} \leqslant (4m)^j C_3 \quad (\forall\, j \leqslant 3). \tag{9}$$

Note that $\phi_m(z) = 1$ for all $z \in (-1 - (1/2m), 1 + (1/2m))$, and $\phi_m(z) = 0$ for all $z \in (-\infty, -1 - (1/m)) \cup (1 + (1/m), +\infty)$. So, if $|z| < 1$ then $|z/\eta_m| < |z| < 1$ and $\phi_m(z) = \phi_m(z/\eta_m) = 1$, i. e., $\Phi_m(z) = 0$.

Further, if $|z| > 1+(3/m)$ then $|z| > |z/\eta_m| > 1+(1/m)$ and then $\phi_m(z) = \phi_m(z/\eta_m) = 0$, i. e., $\Phi_m(z) = 0$.

So, we have

$$\operatorname{supp}\Phi_m \subset [1, 1 + (3/m)] \cup [-1 - (3/m), -1]. \tag{10}$$

Now, if $z \in [1, 1 + (3/m)] \cup [-1 - (3/m), -1]$ then

$$\Big|z - \frac{z}{\eta_m}\Big| = \Big|\frac{(\eta_m - 1)z}{\eta_m}\Big| = \Big|\frac{z}{m\eta_m}\Big| \leqslant \frac{4}{m}. \tag{11}$$

From (9) and (11) we get the following estimates for $z \in [1, 1 + (3/m)] \cup [-1 - (3/m), -1]$

$$\begin{aligned}\Big|\Phi_m(z)\Big| = \Big|\phi_m(z) - \varphi_m(z)\Big| &= \Big|\phi_m(z) - \phi\Big(\frac{z}{\eta_m}\Big)\Big| \\ &\leqslant \Big|z - \frac{z}{\eta_m}\Big|\Big\|\phi'_m\Big\|_{L^\infty} \leqslant \frac{4}{m}4mC_3 = 16C_3 \end{aligned}\tag{12}$$

and

$$\begin{aligned}\Big|\Phi'_m(z)\Big| &= \Big|\phi'_m(z) - \varphi'_m(z)\Big| = \Big|\phi'_m(z) - \Big(\phi_m\Big(\frac{z}{\eta_m}\Big)\Big)'\Big| \\ &= \Big|\phi'_m(z) - \frac{1}{\eta_m}\phi'_m\Big(\frac{z}{\eta_m}\Big)\Big| \leqslant \Big|\phi'_m(z) - \phi'_m\Big(\frac{z}{\eta_m}\Big)\Big| + \Big|\Big(1 - \frac{1}{\eta_m}\Big)\phi'_m\Big(\frac{z}{\eta_m}\Big)\Big| \\ &\leqslant \Big|z - \frac{z}{\eta_m}\Big|\Big\|\phi''_m\Big\|_{L^\infty} + \Big|1 - \frac{1}{\eta_m}\Big|\Big\|\phi'_m\Big\|_{L^\infty} \leqslant \frac{4}{m}(4m)^2C_3 + \Big|1 - \frac{1}{\eta_m}\Big|4mC_3 \leqslant 68mC_3.\end{aligned}\tag{13}$$

Put $\Upsilon(x) = (\mathscr{F}(\Phi_m(z)z^m))(x)$. Then

$$\Upsilon(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} e^{-ixz}\Phi_m(z)z^m\,dz.$$

Therefore, using (10), we obtain

$$\sup_{x\in\mathbb{R}}\Big|\Upsilon(x)\Big| \leqslant \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}\Big|\Phi_m(z)z^m\Big|\,dz = \frac{1}{\sqrt{2\pi}}\int\limits_{1\leqslant|z|\leqslant 1+\frac{3}{m}}\Big|\Phi_m(z)z^m\Big|\,dz$$

and it follows from (9) that

$$\sup_{x\in\mathbb{R}}\big|\Upsilon(x)\big| \leqslant \frac{6}{m\sqrt{2\pi}}\sup_{z\in\mathbb{R}}\big|\Phi_m(z)\big|\Big(1 + \frac{3}{m}\Big)^m \leqslant \frac{96e^3C_3}{m\sqrt{2\pi}}. \tag{14}$$

We also see that

$$\begin{aligned}\sup_{x\in\mathbb{R}}\big|x\Upsilon(x)\big| &= \frac{1}{\sqrt{2\pi}}\Big|\int\limits_{\mathbb{R}} e^{-ixz}\big(\Phi_m(z)mz^{m-1} + \Phi'_m(z)z^m\big)\,dz\Big| \\ &\leqslant \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}\Big|\Phi_m(z)mz^{m-1} + \Phi'_m(z)z^m\Big|\,dz.\end{aligned}$$

Therefore, using (9)–(10), we have

$$\sup_{x\in\mathbb{R}}\Big|x\Upsilon(x)\Big| \leqslant \frac{1}{\sqrt{2\pi}}\int\limits_{1\leqslant|z|\leqslant 1+\frac{3}{m}}\Big|\Phi_m(z)mz^{m-1} + \Phi'_m(z)z^m\Big|dz \tag{15}$$

$$\leqslant \frac{6}{m\sqrt{2\pi}} \sup_{1\leqslant |z|\leqslant 1+\frac{3}{m}} \Big|\Phi_m(z)mz^{m-1} + \Phi'_m(z)z^m\Big|$$
$$\leqslant \frac{6}{m\sqrt{2\pi}} \left[\sup_{z\in\mathbb{R}}\Big|\Phi_m(z)\Big|m\Big(1+\frac{3}{m}\Big)^{m-1} + \sup_{z\in\mathbb{R}}\Big|\Phi'_m(z)\Big|\Big(1+\frac{3}{m}\Big)^{m}\right]$$
$$\leqslant \frac{6}{m\sqrt{2\pi}}\Big[16C_3me^3 + 68C_3me^3\Big] = \frac{504e^3C_3}{\sqrt{2\pi}}.$$

From $0<\gamma+1/r<1$ we have $r-r\gamma>1$ and $\gamma r>-1$. Hence, we get the following estimate

$$\|\Upsilon\|^r_{L^r_\gamma} = \int\limits_{|x|\leqslant m} \big|x^\gamma\Upsilon(x)\big|^r\,dx + \int\limits_{|x|\geqslant m} \big|x^\gamma\Upsilon(x)\big|^r\,dx \tag{16}$$
$$\leqslant \sup_{x\in\mathbb{R}}\big|\Upsilon(x)\big|^r \int\limits_{|x|\leqslant m} |x|^{\gamma r}\,dx + \sup_{x\in\mathbb{R}}\big|x\Upsilon(x)\big|^r \int\limits_{|x|\geqslant m} \frac{1}{|x|^{r-\gamma r}}\,dx$$
$$= \frac{2m^{\gamma r+1}}{\gamma r+1}\sup_{x\in\mathbb{R}}\big|\Upsilon(x)\big|^r + \frac{2m^{\gamma r+1-r}}{r-\gamma r-1}\sup_{x\in\mathbb{R}}\big|x\Upsilon(x)\big|^r.$$

From (14)–(16), we obtain

$$\Big\|\Upsilon\Big\|^r_{L^r_\gamma} \leqslant \frac{2m^{\gamma r+1}}{\gamma r+1}\left(\frac{96e^3C_3}{m\sqrt{2\pi}}\right)^r + \frac{2m^{\gamma r+1-r}}{r-\gamma r-1}\left(\frac{504e^3C_3}{\sqrt{2\pi}}\right)^r$$
$$= 2m^{\gamma r+1-r}\left(\frac{e^3C_3}{\sqrt{2\pi}}\right)^r\left(\frac{504^r}{r-\gamma r-1} + \frac{96^r}{\gamma r+1}\right),$$

and then

$$\Big\|\Upsilon\Big\|_{L^r_\gamma} \leqslant \frac{e^3C_3}{\sqrt{2\pi}}\left(\frac{504^r2}{r-\gamma r-1} + \frac{96^r2}{\gamma r+1}\right)^{\frac{1}{r}} m^{-1+\gamma+\frac{1}{r}} = C_3m^{-\varrho}, \tag{17}$$

where $C_3 = e^3C_3(\frac{504^r2}{r-\gamma r-1}+\frac{96^r2}{\gamma r+1})^{\frac{1}{r}}/\sqrt{2\pi}$.
From (8), (17), we can choose a constant C such that

$$\|D^mf\|_{L^p_q} \leqslant Cm^{-\varrho}\|f\|_{L^u_v}.$$

The proof is complete. $\rhd$

Lemma 2.6. *Let $1<p<\infty$, $0<q+1/p<1$, and $f\in L^p_q(\mathbb{R})$. Then*

$$\liminf_{m\to\infty}\|D^mf\|^{1/m}_{L^p_q} \geqslant \sup\big\{|x|:\, x\in\operatorname{supp}\widehat{f}\,\big\}. \tag{18}$$

$\lhd$ Denote $\sigma := \sup\big\{|x|:\, x\in\operatorname{supp}\widehat{f}\big\}$. If $\sigma=0$ then (18) is obvious. Now, we assume that $\sigma>0$. Without loss of generality we may assume that $\sigma\in\operatorname{supp}\widehat{f}$. For each $\epsilon\in(0,\sigma)$, there exists a function $\varphi\in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\varphi\subset[\sigma-\epsilon,\sigma+\epsilon]$ such that $\big\langle\widehat{f},\varphi\big\rangle\neq 0$. Put

$$\mathscr{Q}_m = \mathscr{F}(\varphi(x)/x^m).$$

Hence, $D^m\mathscr{Q}_m = (-i)^m\mathscr{F}\varphi$ and then

$$0 < \big|\big\langle\widehat{f},\varphi\big\rangle\big| = \big|\big\langle f,\mathscr{F}\varphi\big\rangle\big| = \big|\big\langle f,D^m\mathscr{Q}_m\big\rangle\big| = \big|\big\langle D^mf,\mathscr{Q}_m\big\rangle\big|$$
$$= \left|\int\limits_{\mathbb{R}} D^mf(x)\mathscr{Q}_m(x)\,dx\right| \leqslant \int\limits_{\mathbb{R}} |x^qD^mf(x)||x^{-q}\mathscr{Q}_m(x)|\,dx.$$

Using Hölder inequality, we have

$$0<\big|\langle\widehat{f},\varphi\rangle\big|\leqslant\Big(\int\limits_{\mathbb{R}}|x^qD^mf(x)|^p\,dx\Big)^{1/p}\Big(\int\limits_{\mathbb{R}}|x^{-q}\mathscr{Q}_m(x)|^{\overline{p}}\,dx\Big)^{1/\overline{p}}=\|D^mf\|_{L^p_q}\|\mathscr{Q}_m\|_{L^{\overline{p}}_{-q}},$$

where

$$\frac{1}{p}+\frac{1}{\overline{p}}=1.$$

So,

$$\liminf_{m\to\infty}\|D^mf\|^{1/m}_{L^p_q}\geqslant 1/\limsup_{m\to\infty}\|\mathscr{Q}_m\|^{1/m}_{L^{\overline{p}}_{-q}}. \tag{19}$$

We consider an integer N satisfying $(N+q)\overline{p}>1$. From $0<q+\frac{1}{p}<1$, we deduce that $q\overline{p}<1$, which together with $(N+q)\overline{p}>1$ and

$$\int\limits_{\mathbb{R}}|x^{-q}\mathscr{Q}_m(x)|^{\overline{p}}dx\leqslant\int\limits_{|x|\leqslant 1}|x^{-q}\mathscr{Q}_m(x)|^{\overline{p}}dx+\int\limits_{|x|\geqslant 1}|x^{-q}\mathscr{Q}_m(x)|^{\overline{p}}dx$$
$$\leqslant\sup_{x\in\mathbb{R}}|\mathscr{Q}_m(x)|^{\overline{p}}\int\limits_{|x|\leqslant 1}|x|^{-q\overline{p}}dx+\sup_{x\in\mathbb{R}}\big|x^N\mathscr{Q}_m(x)\big|^{\overline{p}}\int\limits_{|x|\geqslant 1}\big|x^{-q-N}\big|^{\overline{p}}dx$$

imply

$$\int\limits_{\mathbb{R}}|x^{-q}\mathscr{Q}_m(x)|^{\overline{p}}dx\leqslant\frac{2}{1-q\overline{p}}\sup_{x\in\mathbb{R}}|\mathscr{Q}_m(x)|^{\overline{p}}+\frac{2}{(N+q)\overline{p}-1}\sup_{x\in\mathbb{R}}|x^N\mathscr{Q}_m(x)|^{\overline{p}}. \tag{20}$$

Note that

$$\sup_{x\in\mathbb{R}}\Big|\Big(1+|x|^N\Big)\mathscr{Q}_m(x)\Big|\leqslant\int\limits_{[\sigma-\epsilon,\sigma+\epsilon]}\Big(\Big|\varphi(x)/x^m\Big|+\Big|D^N(\varphi(x)/x^m)\Big|\Big)\,dx\leqslant cm^N(\sigma-\epsilon)^{-m},$$

where c is independent of m. Then, by (20), we obtain

$$\limsup_{m\to\infty}\big\|\mathscr{Q}_m\big\|^{1/m}_{L^{\overline{p}}_{-q}}\leqslant 1/(\sigma-\epsilon).$$

So, it follows from (19) that

$$\liminf_{m\to\infty}\big\|D^mf\big\|^{1/m}_{L^p_q}\geqslant\sigma-\epsilon.$$

Letting $\epsilon\to 0$, we confirm (18). The proof is complete. $\rhd$

Lemma 2.7. *Let $1<p<\infty$ and $0<q+1/p$. Then $\mathscr{S}(\mathbb{R})\subset L^p_q(\mathbb{R})$.*

$\lhd$ Let $\varphi\in\mathscr{S}(\mathbb{R})$ and an integer N be satisfying $(N-q)p>1$. From $0<q+\frac{1}{p}$, we deduce $qp>-1$, which together with $(N-q)p>1$ and

$$\int\limits_{\mathbb{R}}|x^q\varphi(x)|^p\,dx\leqslant\int\limits_{|x|\leqslant 1}|x^q\varphi(x)|^p\,dx+\int\limits_{|x|\geqslant 1}|x^q\varphi(x)|^p\,dx$$
$$\leqslant\sup_{x\in\mathbb{R}}|\varphi(x)|^p\int\limits_{|x|\leqslant 1}|x|^{qp}\,dx+\sup_{x\in\mathbb{R}}|x^N\varphi(x)|^p\int\limits_{|x|\geqslant 1}|x^{q-N}|^p\,dx$$

imply that

$$\int_{\mathbb{R}} |x^q\varphi(x)|^p\,dx \leqslant \frac{2}{qp+1}\sup_{x\in\mathbb{R}}|\varphi(x)|^p + \frac{2}{(N-q)p-1}\sup_{x\in\mathbb{R}}|x^N\varphi(x)|^p < \infty.$$

Hence, $\varphi \in L^p_q(\mathbb{R})$. $\rhd$

$\lhd$ Proof of Theorem 2.1. We define

$$_\kappa f(x) = f\left(\frac{x}{\kappa}\right).$$

Since $\operatorname{supp}\widehat{f} \subset [-\kappa,\kappa]$, $\operatorname{supp}\widehat{_\kappa f} \subset [-1,1]$. Then, using Lemma 2.5, we obtain

$$\|D^m {}_\kappa f\|_{L^p_q} \leqslant Cm^{-\varrho}\|_\kappa f\|_{L^u_v}\ . \tag{21}$$

Since $_\kappa f(x) = f\big(\frac{x}{\kappa}\big)$ and Lemma 2.4,

$$\|_\kappa f\|_{L^u_v} = \kappa^{v+\frac{1}{u}}\|f\|_{L^u_v}, \|D^m {}_\kappa f\|_{L^p_q} = \kappa^{-m+q+\frac{1}{p}}\|D^m f\|_{L^p_q}\ .$$

Hence, it follows from (21) that

$$\kappa^{-m+q+\frac{1}{p}}\|D^m f\|_{L^p_q} \leqslant Cm^{-\varrho}\kappa^{v+\frac{1}{u}}\|f\|_{L^u_v}\ .$$

So,

$$\|D^m f\|_{L^p_q} \leqslant Cm^{-\varrho}\kappa^{m+v+\frac{1}{u}-\frac{1}{p}-q}\|f\|_{L^u_v} = Cm^{-\varrho}\kappa^{m+\varrho}\|f\|_{L^u_v}\ ,$$

which confirms (4), and also (3) by using Lemma 2.6.

To complete the proof, it remains to prove that $\operatorname{supp}\widehat{D^m f} = \operatorname{supp}\widehat{f}$ for all $m \in \mathbb{N}$. It is enough to prove this for $m = 1$. Assume the contrary that $\operatorname{supp}\widehat{Df} \neq \operatorname{supp}\widehat{f}$. Since $\widehat{Df} = (ix)\widehat{f}$,

$$\operatorname{supp}\widehat{Df} \subset \operatorname{supp}\widehat{f} \subset \operatorname{supp}\widehat{Df} \cup \{0\}.$$

Hence, by $\operatorname{supp}\widehat{Df} \neq \operatorname{supp}\widehat{f}$, we obtain

$$\operatorname{supp}\widehat{f} = \operatorname{supp}\widehat{Df} \cup \{0\}, \quad 0 \notin \operatorname{supp}\widehat{Df}. \tag{22}$$

Then, it follows from that $\operatorname{supp}\widehat{Df}$ is a compact set, there exists a positive number ϵ such that $B[0,\epsilon] \cap \operatorname{supp}\widehat{f} = \{0\}$. We choose a function $\psi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\psi \subset [-\epsilon,\epsilon]$ satisfying $\psi(x) = 1$ in $[-\epsilon/2,\epsilon/2]$. Then

$$\operatorname{supp}\psi\widehat{f} \subset \{0\},$$

which together with Lemma 2.3 imply

$$\psi\widehat{f} = \sum_{j=0}^{N} c_j D^j\delta,$$

where N is the order of $\psi\widehat{f}$ and δ is the Dirac function ($\langle\delta,\varphi\rangle = \varphi(0)$ for all $\varphi \subset \mathscr{S}(\mathbb{R})$). Therefore, $(\mathscr{F}^{-1}\psi) * f(x)$ is a polynomial and

$$(2\pi)^{-1/2}(\mathscr{F}^{-1}\psi) * f(x) = \sum_{|\alpha|\leqslant N} c_\alpha \mathscr{F}^{-1}(D^\alpha\delta). \tag{23}$$

Using $\gamma + 1/r > 0$ and Lemma 2.7, we deduce $\mathscr{F}^{-1}\psi \in \mathscr{S}(\mathbb{R}) \subset L^r_\gamma(\mathbb{R})$. Combining this, $f \in L^u_v(\mathbb{R})$ and Lemma 2.2, we get $(\mathscr{F}^{-1}\psi) * f \in L^p_q(\mathbb{R})$. This and (23) imply

$$(\mathscr{F}^{-1}\psi) * f(x) = 0.$$

So, $\psi\widehat{f} = 0$. By $0 \in \operatorname{supp}\widehat{f}$, there is a function $\phi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\phi \subset [-\epsilon/2, \epsilon/2]$ such that $\langle \widehat{f}, \phi \rangle \neq 0$. So, it follows from $\psi(x) = 1$ in $[-\epsilon/2, \epsilon/2]$ that

$$0 \neq \langle \widehat{f}, \phi \rangle = \langle \widehat{f}, \psi\phi \rangle = \langle \psi\widehat{f}, \phi \rangle = 0,$$

which is impossible. The proof is complete. ▷

For $\kappa > 0$ we denote

$$L^u_{v,\kappa} = \{ f \in L^u_v(\mathbb{R}) : \operatorname{supp}\widehat{f} \subset [-\kappa, \kappa] \}.$$

The norm of the derivative operator D^m is given by

$$\|D^m\|_{L^u_{v,\kappa} \to L^p_q} = \sup_{\|f\|_{L^u_{v,\kappa}} \leqslant 1} \|D^m f\|_{L^p_q}.$$

From Theorem 2.1, we have the following corollary about the norm of derivative operators.

Corollary 2.8. *Let $1 < u, p < \infty$, $0 < q + 1/p < v + 1/u < 1$, $v - q \geqslant 0$, $\kappa > 0$. Then there exists a constant $C > 0$ independent of m, κ such that*

$$\|D^m\|_{L^u_{v,\kappa} \to L^p_q} \leqslant C m^{-\varrho} \kappa^{m+\varrho},$$

where

$$\varrho = v + \frac{1}{u} - q - \frac{1}{p} > 0.$$

If $p = u$, using Theorem 2.1, we get

Corollary 2.9. *Let $1 < p < \infty, -1/p < q < v < 1 - 1/p$, $\kappa > 0$, $f \in L^p_v(\mathbb{R})$ and $\operatorname{supp}\widehat{f} \subset [-\kappa, \kappa]$. Then $D^m f \in L^p_q(\mathbb{R})$ and there exists a constant $C > 0$ independent of f, m, κ such that*

$$\|D^m f\|_{L^p_q} \leqslant C m^{-\varrho} \kappa^{m+\varrho} \|f\|_{L^p_v},$$

where

$$\varrho = v - q > 0.$$

If $q = v$, it follows from Theorem 2.1 that

Corollary 2.10. *Let $1 < u < p < \infty$, $-1/p < q < 1 - 1/u$, $\kappa > 0$, $f \in L^u_q(\mathbb{R})$ and $\operatorname{supp}\widehat{f} \subset [-\kappa, \kappa]$. Then there exists a constant $C > 0$ independent of f, m, κ such that*

$$\|D^m f\|_{L^p_q} \leqslant C m^{-\varrho} \kappa^{m+\varrho} \|f\|_{L^u_q},$$

where

$$\varrho = \frac{1}{u} - \frac{1}{p} > 0.$$

Using Theorem 2.1 in the case $q = 0$, we have the following:

Corollary 2.11. *Let $1 < u, p < \infty$, $1/p < v + 1/u < 1$, $v \geqslant 0$, $\kappa > 0$, $f \in L^u_v(\mathbb{R})$ and $\operatorname{supp} \widehat{f} \subset [-\kappa, \kappa]$. Then there exists a constant $C > 0$ independent of f, m, κ such that*

$$\|D^m f\|_{L^p} \leqslant C m^{-\varrho} \kappa^{m+\varrho} \|f\|_{L^u_v}, \quad \left(\varrho = v + \frac{1}{u} - \frac{1}{p}\right).$$

In particular,

$$\lim_{m\to\infty} \|D^m f\|_{L^u_v} / \kappa^m = 0, \quad \limsup_{m\to\infty} \|D^m f\|^{1/m}_{L^u_v} \leqslant \kappa.$$

Further, if $v = 0$, we have

Corollary 2.12. *Let $1 < u, p < \infty$, $0 < q + 1/p < 1/u$, $q \leqslant 0$, $\kappa > 0$, $f \in L^u(\mathbb{R})$ and $\operatorname{supp} \widehat{f} \subset [-\kappa, \kappa]$. Then there exists a constant $C > 0$ independent of f, m, κ such that*

$$\|D^m f\|_{L^p_q} \leqslant C m^{-\varrho} \kappa^{m+\varrho} \|f\|_{L^u},$$

where

$$\varrho = \frac{1}{u} - q - \frac{1}{p} > 0.$$

In particular,

$$\lim_{m\to\infty} \|D^m f\|_{L^u} / \kappa^m = 0, \quad \limsup_{m\to\infty} \|D^m f\|^{1/m}_{L^u} \leqslant \kappa.$$

Moreover, if $v = q = 0$ then the following result holds:

Corollary 2.13. *Let $1 < u < p < \infty$, $\kappa > 0$, $f \in L^u(\mathbb{R})$ and $\operatorname{supp} \widehat{f} \subset [-\kappa, \kappa]$. Then $D^m f \in L^p(\mathbb{R})$ and there exists a constant $C > 0$ independent of f, m, κ such that*

$$\|D^m f\|_{L^p} \leqslant C m^{-\varrho} \kappa^{m+\varrho} \|f\|_{L^u} ,$$

where

$$\varrho = \frac{1}{u} - \frac{1}{p} > 0.$$

Using Theorem 2.1 and Bernstein inequality, we can prove the following result.

Corollary 2.14. *Let $1 < u < p < \infty$, $\kappa > 0$. Denote*

$$N_{\kappa,u} := \left\{ f \in \mathscr{S}'(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-\kappa, \kappa],\ f \in L^u(\mathbb{R}) \right\}$$

and

$$\gamma_m := \inf_{f \in N_{\kappa,u}} \frac{\|D^m f\|_{L^p}}{\kappa^m \|f\|_{L^u}}.$$

Then $\gamma_{m+1} \leqslant \gamma_m$ and

$$\lim_{m\to\infty} \gamma_m = 0.$$

Let $1 \leqslant p < \infty$ and $q \in \mathbb{R}$. The weighted Lebesgue space $L^p_q := L^p_q(\mathbb{R}^n)$ consists of all measurable functions such that

$$\|f\|_{L^p_q} = \left(\int_{\mathbb{R}^n} |f(\mathbf{x})|^p \prod_{j=1}^n |x_j|^{pq} \, d\mathbf{x} \right)^{1/p} < \infty,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Consecutively applying Theorem 2.1 to each variable, we get the following result for the n-dimensional case.

Theorem 2.15. *Let* $1 < u, p < \infty$, $0 < q + 1/p < v + 1/u < 1$, $v - q \geqslant 0$, $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n_+$, $f \in L^u_v(\mathbb{R}^n)$ *and* $\operatorname{supp} \widehat{f} \subset [-\kappa_1, \kappa_1] \times \ldots \times [-\kappa_n, \kappa_n]$. *Then* $D^\alpha f \in L^p_q(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *independent of* f, α, κ *such that*

$$\|D^\alpha f\|_{L^p_q} \leqslant C\|f\|_{L^u_v} \prod_{\substack{j=1,\\ \alpha_j \neq 0}}^{n} {\alpha_j}^{-\varrho} \kappa_j^{\alpha_j+\varrho}, \tag{24}$$

where

$$\varrho = v + \frac{1}{u} - q - \frac{1}{p} > 0.$$

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Received March 5, 2020

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Владикавказский математический журнал
2020, Том 22, Выпуск 3, С. 18–29

НЕРАВЕНСТВО БЕРНШТЕЙНА — НИКОЛЬСКОГО В ВЕСОВЫХ ПРОСТРАНСТВАХ ЛЕБЕГА

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Аннотация. В работе устанавливаются результаты, касающиеся неравенства Бернштейна — Никольского в весовых пространствах Лебега. Основной результат содержится в следующем утверждении. Пусть $1 < u, p < \infty$, $0 < q + 1/p < v + 1/u < 1$, $v - q \geqslant 0$, $\kappa > 0$, $f \in L^u_v(\mathbb{R})$ и $\operatorname{supp}\widehat{f} \subset [-\kappa, \kappa]$. Тогда $D^m f \in L^p_q(\mathbb{R}), \operatorname{supp}\widehat{D^m f} = \operatorname{supp}\widehat{f}$ и существует такая постоянная C, независящая от f, m и κ, что $\|D^m f\|_{L^p_q} \leqslant Cm^{-\varrho}\kappa^{m+\varrho}\|f\|_{L^u_v}$ для всех $m = 1, 2, \ldots$, где $\varrho = v + \frac{1}{u} - \frac{1}{p} - q > 0$ и весовое пространство Лебега L^p_q состоит из всех измеримых функций, для которых $\|f\|_{L^p_q} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{pq}\,dx\right)^{1/p} < \infty$. Более того, $\lim_{m\to\infty} \|D^m f\|^{1/m}_{L^p_q} = \sup\{|x| : x \in \operatorname{supp}\widehat{f}\}$. Главным достижением нашего результата является то, что в правой части неравенства содержится множитель $m^{-\varrho}$ ($\varrho > 0$), который ранее никогда не появлялся в аналогичных исследованиях других авторов. Соответствующий результат получен также для n-мерного случая.

Ключевые слова: весовые пространства Лебега, неравенство Бернштейна, неравенство Никольского.

Mathematical Subject Classification (2010): 26D10, 46E30.

Образец цитирования: *Bang, H. H. and Huy, V. N.* A Bernstein–Nikol'skii Inequality for Weighted Lebesgue Spaces // Владикавк. мат. журн.—2020.—Т. 22, № 3.—С. 18–29 (in English). DOI: 10.46698/h8083-6917-3687-w.