Владикавказский математический журнал
2011, Том 13, Выпуск 1, С. 59–70

УДК **512.81**

ON CONSERVATION LAWS IN AFFINE TODA SYSTEMS[1]

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With the help of certain matrix decomposition and projectors of special forms we show that non-Abelian Toda systems associated with loop groups possess infinite sets of conserved quantities following from essentially different conservation laws.

Key words: non-Abelian Toda systems, loop groups, symmetries and conservation laws.

1. Introduction

Classical integrable systems are understood as nonlinear differential equations which can be integrated in one or another sense. According to Liouville [1], the integrability of a system with $2n$ degrees of freedom is related to the existence of an Abelian n-torus. However, it is not always possible to reveal such tori explicitly in the phase space. For some complicated systems, for example, one can construct general solutions explicitly in quadratures, but with no way obvious to find respective "action-angle" variables. Or one can find certain number of integrals of motion, required by the Liouville–Arnold's theorem, but with the question of involutivity of such quantities still to be answered. In any case, the integrability of a nonlinear system is based on its symmetries. The most popular viewpoint at present is that a system is regarded as integrable, if for the equations describing it one can propose a constructive way to find their solutions, and one can prove that it possesses sufficient number of conserved quantities.

Of special interest are two-dimensional systems which can be represented in the form of the zero curvature condition. These are, for example, the Liouville equation, the Nonlinear Schrödinger equation and its modifications, the KdV equation, the sine-Gordon equation that, also describing, as physicists believe, a system equivalent to the Thirring model, and others [2, 3]. In particular, all these systems are very attractive for investigations in mathematical physics and differential geometry. Here we consider Toda systems associated with loop groups, which actually cover most interesting examples of completely integrable systems [4]. In the case of Toda systems associated with finite-dimensional Lie groups one obtains simple conservation laws whose densities give rise to the so-called W-algebras [5].

It is usually claimed that the complete integrability of such systems should be due to the existence of infinite sets of conserved quantities produced by some current conservation laws. This observation has not received a form of a proved statement yet, and so, it still attracts much attention in the scope of the theory of integrable systems. Besides, it is believed that the classification of integrable systems can be performed along the lines of a "symmetry approach", which is also using the conservation laws [6, 7].

[1]Partially supported by a grant from Russian Foundation for Basic Research, project № 08-01-00009.

Hence, the main purpose of our paper is to present, in justifiable detail, current conservation laws for non-Abelian Toda systems, which should produce infinite sets of conserved quantities responsible for the integrability of the nonlinear equations under consideration. In particular, our consideration gives a generalization of what was presented in [8]. Our approach, based upon a detailed consideration of the matrices $c_\pm$ entering the equation under consideration, allows one to separate explicitly proper current conservation laws from the simpler WZNW-type conservation laws and those special relations leading to nonlinear W-algebra extensions of Virasoro algebras.

2. Toda systems and their simplest symmetries

Let G be a complex or real Lie group and $\mathfrak{g}$ its Lie algebra. Consider a trivial principal fiber bundle $M \times G \to M$ with G as the structure group. Here M is a two-dimensional base manifold, being as usual either $\mathbb{C}$ or $\mathbb{R}^2$, where standard coordinates z^- and z^+ are introduced. The Toda equations can be obtained from the zero-curvature condition for a connection on $M \times G \to M$ imposing grading and gauge-fixing constraints on elements of $\mathfrak{g}$. In this, the connection is identified with a $\mathfrak{g}$-valued 1-form on M, and as such, can be decomposed over basis 1-forms,

$$\omega = \omega_- \mathrm{d}z^- + \omega_+ \mathrm{d}z^+,$$

where the components ω_-, ω_+ are $\mathfrak{g}$-valued functions on M. One says that the connection ω is flat, and so, the corresponding curvature is zero, if and only if, in terms of the components, one has

$$\partial_- \omega_+ - \partial_+ \omega_- + [\omega_-, \omega_+] = 0. \tag{2.1}$$

The partial derivatives ∂_- and ∂_+ are taken over the standard coordinates z^- and z^+, respectively.

We study models based on the loop Lie group $G = \mathscr{L}(\mathrm{GL}_n(\mathbb{C}))$ given by the smooth mappings from the circle $\mathbb{S}^1$ to the Lie group $\mathrm{GL}_n(\mathbb{C})$ with the group composition defined point-wise. Its Lie algebra $\mathfrak{g}$ is the loop algebra $\mathscr{L}(\mathfrak{gl}_n(\mathbb{C}))$ formed by the smooth mappings from $\mathbb{S}^1$ to $\mathfrak{gl}_n(\mathbb{C})$. The circle $\mathbb{S}^1$ is parameterized here by the set of complex numbers λ of modulus 1, and extending the unit circle to the whole Riemann sphere we introduce the so-called *spectral parameter* denoted here by the same symbol λ. Here we consider a non-Abelian analogue of the *standard grading* [9, 10] and use a representation where the gradation is over powers of λ.

To obtain nontrivial systems from the zero-curvature condition one should impose on ω grading and gauge-fixing conditions [4, 11]. We assume that $\mathfrak{g}$ is endowed with a $\mathbb{Z}$-gradation,

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m, \quad [\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n},$$

and for some positive integer l the subspaces $\mathfrak{g}_{-m}$ and $\mathfrak{g}_{+m}$ for $0 < m < l$ are trivial. Note that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$. Denote by G_0 the connected Lie subgroup of G which has $\mathfrak{g}_0$ as its Lie algebra. It can be shown that the connection ω can be brought to a form given by the components[2]

$$\omega_-(\lambda) = \gamma^{-1} \partial_- \gamma + \lambda^{-1} c_-, \quad \omega_+ = \lambda \gamma^{-1} c_+ \gamma, \tag{2.2}$$

[2] Hereafter we put $l = 1$, that can be done without any loss of generality.

where γ is a mapping from M to G_0, and c_- and c_+ are some fixed mappings from M to $\mathfrak{g}_{-l}$ and $\mathfrak{g}_{+l}$, respectively, such that $\partial_+ c_- = 0$, $\partial_- c_+ = 0$. The zero-curvature condition for the connection with the components (2.2) produces the Toda system that is the nonlinear matrix differential equation

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma]. \tag{2.3}$$

The Toda equation can also be written in another equivalent form,

$$\partial_- \left(\partial_+\gamma\gamma^{-1}\right) = [\gamma c_-\gamma^{-1}, c_+], \tag{2.4}$$

in which case the connection components are

$$\omega_- = \lambda^{-1}\gamma c_-\gamma^{-1}, \quad \omega_+ = -\partial_+\gamma\gamma^{-1} + \lambda c_+. \tag{2.5}$$

When the Lie group G_0 is Abelian the corresponding Toda system is said to be Abelian, otherwise one deals with a non-Abelian Toda system [11–13]. The complete list of the Toda systems associated with finite-dimensional Lie groups is presented in [14]. For the case of loop Lie groups the respective classification was performed in [15, 16].

Let η_- and η_+ be some mappings from M to G_0 subject to the conditions

$$\partial_+\eta_- = 0, \quad \partial_-\eta_+ = 0.$$

If a mapping γ satisfies the Toda equation (2.3) then the mapping

$$\gamma' = \eta_+^{-1}\gamma\eta_- \tag{2.6}$$

satisfies the Toda equation (2.3) with the functions c_- and c_+ replaced by

$$c'_- = \eta_-^{-1}c_-\eta_-, \quad c'_+ = \eta_+^{-1}c_+\eta_+. \tag{2.7}$$

In this sense the Toda equations defined with the fixed functions $c_\pm$ and $c'_\pm$ related by (2.6), (2.7) are equivalent. It is clear that conservation laws when established in terms of such transformed quantities should be the same as they would be for the original ones.

If the mappings η_- and η_+ satisfy the relations $\eta_-^{-1}c_-\eta_- = c_-$, $\eta_+^{-1}c_+\eta_+ = c_+$, then the mapping γ' satisfies the same Toda equation as the mapping γ. Hence, in such a case the transformation described by (2.6) is a symmetry transformation for the Toda equations. It gives simplest symmetries of the Toda equations, inherited from the WZNW theory [5].

Now, the mapping γ takes values in the Lie group of complex non-degenerate block diagonal $n \times n$ matrices, possessing the partition just according to the $\mathbb{Z}$-gradation under consideration, that is

$$\gamma = \begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_r \end{pmatrix},$$

with Γ_a taking values in the space of complex non-degenerate $k_a \times k_a$ matrices. The fixed matrix-valued mappings c_- and c_+ are explicitly of the forms

$$c_- = \begin{pmatrix} 0 & 0 & \cdots & 0 & C_{-r} \\ C_{-1} & 0 & \cdots & 0 & 0 \\ 0 & C_{-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{-(r-1)} & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_{+1} & 0 & \cdots & 0 \\ 0 & 0 & C_{+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{+(r-1)} \\ C_{+r} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where C_{-a} denotes a $k_{a+1} \times k_a$ submatrix, and the block C_{+a} means a $k_a \times k_{a+1}$ submatrix. It is worthwhile noting that the following relations hold:

$$s^{-1} S c_- S^{-1} = c_-, \quad s S c_+ S^{-1} = c_+, \tag{2.8}$$

where S is a constant diagonal $n \times n$ matrix

$$S = \begin{pmatrix} sI_{k_1} & 0 & \cdots & 0 \\ 0 & s^2 I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{k_r} \end{pmatrix}, \tag{2.9}$$

with s being the rth principal root of unity, $s = \mathrm{e}^{2\pi \mathrm{i}/r}$, so that in terms of the block submatrices

$$S_{a,b} = s^a I_{k_a} \delta_{ab}.$$

The matrix S satisfies the relation

$$S^r = I_n,$$

where I_n is the unit $n \times n$ matrix.

In terms of the submatrices the $n \times n$ matrix Toda equation (2.3) takes the form

$$\partial_+ \left(\Gamma_a^{-1} \partial_- \Gamma_a \right) = C_{-(a-1)} \Gamma_{a-1}^{-1} C_{+(a-1)} \Gamma_a - \Gamma_a^{-1} C_{+a} \Gamma_{a+1} C_{-a}, \tag{2.10}$$

with $a = 1, 2, \ldots, r, \ldots$ and the periodicity condition imposed as follows:

$$\Gamma_{a+r} = \Gamma_a, \quad C_{-(a+r)} = C_{-a}, \quad C_{+(a+r)} = C_{+a}.$$

These submatrices, if transformed according to (2.6), (2.7), would look here as follows:

$$\Gamma'_a = \eta_{+a}^{-1} \Gamma_a \eta_{-a}, \quad C'_{-a} = \eta_{-(a+1)}^{-1} C_{-a} \eta_{-a}, \quad C'_{+a} = \eta_{+a}^{-1} C_{+a} \eta_{+(a+1)},$$

with the block diagonal matrices $\eta_\pm$ defined by $(\eta_\pm)_{ab} = \eta_{\pm a} \delta_{ab}$.

3. The mappings c_- and c_+ as linear operators

Denote by k_* the minimum value of the partition numbers $\{k_a\}$ and suppose that the fixed mappings c_- and c_+ are chosen to be constant. Besides, we assume that the submatrices C_{-a} and C_{+a} are of maximum ranks, and they respect the commutativity between c_- and c_+. Consider the eigenvalue problem for the linear operators c_- and c_+. The corresponding characteristic polynomial is $(-1)^n t^{n-rk_*} (t^r - 1)^{k_*}$, and this gives us the characteristic equation

$$t^{n-rk_*} \prod_{a=1}^{r} (t - s^a)^{k_*} = 0.$$

Therefore, the spectra of the eigenvalue problems corresponding to the mappings c_- and c_+ consist of the zero eigenvalue of algebraic multiplicity $n - rk_*$ and nonzero eigenvalues being powers of the rth root of unity of algebraic multiplicity k_*. To see this, it is sufficient to use the transformations (2.6), (2.7) with a special choice of the mappings η_- and η_+ and recall the fact that eigenvalues of similar matrices do coincide.

The eigenvalue problem relations

$$c_-\psi^{(b)} = s^{-b}\psi^{(b)}, \quad c_+\psi^{(b)} = s^b\psi^{(b)}, \quad b = 1, 2, \dots, r, \tag{3.1}$$

are fulfilled with the eigenvectors represented by $n \times k_*$ matrices $\psi^{(b)}$,

$$\psi^{(b)t} = \left(\psi_1^{(b)t}, \dots, \psi_a^{(b)t}, \dots, \psi_r^{(b)t}\right),$$

where its block submatrices are

$$\psi_a^{(b)} = s^{ab}\xi_a, \quad \psi_a^{(-b)} = s^{-ab}\xi_a,$$

with constant $k_a \times k_*$ submatrices ξ_a satisfying the conditions

$$C_{-a}\xi_a = \xi_{a+1}, \quad C_{+a}\xi_{a+1} = \xi_a.$$

Denote by p the rank of the matrix c_-. Then we have $n - p = \dim\ker c_-$. It means that there are an $n \times (n-p)$ matrix u and an $(n-p) \times n$ matrix $v^\vee$, corresponding to the zero eigenvalue of c_-,

$$c_- u = 0, \quad v^\vee c_- = 0. \tag{3.2}$$

Note that u and $v^\vee$ are orthogonal to $\psi^{(b)}$ for every $b = 1, 2, \dots, r$. In general, however, the algebraic multiplicity of an eigenvalue does not coincide with its geometric multiplicity, the former is just non less than the latter, and so, $n-p \leqslant n-rk_*$. In particular, it is exactly what happens to the zero eigenvalue of c_-, so that its algebraic multiplicity we have received from the characteristic equation, does not coincide with the dimension of the corresponding null subspace. Hence, one should remember that the rank of c_- might be greater than the total number of its nonzero eigenvalues. It is a consequence of the fact that c_- contains a non-diagonalizable part, corresponding to the zero eigenvalue, that is actually the nilpotent part of c_- according to its Jordan normal form. It is clear that for the case under consideration $p = \operatorname{rank} c_- = \sum_{a=1}^r \min(k_a, k_{a+1}) \geqslant rk_*$.

Let v be the vector dual to the left null vector $v^\vee$ of the matrix c_-, and $u^\vee$ the dual of its right null vector u. They are $n \times (n-p)$ and $(n-p) \times n$ matrices subject to the conditions $v^\vee v = I_{n-p}$ and $u^\vee u = I_{n-p}$. Treating c_- as a matrix of a linear operator acting on an n-dimensional vector space V, that is $c_- : V \to V$, we see that the latter can be decomposed into a direct sum as $V = V_0 \oplus V_1$, where V_1 is an rk_*-dimensional subspace spanned by the ψ-eigenvectors of c_- with nonzero eigenvalues, actually, $V_1 = \operatorname{im} c_-$; and V_0 is simply defined to be its orthogonal complement. Besides, we can perform another decomposition of V, namely $V = U_0 \oplus U_1$, where $U_0 = \ker c_-$, and so, it is spanned by the columns of the $n \times (n-p)$ matrix u, while U_1 is just the orthogonal complement to U_0. These decompositions induce dual decompositions $V^* = V_0^* \oplus V_1^*$ and $V^* = U_0^* \oplus U_1^*$, such that V_1^* is spanned by the left eigenvectors of c_- being dual to $\psi^{(b)}$, and V_0^* is determined to be the orthogonal complement to V_1^*; further, U_0^* is spanned by the left null vectors (the rows of $v^\vee$) of c_-, and U_1^* appears to be its orthogonal complement.

For the case under consideration the linear space V is isomorphic to its dual V^*. In view of the above discussion of properties of the eigenvalues of c_-, we see that $n - rk_* = \dim V_0 \geqslant \dim U_0 = n - p$, and $p = \dim U_1 \geqslant \dim V_1 = rk_*$. It is clear also that the following relations hold: $U_0 \subseteq V_0$, $V_1 \subseteq U_1$ (and similarly for the duals). We stress on that U_0 and U_0^* are the right and left null subspaces of the linear operator c_-, while V_1 and V_1^* are subspaces generated by its right and left ψ-eigenspaces. Actually, one can write

$$V_0 \cap U_0 = U_0, \quad V_0^* \cap U_0^* = U_0^*, \quad V_1 \cap U_0 = V_1^* \cap U_0^* = \varnothing, \tag{3.3}$$

$$V_1 \cap U_1 = V_1, \quad V_1^* \cap U_1^* = V_1^*, \quad V_0 \cap U_1 \neq \varnothing, \quad V_0^* \cap U_1^* \neq \varnothing. \tag{3.4}$$

These relations are the basis for the subsequent construction of certain useful projectors. A similar treatment can also be given to the linear operator c_+ as well.

4. General current conservation laws

Let us introduce a decomposition of the identity operator id_V,

$$I_n = \Pi_1 + \Pi_0,$$

where Π_1 is a projector onto the subspace V_1, while Π_0 is the projector onto the complementary subspace V_0,

$$\Pi_0^2 = \Pi_0, \quad \Pi_1^2 = \Pi_1, \quad \Pi_1\Pi_0 = \Pi_0\Pi_1 = 0.$$

We also have

$$[c_-, \Pi_1] = 0, \quad [c_-, \Pi_0] = 0.$$

Note that the generalized non-Abelian analogue of the cyclicity property reads here simply $c_-^r = c_+^r = \Pi_1$. Further, we introduce projectors onto the null subspace U_0 and its dual space U_0^*,

$$R_0 = uu^\vee, \quad L_0 = vv^\vee,$$

such that, together with their orthogonal complements $R_1 = I_n - R_0$ and $L_1 = I_n - L_0$, they reveal the following properties:

$$L_1\Pi_1 = \Pi_1, \quad L_0\Pi_1 = 0, \quad L_0\Pi_0 = L_0, \tag{4.1}$$

$$\Pi_1 R_1 = \Pi_1, \quad \Pi_1 R_0 = 0, \quad \Pi_0 R_0 = R_0, \tag{4.2}$$

and also

$$L_1\Pi_0 = L_1 - \Pi_1 = \Pi_0 - L_0, \quad \Pi_0 R_1 = R_1 - \Pi_1 = \Pi_0 - R_0. \tag{4.3}$$

These relations are a direct consequence of (3.3) and (3.4). The r.h.s. of (4.3) would be identically zero, if the nilpotent part of the matrix c_- is trivial. However, in general, these are some nontrivial projectors onto $V_0^* \cap U_1^*$ and $V_0 \cap U_1$, respectively.

The matrix Toda equation (2.3) decomposes into four sets,

$$\partial_+ \left(\Pi_1\gamma^{-1}\partial_-\gamma\Pi_1\right) = [c_-, \Pi_1\gamma^{-1}c_+\gamma\Pi_1], \tag{4.4}$$

and besides,

$$\partial_+ \left(\Pi_0\gamma^{-1}\partial_-\gamma\Pi_0\right) = [c_-, \Pi_0\gamma^{-1}c_+\gamma\Pi_0], \tag{4.5}$$

$$\partial_+ \left(\Pi_1\gamma^{-1}\partial_-\gamma\Pi_0\right) = [c_-, \Pi_1\gamma^{-1}c_+\gamma\Pi_0], \tag{4.6}$$

$$\partial_+ \left(\Pi_0\gamma^{-1}\partial_-\gamma\Pi_1\right) = [c_-, \Pi_0\gamma^{-1}c_+\gamma\Pi_1]. \tag{4.7}$$

It is well known that the zero-curvature condition (2.1) can be interpreted as the integrability condition imposed on the so-called linear problem [3]

$$\ell_-(\lambda)\Psi = 0, \quad \ell_+(\lambda)\Psi = 0,$$

for some function Ψ taking values in G, with the differential operators ℓ_- and ℓ_+ being explicitly

$$\ell_-(\lambda) = \partial_- + \omega_-(\lambda), \quad \ell_+(\lambda) = \partial_+ + \omega_+(\lambda).$$

Indeed, from the requirement

$$\big[\ell_-(\lambda), \ell_+(\lambda)\big]\,\Psi = 0,$$

which holds at any power of the parameter λ, we derive the Toda equation (2.3).

Consider the parts (4.4)–(4.7) of the Toda equation in their order. The equation (4.4) follows from the integrability condition for the linear problem

$$\big(\partial_- + \Pi_1\gamma^{-1}\partial_-\gamma\Pi_1 + \lambda^{-1}c_-\big)\,\Psi = 0, \quad \big(\partial_+ + \lambda\Pi_1\gamma^{-1}c_+\gamma\Pi_1\big)\,\Psi = 0. \tag{4.8}$$

It is actually this, and only this, part of the full matrix Toda equation that is connected with the diagonalizable part of the matrix c_-. We are interested in conservation laws respecting the Toda equation (4.4), and thus the linear problem (4.8). Choose $k_* \times k_a$ matrices $\xi_a^\vee$ to be dual to the vectors ξ_a, to have

$$\xi_a^\vee \xi_a = I_{k_*}$$

for every value of $a = 1, 2, \ldots, r$. Introduce $n \times n$ matrices D and $D^\vee$ being explicitly of the forms

$$D = r^{-1/2}\left(\psi^{(1)}\xi_1^\vee, \psi^{(2)}\xi_2^\vee, \ldots, \psi^{(r)}\xi_r^\vee\right), \tag{4.9}$$

$$D^\vee = r^{-1/2}\left(\psi^{(-1)}\xi_1^\vee, \psi^{(-2)}\xi_2^\vee, \ldots, \psi^{(-r)}\xi_r^\vee\right), \tag{4.10}$$

and possessing remarkable properties

$$D^\vee D = D D^\vee = \Pi_1,$$

so that one has

$$D^\vee \Pi_1 = D^\vee, \quad \Pi_1 D = D.$$

The matrices D and $D^\vee$ have inherited the block matrix structure induced by the grading condition imposed to obtain the Toda system under consideration. Explicit expressions of the corresponding $k_a \times k_b$ submatrices are

$$D_{ab} = r^{-1/2} s^{ab}\xi_a\xi_b^\vee, \quad D_{ab}^\vee = r^{-1/2} s^{-ab}\xi_a\xi_b^\vee.$$

Then the linear problem (4.8) is equivalent to the relations

$$\big(\partial_- + D^\vee\gamma^{-1}\partial_-\gamma D + \lambda^{-1}D^\vee c_- D\big)\,\widetilde{\Psi} = 0, \quad \big(\partial_+ + \lambda D^\vee\gamma^{-1}c_+\gamma D\big)\,\widetilde{\Psi} = 0, \tag{4.11}$$

where $\widetilde{\Psi} = D^\vee\Psi$. Point out that $\psi^{(b)} = S^b\psi^{(0)}$, where S is the diagonal matrix (2.9). Recalling the relations (3.1) and using the expressions (4.9), (4.10) and the property (2.8) of c_-, such that

$$c_- D = D S^{-1},$$

one can write

$$c_- \to \widetilde{c}_- = D^\vee c_- D = \Pi_1 S^{-1}.$$

We conclude that the action of the matrices D and $D^\vee$ diagonalize the matrix c_-.

Represent the general solution to the linear problem (4.11) as follows (cf. [8]):

$$\widetilde{\Psi} = \Phi\chi\exp\left(-\lambda^{-1}\widetilde{c}_- z^-\right), \tag{4.12}$$

where χ is a block diagonal matrix, and Φ allows for the asymptotic expansion

$$\Phi = \sum_{k\geqslant 0}\lambda^k\Phi_k, \tag{4.13}$$

such that $\Phi_0 = I_n$, while only off-diagonal blocks are nonzero in all other Φ_k. We also assume that Φ and χ are such matrices that

$$[\Phi, \Pi_1] = 0, \quad [\chi, \Pi_1] = 0.$$

It follows from (4.11) that

$$\partial_- \Phi + (D^\vee \gamma^{-1} \partial_- \gamma D)\Phi + \Phi(\partial_- \chi \chi^{-1}) + \lambda^{-1}[\Pi_1 S^{-1}, \Phi] = 0, \tag{4.14}$$

$$\partial_+ \Phi + \Phi(\partial_+ \chi \chi^{-1}) + \lambda(D^\vee \gamma^{-1} c_+ \gamma D)\Phi = 0. \tag{4.15}$$

Further, using the asymptotic expansions

$$\partial_- \chi \chi^{-1} = \sum_{k \geqslant 0} \lambda^k \Sigma_k, \quad \partial_+ \chi \chi^{-1} = \sum_{k \geqslant 0} \lambda^k \Theta_k, \tag{4.16}$$

we obtain recurrent relations

$$\partial_- \Phi_k + \sum_{\substack{i=1, j=0 \\ i+j=k}}^{k} \Phi_i \Sigma_j + D^\vee \gamma^{-1} \partial_- \gamma D \Phi_k + [\Pi_1 S^{-1}, \Phi_{k+1}] = 0, \tag{4.17}$$

$$\partial_+ \Phi_k + \sum_{\substack{i=1, j=0 \\ i+j=k}}^{k} \Phi_i \Theta_j + D^\vee \gamma^{-1} c_+ \gamma D \Phi_{k-1} = 0, \tag{4.18}$$

which allow us to define off-diagonal blocks of Φ_{k+1} (which are, in fact, the only nonzero ones) through those of Φ_l, $l \leqslant k$. And besides, we have

$$\Sigma_k = -\mathrm{Diag}\left(D^\vee \gamma^{-1} \partial_- \gamma D \Phi_k\right), \quad \Theta_k = -\mathrm{Diag}\left(D^\vee \gamma^{-1} c_+ \gamma D \Phi_{k-1}\right),$$

where Diag means taking the block submatrices attached to the main diagonal according to the $\mathbb{Z}$-gradation. In particular, we get from (4.17) an explicit form of Φ_1 in terms of its block $k_a \times k_b$ submatrices,

$$(\Phi_1)_{a,b} = \frac{s^{a+b}}{s^a - s^b} \left(D^\vee \gamma^{-1} \partial_- \gamma D\right)_{a,b}, \quad a \neq b.$$

We see from (4.16) that the block diagonal matrices Σ_k and Θ_k satisfy the equations

$$\partial_+ \Sigma_k - \partial_- \Theta_k + \sum_{\substack{i,j=0 \\ i+j=k}}^{k} [\Sigma_i, \Theta_j] = 0.$$

In terms of the block submatrices the latter equation reads

$$\partial_+ \Sigma_k^a - \partial_- \Theta_k^a + \sum_{\substack{i,j=0 \\ i+j=k}}^{k} [\Sigma_i^a, \Theta_j^a] = 0, \tag{4.19}$$

where we have taken into account that

$$(\Sigma_k)_{a,b} = \Sigma_k^a \delta_{ab}, \quad (\Theta_k)_{a,b} = \Theta_k^a \delta_{ab}.$$

Therefore, introducing the quantities

$$\sigma_k^a = \operatorname{tr} \Sigma_k^a, \quad \theta_k^a = \operatorname{tr} \Theta_k^a,$$

we derive from (4.19) r infinite sets of current conservation laws

$$\partial_+ \sigma_k^a - \partial_- \theta_k^a = 0,$$

where the indices $k = 0, 1, 2, \ldots$, and $a = 1, 2, \ldots, r$. For $k = 0$ we obtain relations which are trivially satisfied due to the Toda equations (2.10). The case $k = 1$ gives us the energy-momentum conservation law for the non-Abelian Toda system under consideration.

There are other r infinite sets of current conservation laws in the system under consideration. They can be obtained along the same way of approach, only that starting with the Toda equation in the form (2.4). There one would face the diagonalization of the matrix c_+ in the corresponding linear problem,

$$c_+ \to \widetilde{c}_+ = D^\vee c_+ D = \Pi_1 S,$$

and work recurrent relations out of asymptotic expansions in λ^{-1}.

Now, it follows from (4.1), (4.2) that nontrivial conservation laws corresponding to the Toda equation (4.4) are exhausted by our consideration above. Note that the semisimple part of the matrix c_- entered this equation alone, while its nilpotent part turned out to be involved and separated into the remaining equations.

The equation (4.5) follows from the integrability condition imposed on the linear problem

$$\left(\partial_- + \Pi_0 \gamma^{-1} \partial_- \gamma \Pi_0 + \lambda^{-1} c_-\right) \varphi = 0, \quad \left(\partial_+ + \lambda \Pi_0 \gamma^{-1} c_+ \gamma \Pi_0\right) \varphi = 0,$$

and to this equation correspond W-symmetry and WZNW-type conservation laws present in the Toda system under consideration. To see this, one can simply use the relations (3.2) and (4.1), (4.2) with the projectors. One derives from (4.5), in particular, that

$$\partial_+ (L_0 \gamma^{-1} \partial_- \gamma R_0) = 0.$$

To the rest of the matrix Toda equation (2.3) Drinfeld–Sokolov techniques are applicable and correspond certain W-symmetry type conservation laws [5]. Consider the equations (4.6) and (4.7). These are obtained from the integrability condition imposed on the linear problems

$$\left(\partial_- + \Pi_1 \gamma^{-1} \partial_- \gamma \Pi_0 + \lambda^{-1} c_-\right) \phi = 0, \quad \left(\partial_+ + \lambda \Pi_1 \gamma^{-1} c_+ \gamma \Pi_0\right) \phi = 0$$

and

$$\left(\partial_- + \Pi_0 \gamma^{-1} \partial_- \gamma \Pi_1 + \lambda^{-1} c_-\right) \phi = 0, \quad \left(\partial_+ + \lambda \Pi_0 \gamma^{-1} c_+ \gamma \Pi_1\right) \phi = 0,$$

respectively. It follows from these conditions also that

$$\partial_- \left(\Pi_1 \gamma^{-1} c_+ \gamma \Pi_0\right) = 0, \quad \partial_- \left(\Pi_0 \gamma^{-1} c_+ \gamma \Pi_1\right) = 0.$$

Other relations can be obtained while using properties of the projectors.

Concluding this part, we have seen that to the equations (4.6) and (4.7) correspond conservation laws with the conserved matrix-valued current $\Pi_1 \gamma^{-1} c_+ \gamma \Pi_0$ and $\Pi_0 \gamma^{-1} c_+ \gamma \Pi_1$ having no dependence on z^+. These quantities produce conserved charges as just it happens in non-Abelian Toda systems associated with finite-dimensional Lie groups [5]. Such a mixture of essentially different types of conservation laws in non-Abelian Toda systems can be explain by certain properties of the matrix-valued mapping entering the Toda equation. Indeed, the matrices c_- and c_+ turn out to be a sum of commuting nilpotent and semi-simple parts, that is just implying the so-called Jordan decomposition of a linear operator, for which one obtains W-symmetry and WZNW-type and "usual" current conservation laws' type relations, respectively.

An interesting special case is also revealed when only submatrices C_{-r} and C_{+r} are nontrivial, thus leading to a combination of WZNW- and W-type symmetries.

5. Abelian Toda system

It is instructive to consider an Abelian affine Toda system being a particular case of the non-Abelian system considered in the preceding sections. To construct one, we put $r = n$ and all $k_a = 1$, reproducing the standard gradation of $\mathscr{L}(\mathfrak{gl}_n(\mathbb{C}))$. The mapping γ here is a diagonal $n \times n$ matrix $\gamma = \|\Gamma_i \delta_{ij}\|$, where Γ_i are ordinary functions of z^- and z^+. The mappings c_- and c_+ can be chosen being proportional to the cyclic elements of $\mathfrak{g}$. Explicitly, it reads $c_- = \|\mu\delta_{i,j+1}\|$ and $c_+ = \|\mu\delta_{i+1,j}\|$, where δ_{ij} is the n-periodic Kronecker symbol and μ some nonzero constant. We see that the matrix Toda equation (2.3) is equivalent to a system of nonlinear partial differential equations, which is convenient to treat as the infinite system

$$\partial_+(\Gamma_i^{-1}\partial_-\Gamma_i) = -\mu^2(\Gamma_i^{-1}\Gamma_{i+1} - \Gamma_{i-1}^{-1}\Gamma_i), \quad i \in \mathbb{Z},$$

with Γ_i subject to the periodicity condition $\Gamma_{i+n} = \Gamma_i$.

Remembering the basic property (2.8) of c_-, where now $S_{ij} = s^i\delta_{ij}$, $s = e^{2\pi \mathrm{i}/n}$, and introducing n-dimensional vectors $\psi^{(l)} = S^l\psi^{(0)}$, for the n-dimensional vector $\psi^{(0)}$ given by $\psi^{(0)t} = n^{-1/2}\,(1, 1, \ldots, 1)$, we obtain

$$c_-\psi^{(l)} = \mu s^{-l} S^l \psi^{(0)} = \mu s^{-l}\psi^{(l)}.$$

Representing the $n \times n$ matrix D as $D = \big(\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(n-1)}, \psi^{(0)}\big)$, that, in terms of the matrix elements, reads $D_{ij} = n^{-1/2}s^{ij}$, we derive the relation

$$c_- D = \mu D S^{-1}.$$

The matrix D is non-degenerate in this Abelian case. Hence, from the relation

$$c_- \to \widetilde{c}_- = D^{-1}c_- D = \mu S^{-1}$$

we conclude that the operator D diagonalizes the cyclic matrix c_-. Now we have the linear problem with transformed operators $\widetilde{\ell}_1(\lambda) = \partial_- + \widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma} + \lambda^{-1}\widetilde{c}_-$, where $\widetilde{\gamma} = \gamma D$, and so, $\widetilde{\gamma}_{ij} = n^{-1/2}\Gamma_i s^{ij}$, $\widetilde{\gamma}^{-1}{}_{ij} = n^{-1/2}s^{-ij}\Gamma_j^{-1}$, and $\widetilde{\ell}_2(\lambda) = \partial_+ + \lambda\widetilde{\gamma}^{-1}c_+\widetilde{\gamma}$. Further, we have $\widetilde{\Psi} = D^{-1}\Psi$.

Following again [8], we separate the diagonal and non-diagonal parts of $\widetilde{\Psi}$ as follows. Represent $\widetilde{\Psi}$ in the form $\widetilde{\Psi} = \Phi\chi\exp(-\lambda^{-1}\widetilde{c}_- z^-)$, where χ is a diagonal $n \times n$ matrix, while Φ allows for the asymptotic expansion in λ (4.13), where again Φ_0 is the unit matrix I_n, and in all other Φ_k, $k \geqslant 1$, only off-diagonal matrix elements are nonzero. Note that $\widetilde{c}_-$ commutes with χ. It is convenient to represent χ as $\chi = \exp\sigma$, where σ can be asymptotically expanded $\sigma = \sum_{k\geqslant 0}\lambda^k\sigma_k$. Substituting the asymptotic expansions of Φ and σ to the integrability condition (4.14), (4.15) and taking into account that in the Abelian case $\Pi_1 \to I_n$ and $D^\vee \to D^{-1}$, we obtain the recursive relations

$$\partial_-\Phi_k + \sum_{\substack{i=1,j=0\\ i+j=k}}^{k} \Phi_i\partial_-\sigma_j + \widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma}\Phi_k + [\widetilde{c}_-, \Phi_{k+1}] = 0, \tag{5.1}$$

$$\partial_+\Phi_k + \sum_{\substack{i=1,j=0\\ i+j=k}}^{k} \Phi_i\partial_+\sigma_j + \widetilde{\gamma}^{-1}c_+\widetilde{\gamma}\Phi_{k-1} = 0 \tag{5.2}$$

for the non-diagonal elements, and

$$\partial_-\sigma_k = -\mathrm{diag}\left(\widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma}\Phi_k\right), \quad \partial_+\sigma_k = -\mathrm{diag}\left(\widetilde{\gamma}^{-1}c_+\widetilde{\gamma}\Phi_{k-1}\right). \tag{5.3}$$

For the matrix elements of σ_k holds the representation $(\sigma_k)_{ij} = \sigma_k^{(i)}\delta_{ij}$. These recurrent relations can be resolved for all Φ_k and σ_k, $k = 0, 1, 2, \ldots$, for which it is sufficient to note that $\Phi_0 = I_n$ and

$$\partial_-\sigma_0 = -\mathrm{diag}\left(\widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma}\right), \quad \partial_+\sigma_0 = 0.$$

The latter is compatible because

$$\partial_+(\widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma})_{ii} = 0$$

thanks to the Toda equations. Taking $k = 1$ as the first nontrivial example, we find

$$\sigma_1^{(i)} = -\int \mathrm{d}z^-(\widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma}\Phi_1)_{ii} - \int \mathrm{d}z^+(\widetilde{\gamma}^{-1}c_+\widetilde{\gamma})_{ii},$$

with explicit forms of the matrix elements under integration

$$\left(\widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma}\right)_{ij} = \frac{1}{n}\sum_{l=1}^{n} s^{-l(i-j)}\Gamma_l^{-1}\partial_-\Gamma_l, \quad \left(\widetilde{\gamma}^{-1}c_+\widetilde{\gamma}\right)_{ij} = \frac{s^j}{n}\sum_{l=1}^{n} s^{-l(i-j)}\Gamma_l^{-1}\Gamma_{l+1},$$

and where

$$(\Phi_1)_{ij} = \frac{s^{i+j}}{\mu n(s^i - s^j)}\sum_{l=1}^{n} s^{-l(i-j)}\Gamma_l^{-1}\partial_-\Gamma_l, \quad i \neq j.$$

Thus resolving the recurrence (5.1), we can determine Φ_k and so obtain σ_k from (5.3). For the latter quantities, imposing $\partial_+\partial_-\sigma_k = \partial_-\partial_+\sigma_k$, we get current conservation laws

$$\partial_+\delta_k^+ - \partial_-\delta_k^- = 0, \tag{5.4}$$

where

$$\delta_k^+ = \mathrm{diag}\left(\widetilde{\gamma}^{-1}\partial_-\widetilde{\gamma}\Phi_k\right), \quad \delta_k^- = \mathrm{diag}\left(\widetilde{\gamma}^{-1}c_+\widetilde{\gamma}\Phi_{k-1}\right). \tag{5.5}$$

But then, the relations (5.4) give n infinite sets of current conservation laws, with the currents components (5.5). So for $k = 0$ these relations are true due to the Toda equations, while for $k = 1$, seeing the explicit forms of σ_1 and Φ_1 given above, we find the energy-momentum conservation law for the Abelian affine Toda system.

There are other n infinite sets of current conservation laws corresponding to the Toda equations written in the right invariant form, which is obtained from the linear problem where the matrix c_+ is being diagonalized, $c_+ \to \widetilde{c}_+ = D^{-1}c_+D = \mu S$.

Acknowledgements. We are grateful to A. A. Makhnev for discussions.

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Received September 29, 2010.

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О ЗАКОНАХ СОХРАНЕНИЯ В АФФИННЫХ ТОДОВСКИХ СИСТЕМАХ

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С помощью некоторого матричного разложения и проекторов мы показываем, что неабелевы тодовские системы, связанные с группами петель, обладают бесконечными наборами сохраняющихся величин, порождаемых существенно различными законами сохранения токов.

Ключевые слова: неабелевы уравнения Тоды, группы петель, симметрии и законы сохранения.