Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 25–41

УДК 510.28, 517.13, 517.98
DOI 10.23671/VNC.2019.21.44619

SOME REMARKS ABOUT NONSTANDARD METHODS IN ANALYSIS. I

E. I. Gordon[1]

[1] Eastern Illinois University,
600 Lincoln Av., Charleston (IL) 61920, USA
E-mail: `yigordon@eiu.edu`

In the blessed memory of Evgeniy Alekseevich Gorin,
a wonderful man and an outstanding mathematician

Abstract. This and forthcoming articles discuss two of the most known nonstandard methods of analysis—the Robinson's infinitesimal analysis and the Boolean valued analysis, the history of their origination, common features, differences, applications and prospects. This article contains a review of infinitesimal analysis and the original method of forcing. The presentation is intended for a reader who is familiar only with the most basic concepts of mathematical logic—the language of first-order predicate logic and its interpretations. It is also desirable to have some idea of the formal proofs and the Zermelo–Fraenkel axiomatics of the set theory. In presenting the infinitesimal analysis, special attention is paid to formalizing the sentences of ordinary mathematics in a first-order language for a superstructure. The presentation of the forcing method is preceded by a brief review of C. Godel's result on the compatibility of the Axiom of Choice and the Continuum Hypothesis with Zermelo–Fraenkel's axiomatics. The forthcoming article is devoted to Boolean valued models and to the Boolean valued analysis, with particular attention to the history of its origination.

Key words: boolean valued analysis, nonstandard analysis, forcing.

Mathematics Subject Classification (2010): 03C90, 46A40.

For citation: *Gordon, E. I.* Some Remarks about Nonstandard Methods in Analysis. I, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 25–41. DOI: 10.23671/VNC.2019.21.44619.

1. Introduction

As noted in the book [1] "Nonstandard methods of analysis in the modern sense consist in attracting two different models of set theory—"standard" and "nonstandard" for the study of specific mathematical objects and problems". Currently, the two nonstandard methods are most widely used in analysis—Robinson's infinitesimal analysis and Boolean valued analysis, each of which have become an independent area of analysis.

Application of the methods of mathematical logic for obtaining new results in pure mathematics, started apparently with the article by A. I. Maltsev [2] in which a general method was developed for obtaining local theorems of group theory. This method was based on Maltsev Compactness Theorem proved in his PhD Thesis in 1936. Further penetration of the methods of logic in various areas of mathematics, mainly in algebra, is associated with the development of model theory—a section of mathematical logic that studies algebraic structures from the point of view of their description by first-order logical languages.

The beginning of application of the methods of mathematical logic in analysis is connected with A. Robinson who made a great contribution to the development of model theory. Using Maltsev Compactness Theorem, he constructed an extension of the standard model of analysis which included a slightly modified version of the basic properties of the standard model, but contained also infinitely large and infinitesimal numbers. In this new analysis, which Robinson called the non-standard analysis, many intuitive mathematical formulations that go back to Leibniz and later to Cauchy, such as, for example, the definition of limit: "$\lim_{x\to a} f(x) = L$ means that if x is infinitely close to a but $x \neq a$, then $f(x)$, is infinitely close to L", received the status of rigorous mathematical statements. This made it possible to simplify significantly the proofs of many theorems of standard analysis and even obtain new results in standard mathematics using nonstandard analysis. After the first edition of Robinson's book [3] was published in 1966, many articles appeared in which nonstandard analysis was used to obtain new results in various fields of standard mathematics, especially in functional analysis, stochastic analysis and mathematical physics (see e.g. [4]). Robinson's nonstandard analysis is briefly discussed in Section 2.

The other of the nonstandard method of analysis, named by G. Takeuti "Boolean valued analysis" originated from P. Cohen's method of forcing*, which was developed to prove the independence of the Continuum Hypothesis (CH). The forcing is a quite complicated method. It requires knowledge not only of the foundations of mathematical logic, but also of the very subtle and deep results in axiomatic set theory. It was impossible for a laymen to understand the proof of independence of CH based on the forcing. At the same time, interest in the result itself, marked with the Fields Prize, was very wide. This served as an incentive for D. Scott and R. Solovay to develop a method of Boolean valued models [5, 6, 7]**. An excellent intuitive explanation of the main ideas of this proof is contained in the article [5]. After reading this article, one cannot fully understand the proof of the independence of CH, but it is quite possible to understand its idea on the basis of Boolean valued models.

In the last lines of this article, the hope is expressed that the Boolean valued models of the field of reals will find application in mathematics not only to prove independence, but also by themselves. This hope was justified. Based on the theory of Boolean valued models, the method of Boolean valued analysis was developed, which found applications in various fields of mathematics. Applications to harmonic analysis and von Neumann algebras are primarily related to G. Takeuti and M. Ozawa [9, 10]; and in the theory of vector lattices to A. G. Kusraev and S. S. Kutateladze. See [1, 11, 12] and the references there. Quite recently Boolean valued analysis found application in Mathematical Economics. See the paper of J. M. Zapata in this issue of the VMJ. The history of origination of Boolean valued analysis will be discussed in the forthcoming paper.

Boolean valued analysis is called nonstandard since it uses not two-valued logic, but one in which truth values form a complete Boolean algebra. The truth of a sentence in such a model means that its truth value is equal to the top of this Boolean algebra. Thus, the objects that simulate $\mathbb{R}$ in Boolean models are significantly more complicated than $\mathbb{R}$. The article [13] shows that the class of Boolean valued fields $\mathbb{R}$ coincides with the class of universally complete Kantorovich spaces. Roughly speaking, this allows us, to reduce many problems about complex objects to problems about simpler objects.

* We use below simply forcing for the method of forcing as it is used in the majority of publications.

** It is mentioned in [8] that [6] is the preliminary version of [7]. However, the article [7] never appeared in print, although it circulated as a preprint and was widely known.

For the most important particular case of Boolean algebras with measure, this characterization of Boolean valued models of $\mathbb{R}$ was actually obtained earlier in [5].

Evgeniy Alekseevich Gorin, who left us a year ago, once attending a seminar, where I gave a talk on Boolean valued analysis, quite accurately characterized this method with his inherent humor: "I understand now what you are doing. You are taking some kind of a theorem about functionals, say something like spells over it and get a theorem about operators."

Those who work in Boolean valued analysis usually do not use the forcing method for the above reasons. There is one more reason. Forcing uses the standard models of ZF. The existence of such a model (SM) cannot be proved in ZF itself by virtue of Gödel's Incompleteness Theorem. For independence proofs the additional hypothesis about the existence of an SM does not matter, because there is a method that allows to convert a deduction using this hypothesis of a contradiction to a deduction of the same contradiction without it. However, if some analysis theorem is proved under the assumption of the existence of a SM, then generally speaking this does not mean that this theorem can be proved without it. For example, the consistency of ZF can be proved in ZF+SM, but it cannot be proved in ZF. There is no such problem in the Boolean valued analysis, since it does not need any SM. Even independence proofs can be carried out in Boolean valued models without resorting to standard models.

The present day books on Boolean valued models do not include even any survey of the method of forcing. However, in my opinion, proofs by the method of forcing are more intuitive than proofs by the method of Boolean valued models, especially in the case of independence proofs. The reader can compare the proofs of independence of CH in the book [11], Section 9.5 and in Section 3.3 of this paper and decide for him/herself which one is more intuitive. CH is a simple case. I think that it would be very hard to implement a proof of Theorem 8 below in the framework of Boolean valued models. At least I tried to do this, when I worked on Theorem 11 below and understood that it is too hard for me. I am sure that this situation may sometimes occur in Boolean valued analysis as well. That is why Section 3 of this paper contains a survey of forcing.

2. A. Robinson's Nonstandard Analysis

Recall some concepts and facts of model theory, on which the nonstandard analysis is based.

Let σ be a signature of a first order logical language L_σ, and let $\mathcal{M} = (M, \sigma)$ and $\mathcal{M}' = (M', \sigma)$ be σ-structures*.

Definition 1. Let $\imath : \mathcal{M} \to \mathcal{M}'$ be a monomorphism. Say that $(\mathcal{M}', \imath)$ is an elementary extension of $\mathcal{M}$, if for every formula $\varphi(x_1, \dots, x_n)$ of L_σ and for every $m_1, \dots, m_n \in M$ one has

$$\mathcal{M} \models \varphi(m_1, \dots, m_n) \Longleftrightarrow \mathcal{M}' \models \varphi(\imath(m_1), \dots, \imath(m_n)). \tag{1}$$

Remark 1. In what follows we assume WLOG that $\imath$ is the identity monomorphism and so $M \subseteq M'$.

Maltsev Compactness Theorem easily implies the following

Theorem 1. *Each structure $\mathcal{M}$ has an elementary extension of an arbitrary cardinality $\leqslant \max(|\sigma|, \aleph_0)$.*

* This means that the basic functions and predicates in $\mathcal{M}$ and $\mathcal{M}'$ are interpretations of the corresponding symbols in the signature σ. In what follows we use the same notations for the signature symbols and their interpretations.

Recall the definition of the superstructure $\mathcal{S}(X)$ over an arbitrary set X following [14] (see also [15]). We assume that X contain all naturals.

Definition 2. Put $\mathcal{S}_0(X) = X$ and for each $n \in \mathbb{N}$, $n > 0$, $\mathcal{S}_n(X) = \mathcal{S}_{n-1}(X) \cup \mathcal{P}(\mathcal{S}_{n-1}(X))$. Then $\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \mathcal{S}_n(X)$. Define the rank of $x \in X$ as follows. For $x \in \mathcal{S}_0(X)$ $\mathrm{rank}(x) = 0$, for $n \geqslant 1$ $\mathrm{rank}(x) = n$, if $x \in \mathcal{S}_n(X) \setminus \mathcal{S}_{n-1}(X)$.

The superstructure $\mathcal{S}(\mathbb{R})$ contains the mathematical objects that are used in the main areas of mathematics—algebra, geometry, analysis, probability theory and others. For example, the book [4] contains only objects from $\mathcal{S}(\mathbb{R})$ and its elementary extension, which, by the way, is also a subfamily of $\mathcal{S}(\mathbb{R})$ (see below). The signature σ of $\mathcal{S}(\mathbb{R})$ is the following one:*

$$\sigma = \langle \varnothing, +, \cdot, \leqslant, \in \rangle. \tag{2}$$

Here $\varnothing$ is a constant symbol, $+$ and $\cdot$ are binary function symbols and $\leqslant$ and $\in$ are binary predicate symbols. The elements of rank 0—real numbers are considered as individuals (not sets). They are the only elements of $\mathcal{S}(\mathbb{R})$ that are not sets. For example $\mathrm{rank}(\varnothing) = 1$ and in the expressions $x+y$, $x \cdot y$, $x \leqslant y$, $t \in z$ we have $\mathrm{rank}(x) = \mathrm{rank}(y) = 0$ and $\mathrm{rank}(t) < \mathrm{rank}(z)$.

Practically any conventional mathematical statement can be formalized by an appropriate formula of the language L_σ. Such a direct formalization is often quite long and difficult to see even for relatively simple mathematical statements. To simplify it, various notations and abbreviations are usually used.

The following abbreviation is generally accepted in all logical languages.

Let $\varphi(x)$ be a formula that contains a free variable x.

$$\exists !\, x \ f(x) := \exists x \forall y \ (\varphi(x) \wedge (\varphi(y) \longrightarrow x = y)).$$

In conventional language we read this formula of L_σ as "There exists a unique x such that $\Phi(x)$", where $\Phi(x)$ is a conventional statement about x whose formalization in L_σ is $\varphi(x)$.

Definition 3. We say that an element $A \in \mathcal{S}(\mathbb{R})$ is definable in the superstructure $\mathcal{S}(\mathbb{R})$ if there exists a formula $\varphi(x)$ of L_σ with the only free variable x, such that $\mathcal{S}(\mathbb{R}) \models \exists !\, x \ \varphi(x) \wedge \varphi(A)$.

If φ contains besides x free variables $p_1, \dots, p_n$, then we say that A is defined in $\mathcal{S}(\mathbb{R})$ via parameters $p_1, \dots p_n$, provided that

$$\mathcal{S}(\mathbb{R}) \models \forall p_1, \dots, p_n \, \exists !\, x \ \varphi(x, p_1, \dots, p_n) \tag{3}$$

and for all $b_1, \dots, b_n \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{S}(\mathbb{R}) \models \varphi(A, b_1, \dots, b_n). \tag{4}$$

If φ satisfies (3), then φ defines a function that assign to each n-tuple $\langle b_1, \dots, b_n \rangle$ the unique element A that satisfies (4). So, the formula φ after assigning an appropriate notation can be added to the signature σ as a function symbol.

For example, consider a formula $\psi(x, x_1, \dots, x_n)$ and let

$$\varphi(X, x, x_1, \dots, x_n) := x \in X \ \longleftrightarrow \ \psi(X, x, x_1, \dots, x_n). \tag{5}$$

Then φ satisfies (4), and so it defines an n-ary function, that is denoted as $X = \{x : \psi(x, x_1, \dots, x_n)\}$ exactly like in the conventional language.

* We assume that the symbol of equality $=$ is an element of any logic language.

In case, when ψ does not contain the variable x, the formula φ define the n-ary predicate $X(x_1, \ldots, x_n)$, $n \geqslant 0$. In case when the truth domain of X is a set that belongs to $\mathcal{S}(\mathbb{R})$ we write $\langle x_1, \ldots x_n\rangle \in X$, identifying the predicate and its domain. We assign some notation to this predicate and include it in the signature σ as the set constant. Sometimes in this situation we say for brevity that a function or a predicate is defined by the formula ψ, not ϕ.

EXAMPLE. 1. The formula $\forall\, y\ (y \notin x \wedge x \neq \varnothing)$ defines the unary predicate "x is a real", whose truth domain is the set of all reals $\mathbb{R}$, so we include the constant $\mathbb{R}$ in the signature σ. Usually we use writing $x \in \mathbb{R}$, not $\mathbb{R}(x)$. The truth domain of the predicate $x \notin \mathbb{R}$ consists of all sets in $\mathcal{S}(\mathbb{R})$. So it is not an element of $\mathcal{S}(\mathbb{R})$. Sometimes this predicate is denoted by Set. Here writing $\mathrm{Set}(x)$ is preferable.

2. All Boolean operations on sets but complementation are obviously definable in L_σ. As it was mentioned above $\mathcal{S}(\mathbb{R})$ is a set, thus $\mathcal{S}(\mathbb{R}) \setminus x$ is a set for any set $x \in \mathcal{S}(\mathbb{R})$. However, this set is not an element of $\mathcal{S}(\mathbb{R})$, since it necessary contains elements of an arbitrarily large rank. As in conventional mathematics, the complementation can be used, when some universe $\mathcal{U} \in \mathcal{S}(\mathbb{R})$ is fixed and we deal only with its subsets.

3. The predicate $x \subseteq y$ is defined by the formula $\forall\, z\ (z \in x \to z \in y)$. The operation $\mathcal{P}(y)$ is defined by the formula $\mathcal{P}(y) = \{x : x \subseteq y\}$

4. The definition of an ordered pair by Kuratowski:

$$\langle a, b\rangle = \{\{a\}, \{a, b\}\}$$

can be considered as a formula of L_σ written with the abbreviations introduced above. Using the definition of an ordered pairs, we can usually formalize the definitions $X \times Y$, $f : X \to Y$, Y^X, etc.

For the further examples of translations from the conventional language to the formal one see Section 3 of Chapter I in [16] and Section 1 of Chapter 0 in [1].

Consider some proper elementary extension ${}^*\mathcal{S}(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$. We use the canonical notations for the elementary extension of $\mathcal{S}(\mathbb{R})$ in the nonstandard analysis. Here ${}^* : \mathcal{S}(\mathbb{R}) \to {}^*\mathcal{S}(\mathbb{R})$ is the monomorphism $\imath$ of the Definition 1. The equivalence (1) is called in the nonstandard analysis the *Transfer principle*.

DEFINITION 4. a) An element ${}^*x \in {}^*\mathcal{S}(\mathbb{R})$, the image of $x \in \mathcal{S}(\mathbb{R})$ under the monomorphism *, is said to be standard. We say that $y \in {}^*\mathcal{S}(\mathbb{R})$ is standard (notation St(y)), if $\exists\, x \in \mathcal{S}(\mathbb{R})$ such that $y = {}^*x$.

b) The elements of ${}^*\mathcal{S}(\mathbb{R})$ are called internal elements.

c) The noninternal sets that belong to $\mathcal{S}({}^*\mathbb{R})$ are called external sets.

d) The elements of ${}^*\mathbb{R}$ are called hyperreal numbers or hyperreals.

In what follows $\forall^{st} x \ldots$ stay for $\forall\, x(\mathrm{St}(x) \to \ldots)$ and $\exists^{st} x \ldots$ for $\exists\, x(\mathrm{St}(x) \wedge \ldots)$ respectively.

REMARK 2. Let us clarify the difference between ${}^*(\mathcal{S}(\mathbb{R}))$, ${}^*\mathcal{S}(\mathbb{R})$ and $\mathcal{S}({}^*\mathbb{R})$. The first one consists of all standard elements, the second one is an elementary extension of $\mathcal{S}(\mathbb{R})$, i. e. it consists of all internal elements, the third one contains all internal and external elements of $\mathcal{S}({}^*\mathbb{R})$. So,

$$ {}^*(\mathcal{S}(\mathbb{R})) \subseteq {}^*\mathcal{S}(\mathbb{R}) \subseteq \mathcal{S}({}^*\mathbb{R}). \tag{6}$$

Both inclusions in (6) are proper. Notice firstly, that since the elementary extension of $\mathcal{S}(\mathbb{R})$ is proper by definition, then ${}^*(R) \neq {}^*R$. Otherwise, it is easy to see that ${}^*(\mathcal{S}(\mathbb{R})) = \mathcal{S}({}^*\mathbb{R})$, etc. We show now that there are internal sets that are nonstandard. Since there exist internal elements $\alpha < \beta \in {}^*\mathbb{R} \setminus {}^*(\mathbb{R})$ and in $\mathcal{S}(\mathbb{R})$ it is true that for any $x < y \in \mathbb{R}$ there exists a set

$[x, y]$; then, by the Transfer Principle, the same statement is true in ${}^*\mathcal{S}(\mathbb{R})$ for ${}^*\mathbb{R}$. Thus, $[\alpha, \beta]$ is an internal set in ${}^*\mathcal{S}(\mathbb{R})$, that is obviously nonstandard.

It is easy to prove the following

Proposition 1. *If a linearly ordered field $\mathcal{R}$ is an arbitrary proper extension of $\mathbb{R}$, then*

1. *There exits $\rho \in \mathcal{R} \setminus \mathbb{R}$ such that $\forall\, r \in (0, \infty) \subseteq \mathbb{R}$ $0 < |\rho| < r$. Notation: $\rho \approx 0$.*

2. *For every $\beta \in \mathcal{R}$ such that $|\beta| < r$ for some $r \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $\beta - b \approx 0$. In this case we write $\beta \approx b$.*

A simple proof of this proposition can be a good exercise for the students who start to study a rigorous course of Analysis. It is just a proposition of standard mathematics. In ${}^*\mathcal{S}(\mathbb{R})$ we apply Proposition 1 to ${}^*\mathbb{R}$ for $\mathcal{R}$ and to ${}^*(\mathbb{R})$ that is an isomorphic copy of $\mathbb{R}$. Here the following definition is used:

Definition 5. a) If $\rho \approx 0$, then ρ is said to be infinitesimal. The set $M_0 = \{\rho \in {}^*\mathbb{R} : \ \rho \approx 0\}$ is called the monad of 0.

b) An element $\Omega \in {}^*\mathbb{R}$ is said to be infinitely large if $\forall^{st}\, r\ |\Omega| > r$.

c) An element $\beta \in {}^*\mathbb{R}$ is said to be limited, or bounded or finite, if $\exists^{st}\, r\ |\beta| < r$. The set of bounded elements is denoted by ${}^*\mathbb{R}_{fin}$. According to Proposition 1 item 2, in this case there exists a standard b such that $b \approx \beta$. This b is called the standard part or the shadow of β and is denoted by ${}^\circ\beta$.

Proposition 2. *The monad M_0 is an external set.*

$\lhd$ Suppose that M_0 is an internal set. Since it is bounded from above (e.g. by number 1), it must have $\sup M_0 = \mu$. It is easy to see that both assumptions $\mu \approx 0$ and $\mu \not\approx 0$ lead to contradiction. $\rhd$

The notions defined in Definition 5 are not formulated in L_σ since in their formulation the unary predicate St is present explicitly or implicitly. This predicate is not included in the signature σ. For the formalization of such statements we need to extend the signature σ by adding to it this predicate. Denote the extended signature σ^{st}. The formulas in $L_{\sigma^{st}}$ are called external formulas, while the formulas of L_σ are called internal formulas. The Transfer Principle is not applicable to external formulas, as we just saw in the proof of Proposition 2.

Mathematicians, who begin to study nonstandard analysis with the aim of applying it in their research, often face the following difficulty. They make mistakes related just to the application of the Transfer Principle to external sets. This is because the definitions of internal and external sets (see Definition 4) are very nonconstructive and require a formalization habit, which usually the mathematicians who work in geometry, ODEs and PDEs do not have. The considerations of the previous paragraph imply the following sufficient condition for a set to be internal (external): *A set defined by a formula of the language L_σ is internal, while a set defined by a formula of the language $L_{\sigma^{st}}$ is external.* This condition makes the difficulty mentioned above somewhat easier.

Nevertheless, a certain difficulty in using nonstandard methods remained. The fact is that the Maltsev Compactness Theorem, on which Theorem 1 is based, is a consequence of Gödel's Completeness Theorem, more precisely, on its generalization to signatures of an arbitrary cardinality belonging to Maltsev. Many mathematicians do not like to use in their research the results whose proofs they do not know, or at least do not even imagine the idea. In order to be sure of the correctness of the results obtained using nonstandard analysis, such mathematicians must study the principles of mathematical logic, at least up to Theorem 1 inclusively. This is a rather extensive material that usually is far away from their scientific interests and does not correspond to their way of thinking. But even after studying the proof of Theorem 1, such mathematicians will not feel completely satisfied, since the proof of this

Theorem based on the Compactness Theorem is a pure proof of existence and does not give any construction of elementary extension.

Fortunately, there is another proof of the existence of an elementary extension of an arbitrary structure, in which this extension is constructed as an ultrapower of this structure. This construction does not rely on Gödel's Completeness Theorem and does not require any knowledge of mathematical logic besides the definition of a first-order language and the truth of the formula of this language in its signature. It can be considered as a construction in the conventional mathematics.

DEFINITION 6. 1) Let $\{\mathcal{M}_i : i \in I\}$ be a family of structures of a signature σ and $\mathcal{F}$ be a free ultrafilter on I. Then the ultraproduct of this family is the structure

$$\prod_{\mathcal{F}} \mathcal{M}_i = \left(\prod_i M_i / \sim_{\mathcal{F}}, \sigma\right), \text{ where } \{m_i\} \sim_{\mathcal{F}} \{m_i'\} := \{i : m_i = m_i'\} \in \mathcal{F}. \tag{7}$$

Let f be a function symbol of the signature σ. Consider the sequence $\{f_i : M_i \to \mathcal{M}_i\}$, where f_i is the interpretation of f in $\mathcal{M}_i$. Then the interpretation $f^{\sim_{\mathcal{F}}}$ of f is defined as follows:

$$f^{\sim_{\mathcal{F}}}(m^{\sim_{\mathcal{F}}}) = n^{\sim_{\mathcal{F}}} := \{i : n_i = f_i(m_i)\} \in \mathcal{F} \tag{8}$$

The definitions of interpretation of k-ary functional symbols for arbitrary $k \in \mathbb{N}$ and k-ary predicate symbols are similar.

2) If all $\mathcal{M}_i = \mathcal{M}$ for a certain σ-structure $\mathcal{M}$, then the ultraproduct defined in 1) is called the ultrapower of $\mathcal{M}$ and is denoted $\mathcal{M}^{\mathcal{F}}$. There exists a monomorphism $j : \mathcal{M} \to \mathcal{M}^{\mathcal{F}}$ such that $j(m) = \{m_i : i \in I\}^{\sim_{\mathcal{F}}}$, where $m_i = m$ for all $i \in I$

Theorem 2 (**Loš**). *Let $\mathcal{M} = (M, \sigma)$ be a structure of the signature σ; $\mathcal{F}$, a free ultrafilter on a set I; $\varphi(x_1, \dots, x_n)$; a formula of L_σ, and $\mu_k = \{m_i^k\}^{\mathcal{F}} \in M^{\mathcal{F}}$, where $k = 1, \dots, n$. Then*

$$\mathcal{M}^{\mathcal{F}} \models \varphi(\mu_1, \dots, \mu_n) \Longleftrightarrow \{i \in I : \mathcal{M} \models \varphi(m_i^1, \dots, m_i^n) \in \mathcal{F}.$$

Corollary 1. *The structure $(\mathcal{M}^{\mathcal{F}}, j)$ is an elementary extension of $\mathcal{M}$.*

The elementary extension of $\mathcal{S}(\mathbb{R})$ is the *bounded ultrapower* of $\mathcal{S}(\mathbb{R})$, that is

$$\mathcal{S}(\mathbb{R})_b^{\mathcal{F}} = \bigcup_{n \in \mathbb{N}} \left(\mathcal{S}_n(\mathbb{R})^{\mathcal{F}} \setminus \mathcal{S}_{n-1}(\mathbb{R})^{\mathcal{F}}\right).$$

Corollary 1 is true for the bounded ultrapower. In what follows we keep the notation $({}^*\mathcal{S}(\mathbb{R}), {}^*)$ for the ultrapower nonstandard extension of $\mathcal{S}(\mathbb{R})$, if an ultrafilter $\mathcal{F}$ is fixed.

We cannot say that at least one nonstandard extension built using an ultrafilter is constructive, if only because the very existence of an ultrafilter cannot be proved without the Axiom of Choice. However, this concept is widely used in conventional mathematics: For example, recall the Stone–Čech compactification or the famous paper [18], which essentially uses the construction of a nonstandard hull known to nonstandard analysis, to which the author comes completely independent of nonstandard analysis and related to it mathematical logic.

The internal sets in an ultraproduct nonstandard extension of $\mathcal{S}(\mathbb{R})$ have very clear description: A set in $\mathcal{S}(\mathbb{R})_b^{\mathcal{F}}$ is internal if and only if it is an ultraproduct of a family of sets $\{X_i : i \in I\} \subseteq \mathcal{S}_k(\mathbb{R})$ for some $k \in \mathbb{N}$.

A hyperreal $\rho \in \mathbb{R}^{\mathcal{F}}$ is limited, if $\rho = \{r_i : i \in I\}^{\mathcal{F}}$ is such that the exists $r \in \mathbb{R}$ such that the set $\{i \in I : |r_i| < r\} \in F$. Then we may assume that the set $\{r_i : i \in I\}$ is a bounded subset of $\mathbb{R}$. The limit of a bounded function over free ultrafilter always exists and it is easy to see, that the standard part of ρ

$$^\circ\rho = \lim_{\mathcal{F}} r_i.$$

Using the technique of ultraproducts allowed mathematicians to easily remake the results obtained by nonstandard analysis into standard ones, without even going into the details of the original nonstandard proofs. This led to a certain drop in interest in nonstandard analysis and decrease in the number of publications and conferences related to nonstandard analysis. I believe that the potential of nonstandard analysis is far from exhausted. The justification of this point of view will be contained in another article.

3. Forcing and Independence Proofs

Another outstanding achievement of the mathematical logic of the 1960s was the proof of independence of the Continuum Hypotheses (CH) and the Axiom of Choice (AC) by P. Cohen and his development of the method of forcing for this and many other proofs of independence of the axioms of set theory. This method was ideologically and technically very complex and accessible for the specialists not only in mathematical logic, but also in its very special field—the axiomatic set theory. We discuss briefly Cohen's method. We deal here with the axiomatic due to Zermelo and Fraenkel (ZF). If AC is included in ZF then this system of axiom is denoted ZFC.

3.1. Axiom of constructivity. The consistency of CH. The consistency of CH with ZFC and AC with ZF was proved by K. Gödel in the late 1930s of the last century. We assume that the reader is aware of the axioms of ZF. However, we remind some notions and notations. Recall that a set x is said to be *transitive*, if $\forall\, y \in x\ z \in y\ (z \in x)$.

A set α is said to be an *ordinal* (notation $\alpha \in \mathrm{On}$), if it is linearly ordered by the membership $\in$. By the axiom of regularity $\in$ is a well-ordering of α.

The formula of ZF $\alpha \in \mathrm{On}$ is absolute with respect to any transitive set in the sense of the following

Definition 7. We say that a formula $\varphi(x)$ is absolute with respect to a set M, if for any $a \in M$

$$\varphi(a) \longleftrightarrow M \models \varphi(a). \tag{9}$$

In this context $\models$ on the right hand side means that all quantifiers are restricted to M. We use the notation φ_M for the formula that is obtained by restriction of all quantifiers in φ to M. If M is a set that is definable by a formula $\psi(x)$ in the sense of (5), then all quantifiers in φ are restricted to ψ and the equivalence (9) means

$$ZF \vdash \forall\, x(\psi(x) \to (\varphi(x) \longleftrightarrow \varphi_M(x))). \tag{10}$$

Obviously, if M is a transitive set and all quantifiers are of the form $\forall\, u \in v$, or $\exists\, u \in v$, then φ is absolute with respect to M.

A *cardinal* is an ordinal that is not in one to one correspondence with any of its elements. This definition is not absolute.

Indeed, there are such extensions of some models of set theory, in which ordinals are the same but the scale of cardinals is compressed. This is achieved by adding to the original model the bijective mapping of the ordinal representing a cardinal in the original model onto an ordinal representing a smaller cardinal. This effect is called the collapse of the cardinals. Exactly such an extension is used in the paper [19] discussed below.

It can be deduced from the Axiom of Regularity that the class of all sets V can be represented as follows:

$$V = \bigcup_{\alpha\in \mathrm{On}} V_\alpha, \quad V_0 = \varnothing, \;\; V_{\alpha+1} = \mathcal{P}(V_\alpha), \;\; V_\alpha = \bigcup_{\beta<\alpha} V_\beta, \;\; \text{if } \alpha \text{ is a limit ordinal.} \tag{11}$$

It is easy to see that $x \in V_\alpha$ can be written as a formula $E(x, \alpha)$ of ZF.

To prove the consistency of CH and AC, Gödel studied the class of all sets definable from ordinals. He called such sets *constructive* and denoted by $\mathcal{L}$. The class $\mathcal{L}$ is defined similar to (11):

Definition 8.

$$\mathcal{L} = \bigcup_{\alpha\in \mathrm{On}} \mathcal{L}_\alpha, \quad L_0 = \varnothing, \;\; \mathcal{L}_{\alpha+1} = \mathcal{P}_{def}(\mathcal{L}_\alpha), \;\; \mathcal{L}_\alpha = \bigcup_{\beta<\alpha} \mathcal{L}_\beta, \;\; \text{if } \alpha \text{ is a limit ordinal.} \tag{12}$$

Here $X \in \mathcal{P}_{def}(\mathcal{L}_\alpha)$ if X is definable like in equivalence (5), but ψ is a formula of ZF, with all quantifiers restricted to L_α and each x_i is either an element of $\mathcal{L}_\alpha$ or $\mathcal{L}_\alpha$ itself.

Theorem 3 (Gödel-1)**.** 1. *There exists a function $F(x, \alpha)$ definable in ZF, which establishes a bijection between classes On and $\mathcal{L}$. This function is absolute with respect to $\mathcal{L}$.*
2. $\mathcal{L} \models ZF$*, which means that, if for any axiom φ of ZF one has $ZF \vdash \varphi_{\mathcal{L}}$.*

In what follows we write $x \in \mathcal{L}_\alpha$ for $F(x, \alpha)$ and $x \in \mathcal{L}$ for $\exists\, \alpha \in \mathrm{On}\; x \in \mathcal{L}$.

The absoluteness of the formula $x \in \mathcal{L}$ implies

Corollary 2. $\mathcal{L} \models \forall\, x\; (x \in \mathcal{L})$.

Theorem 3.2 together with Corollary 2 implies the consistency of the statement $\forall\, x\; (x \in \mathcal{L})$ with ZF. This gave Gödel the basis to call it the *Axiom of Constructivity*: Every set is constructive. This axiom is usually written in the form $V = L$

Theorem 4 (Gödel-2)**.** $ZF + V = L \vdash AC + CH$.

The fact that $ZF + V = \mathcal{L} \vdash AC$ follows immediately from Theorem 3.1. The proof of the statement $ZF + V = \mathcal{L} \vdash CH$ is very subtle and is not discussed here. For the proofs of both Gödel Theorems see [20] or Chapter 10 of [21].

3.2. Standard Transitive Models. The natural way to prove the consistency of a certain sentence φ with the axioms of ZF is to present a model M such that $M \models ZF$. If the class M is definable by a formula $\psi(x)$, then it is called an *inner model*. The class L of constructive sets is an inner model for $V = L$, AC and CH.

The first obstacle to proving the independence of these statements is the theorem that there is no internal model, neither for V=L, nor for AC, nor for CH. See the Introduction to Chapter IV of [22]. So, any proof of the consistency of negations of these statements by presenting a standard model for each of them should be a pure existence proof. Such proof is possible only if a model is a set.

We say that a pair (M, E) is a model of ZF, if M is a set, the binary relation $E \subseteq M \times M$ is an interpretation of $\in$, and $(M, E) \models ZF$. Owing to the Gödel Completeness Theorem the existence of a model of ZF is equivalent to consistency of ZF. Thus, it cannot be proved in ZF due to the Gödel Incompleteness Theorem. The vast majority of mathematicians believe in the consistence of ZF.

An example of a set model of ZF+V=L is the set $(\mathcal{L}_\Omega, \in)$, where Ω is an inaccessible cardinal. Recall that a uncountable cardinal α is said to be *inaccessible*, if 1) it is regular and 2) $\forall\, \lambda < \alpha\; 2^\lambda < \alpha$.

The model, in which the inclusion symbol is interpreted as standard membership $\in$ in V is called a standard model. The conjecture about existence of a standard model of ZF and, thus, the conjecture about existence of an inaccessible cardinal are not provable in ZF. Moreover, they are even stronger, than the conjecture on consistency of ZF. However, there are some natural conditions such that a model (M, E) of ZF satisfying them is isomorphic to a standard model. In what follows we deal only with standard set-models of ZF either. It can be proved also that each standard model of ZF has a countable transitive submodel. The proofs and further discussion of the theorems mentioned in this paragraph can be found in the Chapter II of [22] or in Chapter 10 of [21]). The existence of a countable model is important for a proof of independence of CH, since there does not exist an uncountable standard model of ZFC, in which CH fails (see the Introduction to Chapter IV of [22]).

Since every standard transitive model M contains the set of all constructive sets in M as a minimal submodel with the same ordinals, we can assume from the beginning that M is a countable transitive model and $M \models V = L$. It means that

$$M = \bigcup_{\alpha<\lambda} \mathcal{L}_\alpha,$$

where λ is the minimal countable ordinal that is not in M.

Remark 3. Since for every $\alpha \in \mathrm{On}$ there exists a cardinal $\aleph_\alpha$, for every $\alpha \in M \cap \mathrm{On}$, $M \models \exists\, \beta = \aleph_\alpha$. Since $\beta \in M$, β is countable, thus in $V \not\models \aleph_\alpha$. This example illustrates the mentioned above fact that the formula $\beta = \aleph_\alpha$ is not absolute.

3.3. Independence of V=$\mathcal{L}$ and CH. To obtain a model, where $V = \mathcal{L}$ fails we have to extend the model M by adding some $G \subseteq P$, where $P \in M$, but $G \notin M$. We have to consider the minimal standard transitive model $M[G] \supset M \cup \{G\}$. This model consist of all elements constructible from G.

Definition 9. We say that a set x is constructible from a set G, if

$$x \in \bigcup_{\alpha \in \mathrm{On}} \mathcal{L}_\alpha[G],$$

where the definition of $\{\mathcal{L}_\alpha[G] : \alpha \in \mathrm{On}\}$ is similar to the definition of (12), but $\mathcal{L}_0[G] = \{G\}$

In our case, $M[G]$ is transitive, so the predicate $x \in \mathcal{L}$ is absolute with respect to $M[G]$. Hence, $M[G] \cap \mathcal{L} = M$; therefore, G is not constructive and $M[G] \models V \neq \mathcal{L}$. However, not for every G, that is a subset of a set $P \in M$ it is true that $M[G] \models ZFC$.

We formulate here a sufficient condition for G to have this property, which is the key point of the forcing method. To the end of this subsection the model M is fixed.

Definition 10. Let $(P, \leqslant) \in M$ be a partially ordered set (poset). If

1. A set $Q \subseteq P$ is dense in P, if $\forall p \in P\, \exists q \in Q\ p \leqslant q$.

2. Elements $p, q \in P$ are said to be compatible, if $\exists r \in P\ p \leqslant r \wedge q \leqslant r$. If p and q are incompatible, they are said to be disjoint.

3. We say that $(P, \leqslant)$ is a Boolean-type set (BTS),

a) for every two disjoint $p, q \in P\ \exists r \in P\ p \leqslant r \wedge q \leqslant r$;

b) for all $p, q \in P$ such that $p \not\leqslant q$ there exists $r \leqslant q$ that is incompatible with p.

Remark 4. In P. Cohen's approach to forcing an arbitrary poset P without a top is called a set of forcing conditions. According to the Mac Neille Theorem (see [21]) for any poset $(P, \leqslant)$ that satisfies the condition 3b) of this definition there exist a complete Boolean algebra $\mathbb{B}$ and an embedding $\iota : P \to \mathbb{B}$ that is an anti-monomorphism $(p \leqslant q \longleftrightarrow \iota(q) \leqslant \iota(p))$, and $\iota(P)$ is dense in $\mathbb{B}$. Obviously $\mathbb{B}$ is defined uniquely up to isomorphism. It is called a Dedekind–Mac Neille completion of P and denoted by $RO(P)$. The mapping ι exists even if P does not satisfy 3b), but in this case it is an anti-homomorphism. Condition 3a) follows from the others and from the Mac Neille theorem and is included only for convenience. See [23, § 2.3], and [21, Chapter 16] for proofs and details.

Definition 11. A subset $G \subseteq P$ is said to be M-generic if

1. $\forall p, q \in G\, \exists r \in G\; p \leqslant r \wedge q \leqslant r$;
2. if $p \in G,\ q \leqslant p$, then $q \in G$;
3. for every $Q \subseteq P$, such that $Q \in M$ and Q is dense in P the intersection $G \cap Q \neq \varnothing$.

Theorem 5 (Cohen 1)**.** *If $(P, \leqslant) \in M$ is a Boolean-type set, then there exist an M-generic set $G \subseteq M$, and for every such G, $M[G] \models ZFC + V \neq \mathcal{L}$.*

Proofs of existence of M-generic set and of the statement $M[G] \models V \neq \mathcal{L}$ are very simple.

Since M is countable, it has only countably many dense subsets $P_1, \ldots, P_n, \ldots$. So, there exists a sequence $p_1 \leqslant \ldots \leqslant p_n \leqslant \ldots$, such that $p_n \in P_n$ for all $n \in \mathbb{N}$. Let $G_n = \{p \in P : p \leqslant p_n\}$. Then $G = \bigcup\limits_n G_n$ is an M-generic set.

To prove that $M[G] \models V \neq \mathcal{L}$, it is enough to show that $G \notin M$. This is an immediate corollary of the following

Lemma 1. *If an M-generic set $G \in M$, then $P \setminus G \in M$ is dense.*

The proof of this Lemma is an easy exercise.

The proof of the statement $M[G] \models ZFC$, which belongs to P. Cohen [22] is very complicated and technical. It is not discussed here.

Definition 12. An extension $M[G]$ of G, where G is an M-generic set, is called a generic extension of M.

Definitions 10 and 11 are modifications of the general definitions of a *set of forcing conditions* and generic set in Section 7 of Chapter 4 in [22]. These definitions involve the partially ordered set U of forcing conditions and the set S of all sentences (formulas without free variables) of the language of ZF extended by adding to the signature the constant symbols for all constructible sets and for a generic set G. Nothe that $S \in M$. Also the binary relation $\|{-} \subseteq U \times S$ is defined by induction. The entry $p\| - \varphi$ reads like p forces φ. This relation is crucial for the proof of $M[G] \models ZFC$. In the paper [19] R. Solovay introduced almost the same Definition (I.3) of M-generic sets as Definition 11, which he, keeping in mind to use Boolean valued models, called M-generic filters. The insignificant difference is that Solovay used arbitrary rather than Boolean-type posets.

The concept of an M-generic set and its properties can be considered as definitions and theorems of ZF, if we assume that M is an arbitrary countable family of dense subsets of P. The proof of the existence of an M-generic set and that this set does not belong to M carry over to this case without change. Below the notation $\mathcal{X}$ is used for a countable family in ZF, since the notation M is fixed for the countable standard transitive model $ZF + V = \mathcal{L}$ throughout the entire Section 3.

Given an infinite set Γ put

$$P^\Gamma = \big\{f : f \text{ is a function, } \operatorname{dom}(f) \in \mathcal{P}^{fin}(\Gamma),\ \operatorname{range}(f) \subseteq \{0, 1\}\big\}.$$

We start with considerations in ZF. Put for p, q and $p \leqslant q := \operatorname{graph}(p) \subseteq \operatorname{graph}(q)$. Let $\mathcal{X}$ be a countable family of dense subsets of P^Γ. It is easy to see that $(P^\Gamma, \leqslant)$ is a Boolean type poset and if G is an $\mathcal{X}$-generic set, then G is a linearly ordered subset of P^Γ. Let $f[G] = \bigcup G$, then $f[G] : \Gamma \to \{0,1\}$. Consider the set $C(\Gamma) = \{0,1\}^\Gamma$ endowed with Tychonoff's topology. This a compact set. For $\Gamma = \mathbb{N}$ this is the Cantor's continuum.

Let $\mathcal{O}_p = \{f \in\colon \operatorname{graph}(p) \subseteq \operatorname{graph}(f)\}$. Then the family $\mathcal{T}(\Gamma) = \{\mathcal{O}_p : p \in P^\Gamma\}$ is a base of a topology of $C(\Gamma)$. For $X \subseteq P$ let $O(X) = \bigcup_{p \in X} \mathcal{O}_p$. Obviously, if X is dense, then $\mathcal{O}(X)$ is dense in C in the sense of Tychonoff's topology. So its complement is nowhere dense and, thus the intersection $\bigcap\{\mathcal{O}(X) : X \in \mathcal{X}\}$ is a comeager set. Elements of this set we call *$\mathcal{X}$-generic*. Since, every function $f \in C(\Gamma)$ is a characteristic functions of a subset of Γ or, in case of $\Gamma = \mathbb{N}$ it can be regarded as a binary fraction, so we can speak about X-generic subsets of Γ or about $\mathcal{X}$-generic reals. The following proposition follows easily from definitions.

Proposition 3. *Any $f \in C(\Gamma)$ is $\mathcal{X}$-generic if and only if $f = f[G]$ for some $\mathcal{X}$-generic set G.*

Denote the bottom and the top of any Boolean algebra by $\mathbf{0}$ and $\mathbf{1}$ respectively. Let $\mathbb{B} = \{\mathbf{0},\mathbf{1}\}$. Then the Boolean algebra $\mathcal{T}(\Gamma)$ has the independent system of generators $\{B_\gamma : \gamma \in \mathcal{P}^\Gamma\}$. Here $B_\gamma = \{\mathcal{O}_p, \mathcal{O}_q\}$, where $\operatorname{dom}(p) = \operatorname{dom}(q) = \{\gamma\}, p(\gamma) = \mathbf{0}$, $q(\gamma) = \mathbf{1}$ is isomorphic to $\mathbb{B}$. In this case we write $\mathcal{T}(\Gamma) = \prod_{\gamma\in\Gamma} B_\gamma$ and say that $\mathcal{T}(\Gamma)$ is a free product of Γ copies of B.

Obviously the mapping $p \mapsto \mathcal{O}_p$ is an inverse isomorphism of posets P^Γ and $\mathcal{T}(\Gamma)$ ($p \leqslant q \longleftrightarrow \mathcal{O}_q \subseteq \mathcal{O}_p$). The poset $\mathcal{T}(\Gamma)$ is the Boolean algebra of clopen sets of $C(\Gamma)$. Its Dedekind–MacNeille (DM) completion ([23, § 2.3], and [21, Ch. 16]) is the quotient algebra of the σ-algebra of Borel sets of $C(\Gamma)$ by the ideal of meager sets [23, § 2.4(2)]. In what follows we denote it by $\mathcal{B}(\Gamma)$ and in case when $\Gamma = \mathbb{N}$, simply by $\mathcal{B}$.

All facts about Boolean algebras used below are taken from the book [23].

Proposition 4. *The algebra $\mathcal{T}(\Gamma)$ and, thus, the algebra $\mathcal{B}(\Gamma)$ satisfy the countable chain condition: any subset of each of them that consists of pairwise disjoint elements is at most countable.*

See [11, § 9.5(5)], for a proof. Assume that $|\Gamma| = \aleph_3$ and $2^{\aleph_2} = \aleph_3$ (GCH). Then

$$\aleph_3 = \left(2^{\aleph_0}\right)^{\aleph_2}. \tag{13}$$

So, $\mathcal{B}(\Gamma)$ is isomorphic to $(\mathcal{B})^{\aleph_3}$. This algebra includes as a dense subalgebra the algebra $\mathcal{T}(\aleph_3)$ that is the free product of $\aleph_2$ copies of the algebra $\mathcal{T}(\mathbb{N}) = \mathcal{T}(\aleph_0)$. A similar algebra was used by P. Cohen in the proof of independence of CH as a poset of forcing conditions.

Let's return to our standard transitive model M of ZFC+V=L to present the main ideas of this proof that is based on the original forcing, with improvements by Solovay [19], Section 2. Another proof of this theorem, which uses only Boolean valued models is contained in the book [11, 9.5].

Denote by $\omega_k \in M$ ($k = 1,2,3$) a countable ordinal in M such that $M \models \omega_k = \aleph_k$ and let G be an M-generic filter on $\mathcal{T}(\omega_3)$. Since (13) holds in M, we have in M:

$$\mathcal{T}(\omega_3) = \prod_{\lambda \leqslant \omega_2} \mathcal{T}_\lambda(\omega_0) = \mathcal{T}, \tag{14}$$

where $\mathcal{T}_\lambda(\omega_0)$ is the λth copy of $\mathcal{T}(\omega_0)$. The following theorem (see e.g. [21]) is widely used in forcing.

Theorem 6 (Absoluteness of cardinals)**.** *If a poset $P \in M$ satisfies the countable chain conditions* (*see Proposition* 4), *and G is an M-generic set, then the cardinals in M and in $M[\Gamma]$ are the same.*

Theorem 7 (Cohen 2)**.** *If $G \subseteq \mathcal{T}$ is an M-generic set, then $M[G] \models 2^{\omega_0} > \omega_1$.*

By Theorem 6 $M[G] \models \omega_k = \aleph_k$. Notice, that (14) may not be valid in $M[G]$, since the definition of $\mathcal{P}(X)$ is not absolute. However, the right hand side of this equality is absolute. It is easy to see that $G \cap \mathcal{T}_\lambda(\omega_0) = G_\lambda$ is an M-generic set on $\mathcal{T}_\lambda(\omega_0)$ (see [19, § 2]). So, the subset $f(G_\lambda) = f_\lambda \subseteq \mathbb{N}$ is M-generic. The mapping $\Phi : \omega_2 \to 2^\omega$ such that $\Phi(\lambda) = f_\lambda$ is absolutely definable from G and elements of M, so $\Phi \in M[G]$. It is easy to see that Φ is injective. Indeed, if If $\varphi_\lambda = \varphi_\nu$, $\lambda \neq \mu$, then it belongs to the diagonal of $\left(\{0,1\}^{\mathbb{N}}\right)^2$ and (f_λ, f_ν) is M-generic with respect to the M-generic set $\mathcal{T}_\lambda(\omega_0) \cdot \mathcal{T}_\nu(\omega_0) \cap G$. This is impossible, since the complement of the diagonal is a dense open set. Now $M[G] \models \neg CH$ follows from the obvious inequalities:

$$|\Phi(\omega_1)| = \omega_1 < \omega_2 = |\Phi(\omega_2)| \leqslant 2^\omega. \ \rhd$$

3.4. Solovay's forcing. Random numbers. There is another proof of independence of CH, that starts from the same Boolean algebra of forcing conditions $\mathcal{T}(\Gamma)$. For any set Γ, the product measure μ is defined on the Boolean algebra $\mathcal{T}\ (\Gamma)$ of clopen sets of the compact space $\{0,1\}^\Gamma$ by the formula $\mu(O_p) = 2^{-|\operatorname{dom} p|}$. This measure is extended to a σ-additive measure on the σ-algebra $\mathcal{B}$ of all Borel sets. Let $\mathcal{B}_\mu(\Gamma) = \mathcal{B}/\{A \in \mathcal{B} : \mu(A) = 0\}$. This is a complete Boolean algebra with strictly positive completely additive measure that satisfies countable chain condition. The algebra $\mathcal{B}_\mu(\Gamma)$ is a completion of $\mathcal{T}(\Gamma)$ with respect to the metric $\rho(A_1, A_2) = \mu(A_1 \Delta\, A_2)$. This completion is not isomorphic to the Dedekind–MacNeille completion and we cannot use the algebra $\mathcal{T}(\Gamma)$ for the set of forcing conditions, since $\mathcal{T}(\Gamma)$ is not orderly dense in $\mathcal{B}_\mu(\Gamma)$. In this case we must use the set $\mathcal{B}_\mu(\Gamma) \setminus \{\mathbf{0}\}$ itself as the set of forcing conditions. In order to do so, we have to make sure that this set is absolute. This is true for the case of $|\Gamma| = \aleph_0$. So, as above put $\Gamma = \mathbb{N}$. Since $\mathcal{T}(\mathbb{N})$ is countable and is defined by an absolute formula, the set of all Borel subsets of $\{0,1\}^{\mathbb{N}}$ has cardinality $2^{\aleph_0}$. So, that they can be coded by elements of $\{0,1\}^{\mathbb{N}}$. Moreover, this coding is absolute. If A_β is a Borel set coded by $\beta \in \{0,1\}^{\mathbb{N}}$, then $\alpha \in A_\beta$, $\alpha \notin \mathbb{A}_\beta$, $A_\alpha \subseteq A_\beta$, $\mu(A_\beta) = 0$, etc. are absolute. See Section II of [19] for details. The crucial role for all absoluteness proofs of theorems about Borel coding plays the following.

Lemma 2 (Shoenfield)**.** *Every formula ZF of the form $\exists\, y\, \forall\, x\, \varphi(x, y, c)$ such that x and c range over $\mathbb{N}^{\mathbb{N}}$ (equivalently over $\{0,1\}^{\mathbb{N}}$) and all quantifiers in φ are of the form $\forall\, n \in \mathbb{N}$ or $\exists\, n \in \mathbb{N}$ is absolute with respect to any transitive model.*

Let M be the same model as above and $D \setminus \{\mathbf{0}\}$ is the set of forcing conditions, where $D \in M$ is an arbitrary complete Boolean algebra. Then $G \subseteq D$ is an M-generic set if and only if G is an ultrafilter on D and for any set $E \subseteq G,\ E \in M$ on has $\bigwedge E \in M$ [21]. An M-generic set in a complete Boolean algebra is said to be an M-generic filter.

The analog of M-generic elements (functions, sets, numbers) of the previous subsection are M-random elements. Consider the algebra $\mathcal{B}_\mu$. Given a Borel set $A_\beta \subseteq \{0,1\}^{\mathbb{N}}$, denote by $[A_\beta]$ its equivalence class.

Definition 13. We say that a function $f \in \{0,1\}^{\mathbb{N}}$ is M-random if $f \in A_\beta,\ \beta \in M$ such that $[A_\beta] = \mathbf{1}$ (i. e. $\mu(A_\beta) = 1$). In other words f is M-random, if it avoids every Borel set of measure zero in M.

Proposition 5. *If f is M-random, if and only if $\{[A_\beta] : \beta \in M,\ f \in A_\beta\}$ is an M-generic filter.*

Corollary 3. *If G is an M-generic filter in $\mathcal{B}_\mu$, then almost all elements of $\{0,1\}^{\mathbb{N}}$ in $M[G]$ are M-random.*

Let $\mathcal{B}_\mu(\kappa) = \prod_{\lambda<\kappa} \mathcal{B}_\mu^\lambda$, where κ is a cardinal. Corollary 3 is true for $\kappa = \omega_0$. This is obvious, since $\omega_0^2 = \omega_0$, and the results about coding the of Borel sets are true for this case.

Corollary is not true for the Boolean algebra $\mathcal{B}_\mu(\kappa)$ with uncountable κ, though it also satisfies the countable chains condition. The proof of the fact that $M[G] \models \neg CH$, where G is an M-generic filter in $\mathbb{B}_\mu(\omega_3)$ repeats the proof of this result for algebra 14 almost without changes.

3.5. Solovay's results. The most impressive independence results after P. Cohen were obtained by R. Solovay [19] and by R. Solovay and S. Tennenbaum [24]. In the first of them the following two theorems were proved.

Theorem 8 [19, Theorem 2]**.** *If the existence of an inaccessible cardinal is consistent with ZFC, then there exists a model of ZF+DC,* in which every set of reals is Lebesgue measurable, has the Baire property, and either is at most countable or contains a perfect subset.*

This theorem follows from

Theorem 9 [19, Theorem 1]**.** *If the existence of an inaccessible cardinal is consistent with ZFC, then there exists a model of ZFC, in which all three statements of the previous Theorem hold for the class of sets definable from a sequence of ordinals.*

The class of sets definable from a sequence of ordinals is very big and important. For example it includes all projective sets. There were two long standing problems in descriptive set theory. About a century ago Suslin constructed an example of a set that is a continuous image of a Borel set, but is not a Borel set. He called such sets A-sets.** P. S. Aleksandroff proved that every uncountable Borel set contains a perfect subset and, thus, has the cardinality of continuum. This implied that every A-set has a perfect subset. In attempts to prove CH N. N. Luzin the teacher of Aleksandroff and Suslin suggested to study the hierarchy: A-sets, their complements (CA-sets), the continuous images of CA-sets (PCA-sets), etc. Luzin showed that at each stage of this hierarchy some new sets appear. The sets of this hierarchy are called projective sets. They are studied in descriptive set theory. The first questions about projective set were about cardinality, Lebesgue measurability and Baire property of these sets. The difficulties started at the very first steps stages. It was known that A-sets and, thus, CA-sets are Lebesgue measurable, but the problem of cardinality of CA-sets (does each uncountable set contain perfect subsets?) and the problem of measurability of PCA-sets remained open. In [20] Gödel announced that $V = L$ implies the existence of an uncountable CA-set without a perfect subset and of a nonmeasurable PCA-set. He did not publish a proof of these statements. They were proved later by P. S. Novikov. Thus, from the results of [19, 25] followed the independence of the problems of cardinality and Lebesgue measurability for all projective sets followed, modulo, of course, the hypothesis of an inaccessible cardinal. That is why the following problem formulated in [19] was so important.

*Is it possible to eliminate the assumption about inaccessible cardinals from the statements of this theorem, concerning the Baire property and Lebesgue measurability?****

This problem was solved by S. Shelah [26] who proved that the elimination is possible for the Baire property but impossible for the Lebesgue measurability.

In his proof Solovay used the model $M[G]$, where G is an M-generic filter in the $(\aleph_0, \Omega)$-algebra Lévy (see Model VI in Chapter 20 of [21]), where $M \models \Omega$ is an inaccessible cardinal. This algebra is the Dedekind–MacNeille completion of the following absolute set of forcing conditions

$$P = \bigl\{p : p \subseteq \{\langle \alpha, n, \beta \rangle : \beta, \alpha < \Omega\rangle,\ |p| < \omega_0\}\bigr\}.$$

* DC—the axiom of dependent choice.

** Now they are called Suslin's sets.

*** The impossibility of removing this assumption for the statement concerning perfect subsets follows from the earlier paper of R. Solovay [25].

It is easy to see that, if $G \cap P$ is an M-generic set, then $\bigcup G$ consists of bijections of ω_0 on each infinite ordinal $\alpha \in \Omega$. Thus, $M[G] \models \Omega = \aleph_1$.

The following proposition is needed in the future discussion.

Proposition 6. *Let $s \in M[G]$ be a countable sequence of ordinals. Then the following hold:*

1) *Almost all real numbers in $M[G]$ are $M[s]$-random.*

2) *There exists $M[s]$-generic set $H \subseteq P$ such that $M[G] = M[s][H]$.*

The inaccessible cardinal is crucial for the Item 1 of this Proposition.

Without the hypothesis of the existence of inaccessible cardinal a weaker version of the Solovay's Theorem 1 was proved by G. Saks [8].

Theorem 10. *The existence of an extension of the Lebesgue measure to an invariant σ-additive measure on all sets definable from countable sequences of ordinals is consistent with ZF+DC.*

Notice that the simplest and most known example of a non-measurable set—the Vitali set is non-measurable with respect to any extension of the Lebesgue measure that satisfies the conditions of Theorem 10.

The corresponding version of Solovay's Theorem 2 was not even formulated in [8] and it is not even clear whether it can be proved on the way used in [8].

This theorem was proved in my PhD thesis. The result was announced in the article [13]. The detailed proofs of more general versions of both Solovay's Theorems are published in my PhD thesis and in a preprint [27] deposited in the VINITI (All Union Institute of the Science and Technology Information):

Theorem 11. *Let α be an arbitrary ordinal definable in ZF. Denote by* $\mathrm{Base}(X,\beta)$ *and* $\mathrm{Ext}(X,\beta)$ *the statements*

1. *"X is a σ-compact group with the base of topology of cardinality β";*

2. *"In a σ-compact group X the left Haar measure can be extended to a left invariant σ-additive measure defined on all subsets of X definable by a β-sequence of ordinals"*

respectively. Then the following proposition is consistent with ZFC:

$$\forall X \forall \beta < \aleph_\alpha < |\mathbb{R}| \ \ (\mathrm{Base}(X,\beta) \longrightarrow \mathrm{Ext}(X,\beta))$$

Theorem 12. *Let α be an arbitrary ordinal definable in ZF. Denote by* $\mathrm{Base}(X,\beta)$ *and* $\mathrm{Ext}'(X,\beta)$ *the statements*

1. *"X is a σ-compact group with the base of topology of cardinality β";*

2. *"In a σ-compact group X the left Haar measure can be extended to a left invariant σ-additive measure on all subsets of X"*

respectively. Then the following proposition is consistent with ZF+AD+AC$_\beta$:

$$\forall X \forall \beta < \aleph_\alpha < |\mathbb{R}| \ \ (\mathrm{Base}(X,\beta) \longrightarrow \mathrm{Ext}(X,\beta))\,,$$

where AC$_\beta$ is a the Axiom of Choice for a family of cardinality β.

Both Solovey's theorems almost automatically carry over to the case of Haar measures on locally compact separable groups. In the case of non-separable groups, some problems arise with the absolute coding of Borel sets, due to the fact that the Schonfield Absolute Lemma holds only for countable sequences of positive integers. In the proofs of Theorems 11 and 12 these difficulties are overcome.

The independence of Suslin's hypothesis and Martin's Axiom were proved in [24]. This paper was also very important for the Boolean valued analysis, since the iterated forcing and the technique of ascents and descents were introduced there.

Acknowledgments. I am very grateful to my supervisor in postgraduate studies V. A. Lyubetskiy, who taught me the method of forcing, to my friends and colleagues A. G. Kusraev and S. S. Kutateladze for more, than forty years of friendship and fruitful cooperation in the field of non-standard methods of analysis, as well as my friend and co-author of many papers L. Yu. Glebsky for his great help in preparation of this paper.

References

1. *Kusraev, A. G. and Kutateladze, S. S. Nonstandard Methods of Analysis*, Dordrecht, Kluwer Academic Publishers, 1995, 398 p.
2. *Maltsev, A. I.* A General Method for Obtaining Local Theorems in Group Theory, *Uchen. Zap. Ivanovsk. Gos. Ped. Inst. Uch. Zap. Fiz.-Mat. Fak.*, 1941, vol. 1, no. 1, pp. 3–9 (in Russian).
3. *Robinson, A. Nonstandard Analysis, Revised Edition*, Princeton, Princeton University Press, 1996, 312 p.
4. *Albeverio, S., Fenstad, J. E. and et al. Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, Orlando, etc., 1990, 514 p.
5. *Scott, D.* A Proof of the Independence of the Continuum Hypothesis, *Mathematical System Theory*, 1967, vol. 1, no. 2, pp. 89–111. DOI: 10.1007/BF01705520.
6. *Scott, D. Lectures of Boolean Valued Models of Set Theory*, Amer. Math. Soc., Summer Institute on Axiomatic Set Theory, UCLA, 1967.
7. *Scott, D. and Solovay, R. Boolean Valued Models of Set Theory, Axiomatic Set Theory*, Proc. Sympos. Pure Math., vol. 13, Amer. Math. Soc., Providence, R. I.
8. *Saks, G.* Measure-Theoretical Uniformity in Recursion Theory and Set Theory, *Trans. Amer. Math. Soc.*, 1969, vol. 142, no. 2, pp. 381–420. DOI: 10.1090/S0002-9947-1969-0253895-6.
9. *Takeuti, G. Two Applications of Logic to Mathematics*, Tokio and Princeton, Iwanami and Princeton University Press, 1978, p. 148.
10. *Takeuti, G.* A Transfer Principle in Harmonic Analysis, *Journal of Symbolic Logic*, 1979, vol. 44, no. 3, pp. 417–440.
11. *Kusraev, A. G. and Kutateladze, S. S. Introduction to Boolean Valued Analysis*, Moscow, Nauka, 2005, 526 p. (in Russian).
12. *Kusraev, A. G. and Kutateladze, S. S. Boolean Valued Analysis: Selected Topics*, Vladikavkaz, Southern Mathematical Institute, VSC RAS & RNO-A, 2014, 400 p.
13. *Gordon, E. I.* Real Numbers in Boolean Valued Models of Set Theory, and K-Spaces, *Soviet Mathematics Doklady*, 1977, vol. 18, pp. 1481–1484.
14. *Loeb, P. A. and Wolff, M. Ph. Nonstandard Analysis for the Working Mathematician*, Dordrecht, Kluwer Academic Publishers, 2000, 325 p.
15. *Davis, M. Applied Nonstandard Analysis*, New York, etc., John Wiley & Sons, 1977.
16. *Manin, Yu. I. A Course in Mathematical Logic for Mathematicians. Second Edition*, New York, etc., Springer, 2010, p. 403.
17. *Gordon E. I. Nonstandard Methods in Commutative Harmonic Analysis*, Amer. Math. Soc., Providence, R. I., 1977, 180 p.
18. *McDuff, D.* On Theory of II_1 Factors, *Russian Mathematical Surveys*, 1970, vol. 25, no. 1, pp. 29–51 (in Russian).
19. *Solovay, R.* A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable, *Annals of Mathematics*, 1970, vol. 92, no. 1, pp. 1–56. DOI: 10.2307/1970696.
20. *Gödel, K. Consistency of Continuum Hypothesis*, Princeton University Press, Princeton, 1940.
21. *Jech, T. J. Lectures in Set Theory with Particular Emphasis on the Method of Forcing*, Lecture Notes in Mathematics, vol. 217, Berlin, Springer-Verlag, 1971. DOI: 10.1007/BFb0061131.
22. *Cohen, P. J. Set Theory and the Continuum Hypothesis*, New York, etc., Benjamin, 1966.
23. *Vladimirov, D. A. Boolean Algebras in Analysis*, Dordrecht, Springer Science & Business Media, 2002.
24. *Solovay, R. and Tennenbaum, S.* Iterated Cohen Extensions and Souslin's Problem, *Annals of Mathematics*, 1971, vol. 94, no. 2, pp. 201–245. DOI: 10.2307/2272650.
25. *Solovay, R.* On the Cardinality of $\sum_1^2$-Sets of Reals, *Foundations of Mathematics, Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel*, Berlin, Springer Verlag, 1969, pp.58–73. DOI: 10.1007/978-3-642-86745-3_7.
26. *Shelah, S.* Can you Take Solovay's Inaccessible Away? *Israel Journal of Mathematics*, 1984, vol. 48, pp. 1–47. DOI: 10.1007/BF02760522.
27. *Gordon, E. I. On the Extension of Haar Measure in σ-Compact Groups*, Moscow, Lenin Moscow State Pedagogical Institute, VINITI, no. 1243-81, 1980 (in Russian).

Received 8 November, 2019

Evgeny I. Gordon
Eastern Illinois University,
600 Lincoln Av., Charleston 61920, Illinois, USA,
Professor Emeritus
E-mail: `yigordon@eiu.edu`, `gordonevgeny@gmail.com`

Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 25–41

НЕКОТОРЫЕ ЗАМЕЧАНИЯ О НЕСТАНДАРТНЫХ МЕТОДАХ АНАЛИЗА. I

Гордон Е. И.[1]

[1] Восточный Иллинойский Университет,
США, Иллинойс, 61920, Чарльстон, пр. Линкольна, 600
E-mail: `yigordon@eiu.edu`, `gordonevgeny@gmail.com`

Аннотация. В этой и последующей статьях обсуждаются два наиболее известных нестандартных метода математического анализа — инфинитезимальный анализ А. Робинсона и булевозначный анализ, затрагивается история их возникновения, общие черты и различия, приложения и перспективы. В этой статье содержится обзор инфинитизимального анализа и метода вынуждения. Изложение рассчитано на читателя знакомого лишь с самыми начальными понятиями математической логики — языком логики предикатов 1-го порядка и его интерпретациями. Желательно иметь также некоторое представление о формальных доказательствах и аксиоматике теории множеств Цермело — Френкеля. При изложении инфинитезимального анализа особое внимание уделяется формализации предложений обычной математики в языке первого порядка для суперструктуры. Изложение метода форсинга предваряется кратким обзором результата К. Гёделя о совместимости аксиомы выбора и гипотезы континуума с аксиоматикой Цермело — Френкеля. Следующая статья будет посвящена, булевозначным моделям и булевозначному анализу. Особое внимание будет уделено истории их возникновения.

Ключевые слова: булевозначный анализ, нестандартный анализ, метод вынуждения.

Mathematical Subject Classification (2010): 03C90, 46A40.

Образец цитирования: *Gordon E. I.* Some Remarks about Nonstandard Methods in Analysis. I // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 25–41 (in English). DOI: 10.23671/VNC.2019.21.44619.