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HOMOGENEOUS FUNCTIONS OF REGULAR LINEAR AND BILINEAR OPERATORS[1]

To Yuri G. Reshetnjak on the occasion of his 80th birthday

A. G. Kusraev

Using envelope representations explicit formulae for computing $\widehat{\varphi}(T_1,\dots,T_N)$ for any finite sequence of regular linear or bilinear operators $T_1,\dots,T_N$ on vector lattices are derived.

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1. Introduction

This paper is a continuation of [5]. We apply the upper envelope representation method (or the quasilinearization method) in vector lattices developed in [4, 5] to the homogeneous functional calculus of linear and bilinear operators. Explicit formulae for computing $\widehat{\varphi}(T_1,\dots,T_N)$ for any finite sequence of regular linear or bilinear operators $T_1,\dots,T_N$ are derived.

For the theory of vector lattices and positive operators we refer to the books [1] and [3]. All vector lattices in this paper are real and Archimedean.

Consider conic sets C and K with $K\subset C$ and K closed. Let $\mathscr{H}(C;K)$ denotes the vector lattice of all positively homogeneous functions $\varphi : C\to\mathbb{R}$ with continuous restriction to K. The expression $\widehat{\varphi}(x_1,\dots,x_N)$ can be correctly defined provided that the compatibility condition $[x_1,\dots,x_N]\subset K$ is hold, see [5].

Denote by $\mathscr{H}_\vee(\mathbb{R}^N,K)$ and $\mathscr{H}_\wedge(\mathbb{R}^N,K)$ respectively the sets of all lower semicontinuous sublinear functions $\varphi:\mathbb{R}^N\to\mathbb{R}\cup\{+\infty\}$ and upper semicontinuous superlinear functions $\psi:\mathbb{R}^N\to\mathbb{R}\cup\{-\infty\}$ which are finite and continuous on a fixed cone $K\subset\mathbb{R}^N$. Put $\mathscr{H}_\vee(\mathbb{R}^N):=\mathscr{H}_\vee(\mathbb{R}^N,\{0\})$ and $\mathscr{H}_\wedge(\mathbb{R}^N):=\mathscr{H}_\wedge(\mathbb{R}^N,\{0\})$.

Denote by $\mathscr{G}_\vee(\mathbb{R}^N,K)$ and $\mathscr{G}_\wedge(\mathbb{R}^N,K)$ respectively the sets of all lower semicontinuous gauges $\varphi:\mathbb{R}^N\to\mathbb{R}_+\cup\{+\infty\}$ and upper semicontinuous co-gauges $\psi:\mathbb{R}^N\to\mathbb{R}_+\cup\{-\infty\}$ which are finite and continuous on a fixed cone $K\subset\mathbb{R}^N$. Put $\mathscr{G}_\vee(\mathbb{R}^N):=\mathscr{G}_\vee(\mathbb{R}^N,\{0\})$ and $\mathscr{G}_\wedge(\mathbb{R}^N):=\mathscr{G}_\wedge(\mathbb{R}^N,\{0\})$. Observe that $\mathscr{G}_\vee(\mathbb{R}^N)\subset\mathscr{H}_\vee(\mathbb{R}^N)$ and $\mathscr{G}_\wedge(\mathbb{R}^N)\subset\mathscr{H}_\wedge(\mathbb{R}^N)$, see [4, 5].

Everywhere below E, F, and G denote vector lattices, while $L^r(E,F)$ and $BL^r(E,F;G)$ stand for the spaces of regular linear operators from E to F and regular bilinear operator from $E\times F$ to G, respectively.

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2. Functions of Bilinear Operators

A *partition* of $x \in E_+$ is any finite sequence $(x_1, \dots, x_n)$, $n \in \mathbb{N}$, of elements of E_+ whose sum equals x. Denote by $\operatorname{Prt}(x)$ and $\operatorname{DPrt}(x)$ the sets of all partitions of x and all partitions with pairwise disjoint terms, respectively.

2.1. Lemma. *Let E, F, and G be vector lattices, $b_1, \dots, b_N \in BL^r(E, F; G)$, and $\mathfrak{b} := (b_1, \dots, b_N)$. Let $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$, $\widehat{\varphi}(b_1(x_0, y_0), \dots, b_N(x_0, y_0))$ and $\widehat{\psi}(b_1(x_0, y_0), \dots, b_N(x_0, y_0))$ are well defined in G for all $0 \leqslant x_0 \leqslant x$ and $0 \leqslant y_0 \leqslant y$. Denote $\mathfrak{x} := (x_1, \dots, x_n) \in E^n$ and $\mathfrak{y} := (y_1, \dots, y_m) \in F^m$, $m, n \in \mathbb{N}$. Then the sets*

$$\varphi(\mathfrak{b}; x, y) := \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \widehat{\varphi}(b_1(x_i, y_j), \dots, b_N(x_i, y_j)) :\ n, m \in \mathbb{N}, \mathfrak{x} \in \operatorname{Prt}(x),\ \mathfrak{y} \in \operatorname{Prt}(y) \right\},$$

$$\psi(\mathfrak{b}; x, y) := \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \widehat{\psi}(b_1(x_i, y_j), \dots, b_N(x_i, y_j)) :\ n, m \in \mathbb{N}, \mathfrak{x} \in \operatorname{Prt}(x),\ \mathfrak{y} \in \operatorname{Prt}(y) \right\},$$

are upward directed and downward directed, respectively.

$\lhd$ Assume that $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_{n'})$ are partitions of x while $(y_1, \dots, y_m)$ and $(y'_1, \dots, y'_{m'})$ are partitions of y. By The Riesz Decomposition Property of vector lattices there exist finite double sequences $(u_{i,k})_{i \leqslant n,\, k \leqslant n'}$ in E_+ and $(v_{j,l})_{j \leqslant m,\, l \leqslant m'}$ in F_+ such that

$$\sum\nolimits_{k=1}^{n'} u_{i,k} = x_i, \quad \sum\nolimits_{i=1}^{n} u_{i,k} = x'_k \quad (i := 1, \dots, n,\ k := 1, \dots, n');$$
$$\sum\nolimits_{l=1}^{m'} v_{j,l} = y_j, \quad \sum\nolimits_{j=1}^{m} v_{j,l} = y'_l \quad (j := 1, \dots, m,\ l := 1, \dots, m').$$

In particular, $(u_{i,k})_{i \leqslant n,\, k \leqslant n'}$ and $(v_{j,l})_{j \leqslant m,\, l \leqslant m'}$ are partition of x and y, respectively. Taking subadditivity of φ into consideration we obtain

$$\sum_{i,j=1}^{n,m} \widehat{\varphi}(b_1(x_i, y_j), \dots, b_N(x_i, y_j)) = \sum_{i,j=1}^{n,m} \widehat{\varphi}\bigg(\sum_{k,l=1}^{n',m'} b_1(u_{i,k}, v_{j,l}), \dots, \sum_{k,l=1}^{n',m'} b_N(u_{i,k}, v_{j,l}) \bigg)$$
$$= \sum_{i,j=1}^{n,m} \widehat{\varphi}\bigg(\sum_{k,l=1}^{n',m'} \Big(b_1(u_{i,k}, v_{j,l}), \dots, b_N(u_{i,k}, v_{j,l}) \Big) \bigg) \leqslant \sum_{i,j=1}^{n,m} \sum_{k,l=1}^{n',m'} \widehat{\varphi}(b_1(u_{i,k}, v_{j,l}), \dots, b_N(u_{i,k}, v_{j,l})).$$

In a similar way we get

$$\sum_{k,l=1}^{n',m'} \widehat{\varphi}\left(b_1(x'_k, y'_l), \dots, b_N(x'_k, y'_l)\right) \leqslant \sum_{i,j=1}^{n',m'} \sum_{k,l=1}^{n,m} \widehat{\varphi}(b_1(u_{i,k}, v_{j,l}), \dots, b_N(u_{i,k}, v_{j,l})),$$

so that the first set is upward directed. Similarly, the second set is downward directed. $\rhd$

2.2. Lemma. *Let Let E, F, and G be vector lattices with G Dedekind complete and $\mathscr{B}$ be an order bounded set of regular bilinear operators from $E \times F$ to G. Then for every $x \in E_+$ and $y \in F_+$ we have:*

$$(\sup \mathscr{B})(x, y) = \sup \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} b_{k(i,j)}(x_i, y_j) \right\},$$

$$(\inf \mathscr{B})(x, y) = \inf \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} b_{k(i,j)}(x_i, y_j) \right\},$$

where supremum and infimum are taken over all naturals $n, m, l \in \mathbb{N}$, *functions* $k : \{1, \dots, n\} \times \{1, \dots, m\} \to \{1, \dots, l\}$, *partitions* $(x_1, \dots, x_n) \in \mathrm{Prt}(x)$ *and* $(y_1, \dots, y_m) \in \mathrm{Prt}(y)$, *and arbitrary finite collections* $b_1 \dots, b_l \in \mathscr{B}$.

$\lhd$ See [6, Proposition 2.6]. $\rhd$

2.3. Theorem. *Let* E, F, *and* G *be vector lattices with* G *Dedekind complete,* $b_1, \dots, b_N \in BL^r(E, F; G)$, *and* $\mathfrak{b} := (b_1, \dots, b_N)$. *Assume that* $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$, $\widehat{\varphi}(b_1(x_0, y_0), \dots, b_N(x_0, y_0))$ *and* $\widehat{\psi}(b_1(x_0, y_0), \dots, b_N(x_0, y_0))$ *are well defined in* G *for all* $0 \leqslant x_0 \leqslant x$ *and* $0 \leqslant y_0 \leqslant y$, $\varphi(\mathfrak{b}; x, y)$ *is order bonded above, and* $\psi(\mathfrak{b}; x, y)$ *is order bounded below for all* $x \in E_+$ *and* $y \in F_+$. *Then* $\widehat{\varphi}(b_1, \dots, b_N)$ *and* $\widehat{\psi}(b_1, \dots, b_N)$ *are well defined in* $BL^r(E, F; G)$ *and for every* $x \in E_+$ *and* $y \in F_+$ *the representations*

$$\widehat{\varphi}(b_1, \dots, b_N)(x, y) = \sup \varphi(\mathfrak{b}; x, y),$$
$$\widehat{\psi}(b_1, \dots, b_N)(x, y) = \inf \psi(\mathfrak{b}; x, y)$$

hold with supremum over upward directed set and infimum over downward directed set. If E *and* F *have the strong Freudenthal property (or principal projection property) then* $\mathrm{Prt}(x)$ *and* $\mathrm{Prt}(y)$ *may be replaced by* $\mathrm{DPrt}(x)$ *and* $\mathrm{DPrt}(y)$, *respectively.*

$\lhd$ Denote $b_\lambda := \lambda_1 b_1 + \dots + \lambda_N b_N$ for $\lambda := (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ and observe that if the set $\{b_\lambda : \lambda \in \underline{\partial}\varphi\}$ is order bounded in $BL^r(E, F; G)$, then by [5, Theorem 4.4] $\widehat{\varphi}(b_1, \dots, b_N)$ exists in $BL^r(E, F; G)$ and the upper envelope representation $\widehat{\varphi}(b_1, \dots, b_N) = \sup\{b_\lambda : \lambda \in \underline{\partial}\varphi\}$ holds. Take arbitrary $\lambda^r := (\lambda_1^r, \dots, \lambda_N^r) \in \underline{\partial}\varphi$ $(r := 1, \dots, l)$, $k : \{1, \dots, n\} \times \{1, \dots, m\} \to \{1, \dots, l\}$, $\mathfrak{x} := (x_1, \dots, x_n) \in \mathrm{Prt}(x)$, and $\mathfrak{y} := (y_1, \dots, y_m) \in \mathrm{Prt}(y)$. Making use of Lemma 2.2 and [5, Theorem 4.4] we deduce:

$$\sum_{i,j=1}^{n,m} b_{\lambda^{k(i,j)}}(x_i, y_j) = \sum_{i,j=1}^{n,m} \sum_{s=1}^{N} \lambda_s^{k(i,j)} b_s(x_i, y_j) \leqslant \sum_{i,j=1}^{n,m} \widehat{\varphi}(b_1(x_i, y_j), \dots, b_N(x_i, y_j)) \leqslant a,$$

where a is an upper bound of $\varphi(\mathfrak{b}; x, y)$. Passing to supremum over all $(\lambda^1 \dots, \lambda^l)$, k, $\mathfrak{x}$, and $\mathfrak{y}$ and taking [5, Theorem 4.4] into account we get that $\widehat{\varphi}(b_1, \dots, b_N)$ is well defined and $\widehat{\varphi}(b_1, \dots, b_N)(x, y) \leqslant \varphi(\mathfrak{b}; x, y)$. Surely, in above reasoning we could take $(x_1, \dots, x_n) \in \mathrm{DPrt}(x)$ provided that E has the principal projection property.

Conversely, let $f(x, y)$ stands for the right-hand side of the first equality. Observe that if $(\lambda_1, \dots, \lambda_n) \in \underline{\partial}\varphi$ and $u \in E_+$, $v \in F_+$, then by [5, Theorem 4.4] we have

$$\sum_{k=1}^{N} \lambda_k b_k(u, v) = \Big(\sum_{k=1}^{N} \lambda_k b_k\Big)(u, v) \leqslant \widehat{\varphi}(b_1, \dots, b_N)(u, v)$$

and again $\widehat{\varphi}(b_1(u, v), \dots, b_N(u, v)) \leqslant \widehat{\varphi}(b_1, \dots, b_N)(u, v)$ by [5, Theorem 4.4]. Now, given $(x_1, \dots, x_n)$ in $\mathrm{Prt}(x)$ or $\mathrm{DPrt}(x)$ and $(y_1, \dots, y_n)$ in $\mathrm{Prt}(y)$ or $\mathrm{DPrt}(y)$, we can estimate

$$\sum_{i,j=1}^{n,m} \widehat{\varphi}(b_1(x_i, y_j), \dots, b_N(x_i, y_j)) \leqslant \sum_{i,j=1}^{n,m} \widehat{\varphi}(b_1, \dots, b_N)(x_i, y_j) \leqslant \widehat{\varphi}(b_1, \dots, b_N)(x, y)$$

and thus $f(x, y) \leqslant \widehat{\varphi}(b_1, \dots, b_N)(x, y)$. Thus the first equality is hold true. By Lemma 2.1 the supremum on the right-hand side of the required formula is taken over upward directed set.

The second representation is proved in a similar way. $\rhd$

2.4. Corollary. *Let E, F, G, φ, ψ, $b_1,\dots,b_N$ be the same as in 2.1, $\overline{b}:=\widehat{\varphi}(b_1,\dots,b_N)$ and $\underline{b}:=\widehat{\psi}(b_1,\dots,b_N)$. Assume that, in addition, $E=F$ has the strong Freudenthal property and $b_1,\dots,b_N$ are orthosymmetric. Then for every $x\in E$ the representations*

$$\overline{b}(x,x)=\sup\left\{\sum_{i=1}^{n}\varphi(b_1(x_i,|x|),\dots,b_N(x_i,|x|)):\ (x_1,\dots,x_n)\in\mathrm{DPrt}(|x|)\right\},$$

$$\underline{b}(x,x)=\inf\left\{\sum_{i=1}^{n}\psi(b_1(x_i,|x|),\dots,b_N(x_i,|x|)):\ (x_1,\dots,x_n)\in\mathrm{DPrt}(|x|)\right\},$$

hold with supremum and infimum over upward and downward directed sets, respectively.

$\lhd$ It is sufficient to check the first formula. We can assume $x\in E_+$. Denote by $g(x)$ the right-hand side of the desired equality. From Theorem 2.3 we have $g(x)\leqslant\widehat{\varphi}(b_1,\dots,b_N)(x,x)$. To prove the reverse inequality take two disjoint partitions of x, say $\mathfrak{x}':=(x'_1,\dots,x'_l)$ and $\mathfrak{x}'':=(x''_1,\dots,x''_m)$, and let $(x_1,\dots,x_n)\in\mathrm{DPrt}(x)$ be their common refinement. Since $b_1,\dots,b_N$ are orthosymmetric we deduce

$$\sum_{r,s=1}^{l,m}\widehat{\varphi}(b_1(x'_r,x''_s),\dots,b_N(x'_r,x''_s))$$

$$=\sum_{i=1}^{n}\widehat{\varphi}(b_1(x_i,x_i),\dots,b_N(x_i,x_i))=\sum_{i=1}^{n}\widehat{\varphi}(b_1(x_i,x),\dots,b_N(x_i,x)).$$

Passing to supremum over all $\mathfrak{x}'$ and $\mathfrak{x}''$ we get the desired inequality. $\rhd$

3. Functions of Linear Operators

The above machinery is applicable to the calculus of order bounded operators.

3.1. Theorem. *Let E and F be vector lattices with F Dedekind complete, $T_1,\dots,T_N\in L^r(E,F)$, and $\mathfrak{T}:=(T_1,\dots,T_N)$. Let $\varphi\in\mathscr{H}_\vee(\mathbb{R}^N)$, $\psi\in\mathscr{H}_\wedge(\mathbb{R}^N)$, $\widehat{\varphi}(T_1x_0,\dots,T_Nx_0)$ and $\widehat{\psi}(T_1x_0,\dots,T_Nx_0)$ are well defined in F for all $0\leqslant x_0\leqslant x$. If for every $x\in E_+$ the sets*

$$\varphi(\mathfrak{T};x)=\left\{\sum_{k=1}^{n}\widehat{\varphi}(T_1x_k,\dots,T_Nx_k):\ (x_1,\dots,x_n)\in\mathrm{Prt}(x)\right\},$$

$$\psi(\mathfrak{T};x)=\left\{\sum_{k=1}^{n}\widehat{\psi}(T_1x_k,\dots,T_Nx_k):\ (x_1,\dots,x_n)\in\mathrm{Prt}(x)\right\}$$

are order bounded from above and from below respectively, then $\widehat{\varphi}(T_1,\dots,T_N)$ and $\widehat{\psi}(T_1,\dots,T_N)$ exist in $L^r(E,F)$, and the representations

$$\widehat{\varphi}(T_1,\dots,T_N)x=\sup\varphi(\mathfrak{T};x),$$

$$\widehat{\psi}(T_1,\dots,T_N)x=\inf\psi(\mathfrak{T};y)$$

hold with supremum over upward directed set and infimum over downward directed set. If E has the principal projection property then $\mathrm{Prt}(x)$ *may be replaced by* $\mathrm{DPrt}(x)$.

$\lhd$ Follows immediately from 2.3. $\rhd$

3.2. Remark. **(1)** Assume that E, F, $T_1,\dots,T_N$, φ, and ψ are the same as in [4, Theorem

5.2]. Then $\widehat{\varphi}(T_1,\dots,T_N)x \geqslant \widehat{\varphi}(T_1x,\dots,T_Nx)$ and $\widehat{\psi}(T_1,\dots,T_N)x \leqslant \widehat{\psi}(T_1x,\dots,T_Nx)$ for all $x \in E_+$. In particular, if $\mathbb{R}^N_+ \subset \operatorname{dom}(\varphi) \cap \operatorname{dom}(\psi)$ and $\widehat{\varphi}(T_1x,\dots,T_Nx) \geqslant \widehat{\psi}(T_1x,\dots,T_Nx)$ for all $x \in E_+$, then $\widehat{\varphi}(T_1,\dots,T_N) \geqslant \widehat{\psi}(T_1,\dots,T_N)$.

(2) Assume that $\varphi \in \mathscr{H}(C;[\mathfrak{x}])$ and $\varphi(0,t_2,\dots,t_N) = 0$ for all $(t_1,\dots,t_N) \in \operatorname{dom}(\varphi)$. Then evidently $\widehat{\varphi}(x_1,\dots,x_N) \in \{x_1\}^{\perp\perp}$ provided that $[\mathfrak{x}] \subset \operatorname{dom}(\varphi)$. This simple observation together with [4, Theorem 5.2] enables one to attack the nonlinear majorization problem for wider variety of majorants $\widehat{\varphi}(T_1,\dots,T_N)$, cp. [2].

3.3. *Let E and F be vector lattices with E relatively uniformly complete and F Dedekind complete. Then for $T_1,\dots,T_N \in L^r_+(E,F)$, $x_1,\dots,x_N \in E_+$, and $\alpha_1,\dots,\alpha_N \in \mathbb{R}_+$ with $\alpha_1+\dots+\alpha_N = 1$ we have*

$$(T_1^{\alpha_1}\dots T_N^{\alpha_N})(x_1^{\alpha_1}\dots x_N^{\alpha_N}) \leqslant (T_1x_1)^{\alpha_1}\dots(T_Nx_N)^{\alpha_N}.$$

The reverse inequality holds provided that $\alpha_1+\dots+\alpha_N = 1$, $(-1)^k(1-\alpha_1-\dots-\alpha_k)\alpha_1\cdot\ldots\cdot\alpha_k \geqslant 0$ $(k:=1,\dots,N-1)$, and $x_i \gg 0$, $f(x_i) \gg 0$ for all i with $\alpha_i < 0$.

$\lhd$ Apply [4, Corollary 6.7] with $K = \mathbb{R}^N_+$, $C = 1$, $\varphi_0(t) = \varphi_1(t) = \varphi_2(t) = t_1^{\alpha_1}\dots t_N^{\alpha_N}$. $\rhd$

3.4. Theorem. *Let E and F be vector lattices with F Dedekind complete and $T_1,\dots,T_N \in L^r_+(E,F)$. Suppose that $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N,\mathbb{R}^N_+)$ and $\psi \in \mathscr{G}_\wedge(\mathbb{R}^N,\mathbb{R}^N_+)$ are increasing and $[T_1,\dots,T_N] \subset \operatorname{dom}(\varphi) \cap \operatorname{dom}(\psi)$. Then for every $x \in E_+$ the representations hold*

$$\widehat{\varphi}(T_1,\dots,T_N)x = \sup\left\{\sum_{k=1}^N T_kx_k :\ x_1,\dots,x_N \in E_+,\ \widehat{\varphi^\circ}(x_1,\dots,x_N) \leqslant x\right\},$$

$$\widehat{\psi}(T_1,\dots,T_N)x = \inf\left\{\sum_{k=1}^N T_kx_k :\ x_1,\dots,x_N \in E_+,\ \widehat{\psi_\circ}(x_1,\dots,x_N) \geqslant x\right\},$$

with supremum over upward directed set and infimum over downward directed set.

$\lhd$ Suppose that $\widehat{\varphi}(T_1,\dots,T_N)$ exists and $x \in E_+$. If $x_1,\dots,x_N \in E_+$ and $\widehat{\varphi^\circ}(x_1,\dots,x_N) \leqslant x$, then making use of the Bipolar Theorem, positivity of $\widehat{\varphi}(T_1,\dots,T_N)$, and [4, Corollary 6.8] we deduce

$$\sum_{k=1}^N T_kx_k \leqslant \widehat{\varphi}(T_1,\dots,T_N)(\widehat{\varphi^\circ}(x_1,\dots,x_N)) \leqslant \widehat{\varphi}(T_1,\dots,T_N)x.$$

To prove the reverse inequality take $(x_1,\dots,x_n) \in \operatorname{Prt}(x)$, $\lambda^k = (\lambda^k_1,\dots,\lambda^k_N) \in \underline{\partial}\varphi = \{\varphi^\circ \leqslant 1\}$ $(k:=1,\dots,n)$, and put $u_i := \sum_{k=1}^n \lambda^k_i x_k$. If $\alpha := (\alpha_1,\dots,\alpha_N) \in \underline{\partial}\varphi^\circ = \{\varphi \leqslant 1\}$, then $\langle\alpha,\lambda^k\rangle \leqslant \varphi(\alpha)\varphi^\circ(\lambda^k) \leqslant 1$ and thus

$$\sum_{i=1}^N \alpha_iu_i = \sum_{i=1}^N \alpha_i \sum_{k=1}^n \lambda^k_i x_k = \sum_{k=1}^n \langle\alpha,\lambda^k\rangle x_k \leqslant x.$$

It follows from [5, Theorem 5.4] that $\widehat{\varphi^\circ}(u_1,\dots,u_N) \leqslant x$.

Denote $S(\lambda) := \lambda_1T_1+\dots+\lambda_NT_N$ with $\lambda := (\lambda_1,\dots,\lambda_N)$. Let $f(x)$ is the right-hand side of the first equality. Then

$$\sum_{k=1}^n S(\lambda^k)(x_k) = \sum_{i=1}^N T_iu_i \leqslant f(x).$$

It remains to observe that $\varphi(T_1,\dots,T_N) = \sup\{S(\lambda) :\ \lambda \in \underline{\partial}\varphi\}$ by [5, Theorem 4.4]. $\rhd$

3.5. Proposition. *Let E, F, and G be vector lattices with F Dedekind complete, $R : E \to G$ an order interval preserving operator, $T : G \to F$ an order continuous lattice homomorphism, and $\varphi \in \mathscr{H}(C,K)$. Assume that $S_1, \dots, S_N \in L^r(E,F)$ and $[S_1, \dots, S_N] \subset K$. Then $[S_1 \circ R, \dots, S_N \circ R] \subset K$ and*

$$\widehat{\varphi}(S_1, \dots, S_N) \circ R = \widehat{\varphi}(S_1 \circ R, \dots, S_N \circ R).$$

If, in addition, G is Dedekind complete, then $[T \circ S_1, \dots, T \circ S_N] \subset K$ and

$$T \circ \widehat{\varphi}(S_1, \dots, S_N) = \widehat{\varphi}(T \circ S_1, \dots, T \circ S_N).$$

$\lhd$ Under the indicated hypotheses the operators $S \mapsto S \circ R$ from $L^r(G,F)$ to $L^r(E,F)$ and $S \mapsto T \circ S$ from $L^r(E,G)$ to $L^r(E,F)$ are lattice homomorphisms, see [1, Theorem 7.4 and 7.5]. Therefore, it is sufficient to apply [5, Proposition 2.6]. $\rhd$

3.6. Proposition. *Let E and F be vector lattices with F Dedekind complete. Assume that $\varphi \in \mathscr{H}(C,K)$, $S_1, \dots, S_N \in L^r(E,F)$, and $[S_1, \dots, S_N] \subset K$. If S^* denotes the restriction of the order dual S' to $F_n^\sim$, the order continuous dual of F, then $[S_1^*, \dots, S_N^*] \subset K$ and*

$$\widehat{\varphi}(S_1, \dots, S_N)^* = \widehat{\varphi}(S_1^*, \dots, S_N^*).$$

$\lhd$ By Krengel–Synnatschke Theorem [1, Theorem 5.11] the map $S \mapsto S^*$ is a lattice homomorphism from $L^r(E,F)$ into $L^r(F_n^\sim, E^\sim)$, see [1, Theorem 7.6]. Thus, we need only to apply [5, Proposition 2.6]. $\rhd$

3.7. Proposition. *The second formula in Theorem* 3.4 *and Proposition* 3.6 *were obtained by A. V. Bukhvalov* [2] *under some additional restrictions.*

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Kusraev Anatoly Georgievich
South Mathematical Institute
Vladikavkaz Science Center of the RAS, *Director*
22 Markus Street, Vladikavkaz, 362027, Russia
E-mail: `kusraev@smath.ru`