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WEAKLY $\aleph_1$-SEPARABLE QUASI-COMPLETE ABELIAN p-GROUPS ARE BOUNDED

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We prove that each weakly $\aleph_1$-separable quasi-complete abelian p-group is bounded, thus extending recent results of ours in (Vladikavkaz Math. J., 2007 and 2008).

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Throughout the text of this brief article, suppose that all our groups are abelian p-torsion groups where p is an arbitrary prime fixed for the duration. We first of all fix two basic concepts:

Definition 1. A separable group G is called *weakly $\aleph_1$-separable* if its each countable subgroup can be embedded in a countable pure and nice subgroup of G.

Referring to [6], G is weakly $\aleph_1$-separable if and only if for every $C \leqslant G$ with $|C| \leqslant \aleph_0$ the inequality $|(G/C)^1| \leqslant |C|$ holds. If it is fulfilled for all infinite subgroups C, then G is said to be *a Q-group*.

Definition 2. A reduced group G is called *quasi-complete* if for every pure subgroup P of G the quotient $(G/P)^1$ is divisible.

It is straightforward that quasi-complete groups are themselves separable.

All other notions and notations used here are standard and follow essentially the existing classical literature [4] (see also [1] and [2]).

In [1], we showed that every weakly $\aleph_1$-separable group of cardinal $\aleph_1$ is quasi-complete if and only if it is bounded and, in particular, the same holds for Q-groups of cardinal $\aleph_1$. Specifically, the following is true:

Theorem (2007). *Let G be a group of cardinality $\aleph_1$. Then G is quasi-complete weakly $\aleph_1$-separable if and only if G is bounded.*

Next, we extended in [2] that result for Q-groups of arbitrary cardinality. In fact, the following is valid:

Theorem (2008). *Let G be a group. Then G is a quasi-complete Q-group if and only if G is bounded.*

It is the purpose of this short note to generalize both the results presented to weakly $\aleph_1$-separable groups of any cardinality.

So, we are in a position to formulate the following assertion, provided that the *Continuum Hypothesis* (CH), that is, $\aleph_1 = 2^{\aleph_0}$ holds.

Theorem (CH). *Let G be a group. Then G is quasi-complete weakly $\aleph_1$-separable if and only if G is bounded.*

$\lhd$ The sufficiency is self-evident.

As for the necessity, first of all, suppose G is of cardinal $\aleph_1$. Then the claim follows in virtue of the quoted above Theorem (2007).

Let us assume that $|G| > \aleph_1$. Note that since any group G is isomorphic to a direct sum $B \oplus G'$, where B is bounded and the rank and final rank of G' agree, we will use in our situation the terms «cardinality» and «final rank» interchangeable. So, $\mathrm{fin}r(G) > \aleph_1$ and applying [4, Theorem 74.8] G has to be torsion-complete. In accordance with [5] (see also [4, p. 24, Theorem 68.1]) one may write that $G = \overline{B}$ for some basic subgroup B of G, where $\overline{B}$ is its torsion completion in the p-adic topology of G. Let B_1 be a countable unbounded summand of B, that is, $B = B_1 \oplus B_2$ and $|B_1| \leqslant \aleph_0$. It is not hard to verify that $G = \overline{B_1} \oplus \overline{B_2}$, where $\overline{B_1}$ and $\overline{B_2}$ are the torsion completions of B_1 and B_2, respectively, in the p-adic topology of G (see [4, p. 23]). It is simple to check that a subgroup of a weakly $\aleph_1$-separable group is again weakly $\aleph_1$-separable as well. Therefore, $\overline{B_1}$ must be weakly $\aleph_1$-separable. However, utilizing [4, p. 29, Exercise 7] we derive that $|\overline{B_1}| = |B_1|^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph_1 > \aleph_0 = |B_1|$. But, it is well known that $\overline{B_1}/B_1$ is divisible (see, e.g., [4]) and hence $|(\overline{B_1}/B_1)^1| = |\overline{B_1}/B_1| \leqslant |B_1| = \aleph_0$. That is why, $|\overline{B_1}| = |B_1|$ which contradicts the inequality given above. Consequently, B does not possess an unbounded summand and thus it is bounded, hence so is G because it is reduced itself, as required. $\rhd$

In closing, we state the following.

Remark. Regarding [3, p. 136, Proposition 7.1] and [3, p. 137, Conjecture 7.1], by what we have shown above, it is simply impossible in any version of set theory for a weakly $\aleph_1$-separable group to have a subgroup which is an unbounded torsion-complete group. Indeed, as observed above, such a subgroup will always have a torsion-complete summand of cardinal $2^{\aleph_0}$ with a countable basic subgroup. So, we will always contradict the idea of weak $\aleph_1$-separability which is inherited by arbitrary subgroups.

So, in conclusion, Proposition 7.1 can be extended to groups with basic subgroups of arbitrary cardinality and thereby Conjecture 7.1 is settled in the negative.

Corrections: In [2] there are three misprints. First, in Theorem 2 and Corollary 3 the abbreviation «[2]» should be written and read as «(CH)», as well as on line 3 of the proof of Theorem 2 the symbol «α_1» should be «$\aleph_1$».

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