Vladikavkaz Mathematical Journal
2009, Vol. 11, No 2, pp. 31–42

UDC 517.98

FUNCTIONAL CALCULUS AND MINKOWSKI DUALITY ON VECTOR LATTICES

To Şafak Alpay on his sixtieth birthday

A. G. Kusraev

The paper extends homogeneous functional calculus on vector lattices. It is shown that the function of elements of a relatively uniformly complete vector lattice can naturally be defined if the positively homogeneous function is defined on some conic set and is continuous on some closed convex subcone. An interplay between Minkowski duality and homogeneous functional calculus leads to the envelope representation of abstract convex elements generated by the linear hull of a finite collection in a uniformly complete vector lattice.

Mathematics Subject Classification (2000): 46A40, 47A50, 47A60, 47A63, 47B65.

Key words: vector lattices, functional calculus, Minkowski duality, sublinear and superlinear operators, envelope representation.

1. Introduction

For any finite sequence $(x_1, \dots, x_N)$ $(N \in \mathbb{N})$ in a relatively uniformly complete vector lattice the expression of the form $\widehat{\varphi}(x_1, \dots, x_N)$ can be correctly defined provided that φ is a positively homogeneous continuous function on $\mathbb{R}^N$. The study of such expressions, called *homogeneous functional calculus*, provides a useful tool in a variety of areas, see [4, 9, 10, 14, 15, 16, 21]. At the same time it is of importance in certain problems to deal with $\widehat{\varphi}(x_1, \dots, x_N)$ even if φ is defined on a conic subset of $\mathbb{R}^N$ [2, 16, 17]. The first aim of this paper is to extend homogeneous functional calculus on uniformly complete vector lattices.

Let H be a linear (or semilinear) subset of a vector lattice E. The *support set* $\partial_H x$ of $x \in E$ with respect to H is the set of all H-minorants of x: $\partial_H x := \{h \in H : h \leqslant x\}$. The *$H$-convex hull* of $x \in E$ is defined by $\mathrm{co}_H\, x := \sup\{h \in H : h \in \partial_H x\}$. An element x is called *H-convex* (abstract convex with respect to H) if $\mathrm{co}_H\, x = x$. Now the problem is to examine abstract convex elements, that is elements which can be represented as upper envelopes of subsets of a given set H of elementary elements. (For this abstract convexity see [13, 20]). The second aim of the paper is the description of H-convex elements in E in the event that H is the linear hull of a finite collection $\{x_1, \dots, x_N\} \subset E$ of a vector lattice E. It turns out that under some conditions an element in E is H-convex if and only if it is of the form $\widehat{\varphi}(x_1, \dots, x_N)$ for some lower semicontinuous sublinear function φ.

Section 2 collects some auxiliary results. In Section 3 the extended homogeneous functional calculus is defined. It is shown that the expression $\widehat{\varphi}(x_1, \dots, x_N)$ can naturally be defined in any relatively uniformly complete vector lattice if a positively homogeneous function φ is defined on some conic set $\mathrm{dom}(\varphi) \subset \mathbb{R}^N$ and is continuous on some closed subcone of $\mathrm{dom}(\varphi)$. Section 4 contains some examples of computing $\widehat{\varphi}(u_1, \dots, u_N)$ whenever $u_1, \dots, u_N$ are continuous or measurable vector-valued functions, or φ is a Kobb–Duglas type function and $u_i := b(x_i, y_i)$ $(i = 1, \dots, N)$ for some lattice bimorphism b. In Section 5 Minkowski duality is transplanted to vector lattice by means of extended functional calculus.

There are different ways to define homogeneous functional calculus on vector lattices [3, 9, 14, 18]. We follow the approach of G. Buskes, B. de Pagter, and A. van Rooij [3] going back to G. Ya. Lozanovskiĭ [18]. Theorem 2.1 below see in [3, 10, 14, 21]. For the theory of vector lattices and positive operators we refer to the books [1] and [10]. All vector lattices in this paper are real and Archimedean.

2. Auxiliary results

Denote by $\mathscr{H}(\mathbb{R}^N)$ the vector lattice of all continuous functions $\varphi : \mathbb{R}^N \to \mathbb{R}$ which are *positively homogeneous* ($\equiv \varphi(\lambda t) = \lambda\varphi(t)$ for $\lambda \geqslant 0$ and $t \in \mathbb{R}^N$). Let dt_k stands for the kth coordinate function on $\mathbb{R}^N$, i. e. $dt_k : (t_1, \ldots, t_N) \mapsto t_k$.

2.1. Theorem. *Let E be a relatively uniformly complete vector lattice. For any $\mathfrak{x} := (x_1, \ldots, x_N) \in E^N$ there exists a unique lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \ldots, x_N) \quad \big(\varphi \in \mathscr{H}(\mathbb{R}^N)\big)$$

of $\mathscr{H}(\mathbb{R}^N)$ into E with $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1, \ldots, N)$.

If the vector lattice E is *universally σ-complete* ($\equiv$ Dedekind σ-complete and laterally σ-complete) and has an order unit, then Borel functional calculus is also available on E. Let $\mathscr{B}(\mathbb{R}^N)$ denotes the vector lattice of all Borel measurable functions $\varphi : \mathbb{R}^N \to \mathbb{R}$. The following result can be found in [10, Theorem 8.2.14].

2.2. Theorem. *Let E be a universally σ-complete vector lattice with a fixed weak order unit $\mathbb{1}$. For any $\mathfrak{x} := (x_1, \ldots, x_N) \in E^N$ there exists a unique sequentially order continuous lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \ldots, x_N) \quad \big(\varphi \in \mathscr{B}(\mathbb{R}^N)\big)$$

of $\mathscr{B}(\mathbb{R}^N)$ into E such that $\widehat{\mathfrak{x}}(1_{\mathbb{R}^N}) = \mathbb{1}$ and $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1, \ldots, N)$.

Let $\mathscr{H}_{\rm Bor}(\mathbb{R}^N)$ denote the vector sublattice of $\mathscr{B}(\mathbb{R}^N)$ consisting of all positively homogeneous Borel functions $\varphi : \mathbb{R}^N \to \mathbb{R}$.

2.3. Theorem. *Let E be a universally σ-complete vector lattice with an order unit. For any $\widehat{\mathfrak{x}} := (x_1, \ldots, x_n) \in E^N$ there exists a unique sequentially order continuous lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \ldots, x_N) \quad \big(\varphi \in \mathscr{H}_{\rm Bor}(\mathbb{R}^N)\big)$$

of $\mathscr{H}_{\rm Bor}(\mathbb{R}^N)$ into E such that $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1, \ldots, N)$.

$\lhd$ Fix an order unit $\mathbb{1}$ in E and take $\widehat{\mathfrak{x}}$ as in Theorem 2.2. Since $\mathscr{H}_{\rm Bor}(\mathbb{R}^N)$ is an order σ-closed vector sublattice of $\mathscr{B}(\mathbb{R}^N)$, the restriction of $\widehat{\mathfrak{x}}$ onto $\mathscr{H}_{\rm Bor}(\mathbb{R}^N)$ is also an order σ-continuous lattice homomorphism. If $h : \mathscr{H}_{\rm Bor}(\mathbb{R}^N) \to E$ is another order σ-continuous lattice homomorphism with $h(dt_k) = \widehat{\mathfrak{x}}(dt_k)$ $(k := 1, \ldots, N)$, then h and $\widehat{\mathfrak{x}}(\cdot)$ coincide on $\mathscr{H}(\mathbb{R}^N)$ by Theorem 2.1. Afterwards, we infer that h and $\widehat{\mathfrak{x}}(\cdot)$ coincide on the whole $\mathscr{H}_{\rm Bor}(\mathbb{R}^N)$ due to order σ-continuity. $\rhd$

3. Functional calculus

In this section we define extended homogeneous functional calculus on relatively uniformly complete vector lattices. Everywhere below $\mathfrak{x} := (x_1, \ldots, x_N) \in E^N$.

3.1. Consider a finite collection $x_1, \ldots, x_N \in E$ and a vector sublattice $L \subset E$. Denote by $\langle x_1, \ldots, x_N\rangle$ and $\mathrm{Hom}(L)$ respectively the vector sublattice of E generated by $\{x_1, \ldots, x_N\}$ and the set of all $\mathbb{R}$-valued lattice homomorphisms on L. Put

$$[\mathfrak{x}] := [x_1, \ldots, x_N] := \big\{(\omega(x_1), \ldots, \omega(x_N)) \in \mathbb{R}^N : \ \omega \in \mathrm{Hom}(\langle x_1, \ldots, x_N\rangle)\big\}\,.$$

Let $e := |x_1| + \ldots + |x_N|$ and $\Omega := \{\omega \in \mathrm{Hom}(\langle x_1, \ldots, x_N\rangle) : \ \omega(e) = 1\}$. Then e is a strong order unit in $\langle x_1, \ldots, x_N\rangle$ and Ω separates the points of $\langle x_1, \ldots, x_N\rangle$. Moreover, Ω may be endowed with a compact Hausdorff topology so that the functions $\widehat{x}_k : \Omega \to \mathbb{R}$ defined by $\widehat{x}_k(\omega) := \omega(x_k)$ $(k := 1, \ldots, N)$ are continuous and $x \mapsto \widehat{x}$ is a lattice isomorphism of $\langle x_1, \ldots, x_N\rangle$ into $C(\Omega)$. Put

$$\Omega(x_1, \ldots, x_N) := \big\{(\omega(x_1), \ldots, \omega(x_N)) \in \mathbb{R}^N : \ \omega \in \Omega\big\},$$

and observe that $[x_1, \ldots, x_N] := \mathrm{cone}(\Omega(x_1, \ldots, x_N))$, where $\mathrm{cone}(A)$ is the conic hull of A defined as $\bigcup\{\lambda A : \ 0 \leqslant \lambda \in \mathbb{R}\}$. Evidently, $\Omega(x_1, \ldots, x_N)$ is a compact subset of $\mathbb{R}^N$, since it is the image of the compact set Ω under the continuous map $\omega \mapsto (\widehat{x}_1(\omega), \ldots, \widehat{x}_N(\omega))$. Therefore, $[x_1, \ldots, x_N]$ is a *compactly generated* conic set in $\mathbb{R}^N$. (The conic set $[x_1, \ldots, x_N]$ is closed if $0 \notin \Omega(x_1, \ldots, x_N)$.) A set $C \subset \mathbb{R}^N$ is called *conic* if $\lambda C \subset C$ for all $\lambda \geqslant 0$ while a convex conic set is referred to as a *cone*. The reasoning similar to [3, Lemma 3.3] shows that $[x_1, \ldots, x_N]$ is uniquely determined by any point separating subset Ω_0 of $\mathrm{Hom}(\langle x_1, \ldots, x_N\rangle)$. Indeed, if $\Omega_0' := \{\omega(e)^{-1}\omega : \ 0 \neq \omega \in \Omega_0\}$, then Ω_0' is a dense subset of Ω and $[x_1, \ldots, x_N] = \mathrm{cone}\big(\mathrm{cl}(\Omega_0'(x_1, \ldots, x_N))\big)$, where $\Omega_0'(x_1, \ldots, x_N)$ is the set of all $(\omega(x_1), \ldots, \omega(x_N)) \in \mathbb{R}$ with $\omega \in \Omega_0'$.

3.2. For a conic set C in $\mathbb{R}^N$ denote by $\widehat{C} \subset E^N$ the set of all $\mathfrak{x} := (x_1, \ldots, x_N) \in E^N$ with $[\mathfrak{x}] \subset C$. Consider a conic set $K \subset C$. Let $\mathscr{H}(C; K)$ denotes the vector lattice of all positively homogeneous functions $\varphi : C \to \mathbb{R}$ with continuous restriction to K. Fix $(x_1, \ldots, x_N) \in \widehat{C}$ and take $\varphi \in \mathscr{H}(C; [\mathfrak{x}])$. We say that $\widehat{\varphi}(x_1, \ldots, x_N)$ *exists* or *is well defined* in E and write $y = \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \ldots, x_N)$ if there is an element $y \in E$ such that $\omega(y) = \varphi(\omega(x_1), \ldots, \omega(x_N))$ for every $\omega \in \mathrm{Hom}(\langle x_1, \ldots, x_N, y\rangle)$. This definition is correct, since for any given $(x_1, \ldots, x_N) \in \widehat{C}$ and $\varphi \in \mathscr{H}(C; [\mathfrak{x}])$ there exists at most one $y \in E$ such that $y = \widehat{\varphi}(x_1, \ldots, x_N)$. It is immediate from the definition that $\widehat{\varphi}(\lambda_1 x, \ldots, \lambda_N x)$ is well defined for any $(\lambda_1, \ldots, \lambda_N) \in C$ and $\widehat{\varphi}(\lambda_1 x, \ldots, \lambda_N x) = \widehat{\varphi}(\lambda_1, \ldots, \lambda_N)x$ whenever $0 \leqslant x \in E$. The following proposition can be proved as [3, Lemma 3.3].

Assume that L is a vector sublattice of E containing $\{x_1, \ldots, x_N, y\}$ and $\varphi \in \mathscr{H}(C; [x_1, \ldots, x_N])$. If $\omega(y) = \varphi(\omega(x_1), \ldots, \omega(x_N))$ $(\omega \in \Omega_0)$ for some point separating set Ω_0 of $\mathbb{R}$-valued lattice homomorphisms on L, then $y = \widehat{\varphi}(x_1, \ldots, x_N)$.

3.3. Theorem. *Let E be a relatively uniformly complete vector lattice and $\mathfrak{x} \in E^N$, $\mathfrak{x} = (x_1, \ldots, x_N)$. Assume that $C \subset \mathbb{R}^N$ is a conic set and $[\mathfrak{x}] \subset C$. Then $\widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \ldots, x_N)$ exists for every $\varphi \in \mathscr{H}(C; [\mathfrak{x}])$ and the mapping*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \ldots, x_N) \quad \big(\varphi \in \mathscr{H}(C; [\mathfrak{x}])\big)$$

is a unique lattice homomorphism from $\mathscr{H}(C; [\mathfrak{x}])$ into E with $\widehat{dt_j}(x_1, \ldots, x_N) = x_j$ for $j := 1, \ldots, N$.

$\triangleleft$ Let $\mathscr{H}([\mathfrak{x}])$ denotes the vector lattice of all positively homogeneous continuous functions defined on $[\mathfrak{x}]$. Then $\mathscr{H}([\mathfrak{x}])$ is isomorphic to $C(Q)$, where $Q := [\mathfrak{x}] \cap \mathbb{S}$ and $\mathbb{S} := \{s \in \mathbb{R}^N : \ \|s\| := \max\{|s_1|, \ldots, |s_N|\} = 1\}$. Much the same reasoning as in [3, Proposition 3.6, Theorem 3.7] shows the existence of a unique lattice homomorphism h from $\mathscr{H}([\mathfrak{x}])$ into E such that $\widehat{dt_j}(x_1, \ldots, x_N) = x_j$ $(j := 1, \ldots, N)$. Denote by ρ the restriction operator $\varphi \mapsto \varphi|_{[\mathfrak{x}]}$ $\big(\varphi \in \mathscr{H}(C; [\mathfrak{x}])\big)$. Then $\rho \circ h$ is the required lattice homomorphism. $\triangleright$

Observe that if $\varphi, \psi \in \mathscr{H}(C; [\mathfrak{x}])$ and $\varphi(t) \leqslant \psi(t)$ for all $t \in [\mathfrak{x}]$, then $\widehat{\varphi}(x_1, \ldots, x_N) \leqslant \widehat{\psi}(x_1, \ldots, x_N)$. Evidently, $|\varphi(t)| \leqslant |||\varphi||| \cdot \|t\|$ for all $t \in [\mathfrak{x}]$ with $|||\varphi||| := \sup\{\varphi(t) : \ t \in Q\}$ and hence

$$|\widehat{\varphi}(x_1,\dots,x_N)| \leqslant |||\varphi||| \left(|x_1| \vee \dots \vee |x_N|\right).$$

In particular, the kernel $\ker(\widehat{\mathfrak{x}})$ of $\widehat{\mathfrak{x}}$ consists of all $\varphi \in \mathscr{H}(C;[\mathfrak{x}])$ vanishing on $[\mathfrak{x}]$.

3.4. Let $K, M, N \in \mathbb{N}$ and consider two conic sets $C \subset \mathbb{R}^N$ and $D \subset \mathbb{R}^M$. Let $x_1,\dots,x_N \in E$, $\mathfrak{x} := (x_1,\dots,x_N)$, $[\mathfrak{x}] \subset C$, $\varphi_1,\dots,\varphi_M \in \mathscr{H}(C;[\mathfrak{x}])$, and denote $\varphi := (\varphi_1,\dots,\varphi_M)$ and $\mathfrak{y} := (y_1,\dots,y_N)$ with $y_k = \widehat{\varphi}_k(x_1,\dots,x_N)$ $(k := 1,\dots,M)$. Suppose that $[\mathfrak{y}] \subset D$, $\varphi(C) \subset D$, and $\varphi([\mathfrak{x}]) \subset [\mathfrak{y}]$. If $\psi := (\psi_1,\dots,\psi_K)$ with $\psi_1,\dots,\psi_K \in \mathscr{H}(D;[\mathfrak{y}])$, then $\psi_1 \circ \varphi,\dots,\psi_K \circ \varphi \in \mathscr{H}(C;[\mathfrak{x}])$. Moreover, $\widehat{\varphi}(\mathfrak{x}) := (\widehat{\varphi}_1(\mathfrak{x}),\dots,\widehat{\varphi}_M(\mathfrak{x})) \in E^M$, $\widehat{\psi}(\mathfrak{y}) := (\widehat{\psi}_1(\mathfrak{y}),\dots,\widehat{\psi}_K(\mathfrak{y})) \in E^K$, and $\widehat{\psi \circ \varphi}(\mathfrak{x}) := (\widehat{\psi_1 \circ \varphi}(\mathfrak{x}),\dots,\widehat{\psi_K \circ \varphi}(\mathfrak{x})) \in E^K$ are well defined and

$$\widehat{(\psi \circ \varphi)}(\mathfrak{x}) = \widehat{\psi}(\widehat{\varphi}(\mathfrak{x})).$$

3.5. Theorem. *Let C and K are conic sets in $\mathbb{R}^N$ with K closed and $K \subset C$ and let $\varphi \in \mathscr{H}(C;K)$. Then for every $\varepsilon > 0$ there exists a number $R_\varepsilon > 0$ such that*

$$|\widehat{\varphi}(\mathfrak{x}+\mathfrak{y}) - \widehat{\varphi}(\mathfrak{x})| \leqslant \varepsilon |||\mathfrak{x}||| + R_\varepsilon |||\mathfrak{y}|||$$

for any finite collections $\mathfrak{x} = (x_1,\dots,x_N) \in E^N$ and $\mathfrak{y} = (y_1,\dots,y_N) \in E^N$, provided that $\mathfrak{x},\mathfrak{y} \in \widehat{K}$, $\mathfrak{x}+\mathfrak{y} \in \widehat{K}$ and $|||(u_1,\dots,u_N)|||$ stands for $|u_1| \vee \dots \vee |u_N|$.

$\lhd$ The proof is a duly modification of arguments from [4, Theorem 7]. Denote $K^\times := \{(s,t) \in K \times K :\ s+t \in K\}$ and define A as the set of all $(s,t) \in K^\times$ with $\max\{\|s\|,\|t\|\} = 1$ and $\tau(s,t) := |\varphi(s+t) - \varphi(s)| \geqslant \varepsilon\|s\|$, where $\|s\| := \max\{|s_1|,\dots,|s_N|\}$. Then A is a compact subset of $K \times K$ and $(s,t) \mapsto (\tau(s,t) - \varepsilon\|s\|)/\|t\|$ is a continuous function on A, since $\|t\| \neq 0$ for $(s,t) \in A$. Therefore,

$$R_\varepsilon := \sup\left\{\frac{\tau(s,t) - \varepsilon\|s\|}{\|t\|} :\ (s,t) \in A\right\} < \infty.$$

Hence $\tau(s,t) \leqslant \varepsilon\|s\| + R_\varepsilon\|t\| =: \sigma(s,t)$ for all $(s,t) \in K^\times$. Evidently, $\tau \in \mathscr{H}(C^\times, K^\times)$, $\sigma \in \mathscr{H}(\mathbb{R}^N \times \mathbb{R}^N)$, and $\tau \leqslant \sigma$ on $K^\times$. It remains to observe that $(\mathfrak{x},\mathfrak{y}) \in \widehat{K^\times}$ and apply 3.3 and the desired inequality follows. $\rhd$

3.6. Proposition. *Let E and F be uniformly complete vector lattices, E_0 a uniformly closed sublattice of E, and $h : E_0 \to F$ a lattice homomorphism. Let C be a conic set in $\mathbb{R}^N$, $x_1,\dots,x_N \in E_0$, and $\varphi \in \mathscr{H}(C;[x_1,\dots,x_N])$. Then $[h(x_1),\dots,h(x_N)] \subset [x_1,\dots,x_N]$ and*

$$h(\widehat{\varphi}(x_1,\dots,x_N)) = \widehat{\varphi}(h(x_1),\dots,h(x_N)).$$

In particular, if h is the inclusion map $E \hookrightarrow F$ and $x_1,\dots,x_N \in E$, then the element $\widehat{\varphi}(x_1,\dots,x_N)$ relative to F is contained in E and its meaning relative to E is the same.

$\lhd$ Put $y_i := h(x_i)$ $(i := 1,\dots,N)$. If $\omega \in \operatorname{Hom}(\langle y_1,\dots,y_N\rangle)$, then $\bar\omega := \omega \circ h$ belongs to $\operatorname{Hom}(\langle x_1,\dots,x_N\rangle)$ and $(\omega(y_1),\dots,\omega(y_N)) = (\bar\omega(x_1),\dots,\bar\omega(x_N)) \in [x_1,\dots,x_N]$. Therefore, $[y_1,\dots,y_N]$ is contained in $[x_1,\dots,x_N]$. Now, if $y = \widehat{\varphi}(y_1,\dots,y_N)$, $x = \widehat{\varphi}(x_1,\dots,x_N)$, and $\omega \in \operatorname{Hom}(\langle y,y_1,\dots,y_N\rangle)$, then $\bar\omega \in \operatorname{Hom}(\langle x,x_1,\dots,x_N\rangle)$ and by definition

$$\omega(y) = \varphi(\bar\omega(x_1),\dots,\bar\omega(x_N)) = \bar\omega(\widehat{\varphi}(x_1,\dots,x_N) = \omega(h(x)),$$

so that $y = h(x)$. $\rhd$

Denote $\mathscr{H}^\infty_{\mathrm{Bor}}(\mathbb{R}^N,[\mathfrak{x}]) := \big\{\varphi \in \mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N) :\ \sup\{|\varphi(s)| :\ s \in \mathbb{S} \cap [\mathfrak{x}]\} < +\infty\big\}$.

3.7. Theorem. *Let E be a Dedekind σ-complete vector lattice. For $\widehat{\mathfrak{x}} := (x_1,\dots,x_n)$ in E^N there exists a unique sequentially order continuous lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1,\dots,x_N) \quad \big(\varphi \in \mathscr{H}^\infty_{\mathrm{Bor}}(\mathbb{R}^N,[\mathfrak{x}])\big)$$

of $\mathscr{H}^\infty_{\mathrm{Bor}}(\mathbb{R}^N,[\mathfrak{x}])$ into E such that $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1,\dots,N)$.

◁ Let E_0 be the order ideal in E generated by $x_1, \dots, x_N$. According to 1.3 there exists a unique sequentially order continuous lattice homomorphism $\widehat{\mathfrak{x}}$ of $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ into $(E_0)^{u\sigma}$, a universal σ-completion of E_0, with $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1, \dots, N)$. Clearly, the image of $\mathscr{H}^{\infty}_{\mathrm{Bor}}(\mathbb{R}^N, [\,\mathfrak{x}\,])$ under $\widehat{\mathfrak{x}}$ is contained in E_0. ▷

4. Examples

Now, we consider extended functional calculus on some special vector lattices E and for some special functions φ. Everywhere in the section $\varphi \in \mathscr{H}(C; K)$.

4.1. Proposition. *Let Q be a Hausdorff topological space, X a Banach lattice, and $C_b(Q, X)$ the Banach lattice of norm bounded continuous functions from Q to X. Assume that $u_1, \dots, u_N \in C_b(Q, X)$ and $[u_1, \dots, u_N] \subset K$. Then $[u_1(q), \dots, u_N(q)] \subset K$ for all $q \in Q$ and*

$$\widehat{\varphi}(u_1, \dots, u_N)(q) = \widehat{\varphi}(u_1(q), \dots, u_N(q)) \quad (q \in Q).$$

◁ Indeed, for $q \in Q$ the map $\widehat{q} : C_b(Q, X) \to X$ defined by $\widehat{q} : u \mapsto u(q)$ is a lattice homomorphism. Therefore, given $u_1, \dots, u_N \in C_b(Q, X)$, by Proposition 3.6 we have $[\widehat{q}(u_1), \dots, \widehat{q}(u_N)] \subset [u_1, \dots, u_N]$ and $\widehat{q}(\widehat{\varphi}(u_1, \dots, u_N)) = \widehat{\varphi}(\widehat{q}(u_1), \dots, \widehat{q}(u_N))$ from which the required is immediate. ▷

4.2. Suppose now that Q is compact and extremally disconnected. Let $u : D \to X$ be a continuous function defined on a dense subset $D \subset Q$. Denote by $\bar{D}$ the totality of all points in Q at which u has limit and put $\bar{u}(q) := \lim_{p \to q} u(p)$ for all $q \in \bar{D}$. Then the set $\bar{D}$ is comeager in Q and the function $\bar{u} : \bar{D} \to X$ is continuous. Recall that a set is called *comeager* if its complement is meager. Thus, the function $\bar{u}$ is the «widest» continuous extension of u i. e., the domain of every continuous extension of u is contained in $\bar{D}$ and, moreover, $\bar{u}$ is an extension of every continuous extension of u. The function $\bar{u}$ is called the *maximal extension* of u and denoted by $\mathrm{ext}(u)$, see [6]. A continuous function $u : D \to X$ defined on a dense subset $D \subset Q$ is said to be *extended*, if $\mathrm{ext}(u) = u$. Note that all extended functions are defined on comeager subsets of Q.

Let $C_\infty(Q, X)$ stands for the set of all extended X-valued functions. The totality of all bounded extended functions is denoted by $C^b_\infty(Q, X)$. Observe that $C_\infty(Q, X)$ can be represented also as the set of cosets of continuous functions u that act from comeager subsets $\mathrm{dom}(u) \subset Q$ into X. Two vector-valued functions u and v are equivalent if $u(t) = v(t)$ whenever $t \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$.

The set $C_\infty(Q, X)$ is endowed, in a natural way, with the structure of a lattice ordered module over the f-algebra $C_\infty(Q)$. Moreover, $C_\infty(Q, X)$ is uniformly complete and for any $u_1, \dots, u_N \in C_\infty(Q, X)$ the element $\widehat{\varphi}(u_1, \dots, u_N)$ is well defined in $C_\infty(Q, X)$ provided that $[u_1, \dots, u_N] \subset K$.

4.3. Proposition. *Let Q be a extremally disconnected conpact space and X a Banach lattice. Let $u_1, \dots, u_N \in C_\infty(Q, X)$ and $[u_1, \dots, u_N] \subset K$. Then there exists a comeager subset $Q_0 \subset Q$ such that $Q_0 \subset \mathrm{dom}(u_i)$ for all $i := 1, \dots, N$, $[u_1(q), \dots, u_N(q)] \subset K$ for every $q \in Q_0$, and $\widehat{\varphi}(u_1, \dots, u_N)$ is the maximal extension of the continuous function $q \mapsto \widehat{\varphi}(u_1(q), \dots, u_N(q))$ $(q \in Q_0)$, i. e.*

$$\widehat{\varphi}(u_1, \dots, u_N)(q) = \widehat{\varphi}(u_1(q), \dots, u_N(q)) \quad (q \in Q_0).$$

◁ Put $Q' := \mathrm{dom}(u_1) \cap \dots \cap \mathrm{dom}(u_N)$ and observe that Q' is comeager. There exists a unique function $e \in C_\infty(Q)$ such that $e'(q) := \|u_1(q)\| + \dots + \|u_N(q)\|$ $(q \in Q')$. Let E be the order ideal in $C_\infty(Q)$ generated by e and define the sublattice $E(X) \subset C_\infty(Q,X)$ by

$$E(X) := \big\{u \in C_\infty(Q,X) :\ (\exists\, 0 \leqslant C \in \mathbb{R})\, (\forall\, q \in \mathrm{dom}(u))\, \|u(q)\| \leqslant Ce(q)\big\}.$$

In the Boolean algebra of clopen subsets of Q there exists a partition of unity $\big(Q(\xi)\big)_{\xi\in\Xi}$ with $\chi_{Q(\xi)}e \in C(Q)$ for all $\xi \in \Xi$. Put $Q'_\xi := Q' \cap Q_\xi$ and $Q_0 := \bigcup_{\xi\in\Xi} Q'_\xi$ and observe that Q_0 is comeager in Q. Let π_ξ stands for the band projection in $C_\infty(Q,X)$ defined by $\pi_\xi : u \mapsto \chi_{Q(\xi)}u$. Then $\pi_\xi(E(X)) \subset C_b(Q,X)$ and $(\pi_\xi u_i)(q) = u_i(q)$ $(q \in Q'_\xi;\ i = 1,\dots,N)$. Finally, given $q \in Q'_\xi$, in view of Propositions 3.6 and 4.1 we have $[u_1(q),\dots,u_N(q)] = [(\pi_\xi u_1)(q),\dots,(\pi_\xi u_N)(q)] \subset K$ and

$$(\pi_\xi\widehat{\varphi}(u_1,\dots,u_N))(q)\widehat{\varphi}((\pi_\xi u_1)(q),\dots,(\pi_\xi u_N)(q)) =$$
$$= \widehat{\varphi}(\pi_\xi u_1,\dots,\pi_\xi u_N)(q) = \widehat{\varphi}(u_1(q),\dots,u_N(q))$$

and the proof is complete. $\triangleright$

4.4. Let (Ω,Σ,μ) be a measure space with the direct sum property and X be a Banach lattice. Let $\mathscr{L}^0(\mu,X) := \mathscr{L}^0(\Omega,\Sigma,\mu,X)$ be the set of all Bochner measurable functions defined almost everywhere on Ω with values in X and $L^0(\mu,X) := \mathscr{L}^0(\mu,X)/\sim$ the space of all equivalence classes (of almost everywhere equal) functions from $\mathscr{L}^0(\mu,X)$. Then $L^0(\mu,X)$ is a Banach lattice and hence $\widehat{\varphi}(u_1,\dots,u_N)$ is well defined in $L^0(\mu,X)$ for $\varphi \in \mathscr{H}(C;K)$ and $u_1,\dots,u_N \in L^0(\mu,X)$ with $[u_1,\dots,u_N] \subset K$. Denote by $\tilde{u}$ the equivalence class of $u \in \mathscr{L}^0(\mu,X)$.

Let $\mathscr{L}^\infty(\mu,X)$ stand for the part of $\mathscr{L}^0(\mu,X)$ comprising all essentially bounded functions and $L^\infty(\mu,X) := \mathscr{L}^\infty(\mu,X)/\sim$. Put $\mathscr{L}^\infty(\mu) := \mathscr{L}^\infty(\mu,\mathbb{R})$ and $L^\infty(\mu) := L^\infty(\mu,\mathbb{R})$. Denote by $\mathbb{L}^\infty(\mu)$ the part of $\mathscr{L}^\infty(\mu)$ consisting of all function defined everywhere on Ω. Then $\mathbb{L}^\infty(\mu)$ is a vector lattice with point-wise operations and order. Recall that a lattice homomorphism $\rho : L^\infty(\mu) \to \mathbb{L}^\infty(\mu)$ is said to be a *lifting* of $L^\infty(\mu)$ if $\rho(f) \in f$ for every $f \in L^\infty(\mu)$ and $\rho(\mathbb{1})$ is the identically one function on Ω. (Here $\mathbb{1}$ is the coset of the identically one function on Ω). Clearly, a lifting is a right-inverse of the quotient homomorphism $\phi : f \mapsto \tilde{f}$ $(f \in \mathscr{L}^\infty(\mu)$. The space $L^\infty(\mu)$ admits a lifting if and only if (Ω,Σ,μ) possesses the direct sum property. If $f \in \mathscr{L}^\infty(\mu)$, then the function $\rho(\tilde{f})$ is also denoted by $\rho(f)$.

4.5. Proposition. *Let $u_1,\dots,u_N \in \mathscr{L}^0(\Omega,\Sigma,\mu,F)$, and $[\widetilde{u}_1,\dots,\widetilde{u}_N] \subset K$. Then there exists a measurable set $\Omega_0 \subset \Omega$ such that $\mu(\Omega\setminus\Omega_0) = 0$, $[u_1(\omega),\dots,u_N(\omega)] \subset K$ for all $\omega \in \Omega_0$, and $\widehat{\varphi}(\tilde{u}_1,\dots,\tilde{u}_N)$ is the equivalence class of the measurable function $\omega \mapsto \widehat{\varphi}(u_1(\omega),\dots,u_N(\omega))$ $(\omega \in \Omega_0)$.*

$\triangleleft$ The problem can be reduced to Proposition 4.2 by means of Gutman's approach to vector-valued measurable functions. Let ρ be a lifting of $L^\infty(\Omega,\Sigma,\mu)$ and $\tau : \Omega \to Q$ be the corresponding canonical embedding of Ω into the Stone space Q of the Boolean algebra $B(\Omega,\Sigma,\mu)$, see [?]. The preimage $\tau^{-1}(V)$ of any meager set $V \subset Q$ is measurable and μ-negligible. Moreover τ is Borel measurable and $v \circ \tau$ is Bochner measurable for every $v \in C_\infty(Q,X)$. Denote by τ^* the mapping which sends each function $v \in C_\infty(Q,X)$ to the equivalence class of the measurable function $v \circ \tau$. The mapping τ^* is a linear and order isomorphism of $C_\infty(Q,X)$ onto $L^0(\Omega,\Sigma,\mu,X)$. If σ is the inverse of τ^*, then $[\sigma(\tilde{u}_1),\dots,\sigma(\tilde{u}_N)] \subset K$ and $\sigma\widehat{\varphi}(\tilde{u}_1,\dots,\tilde{u}_N) = \widehat{\varphi}(\sigma(\tilde{u}_1),\dots,\sigma(\tilde{u}_N))$ by Proposition 3.6. According to Proposition 4.3 there exists a comeager subset $Q_0 \subset Q$ such that $[\sigma(\tilde{u}_1)(q),\dots,\sigma(\tilde{u}_N)(q)] \subset K$ for all $q \in Q_0$ and

$$\widehat{\varphi}(\sigma(\tilde{u}_1),\dots,\sigma(\tilde{u}_N))(q) = \widehat{\varphi}\big(\sigma(\tilde{u}_1)(q),\dots,\sigma(\tilde{u}_N)(q)\big) \quad (q \in Q_0).$$

Clearly, the functions $u'_i := \sigma(\tilde{u}_i)\circ\tau$ and u_i are equivalent and $\widehat{\varphi}(\tilde{u}_1,\dots,\tilde{u}_N)$ is the equivalence class of $\sigma(\widehat{\varphi}(\tilde{u}_1,\dots,\tilde{u}_N))\circ\tau$. Let Ω' stands for the set of all $\omega\in\Omega$ with $u'_i(\omega)=u_i(\omega)$ for all $i=1,\dots,N$. Then $\Omega_0:=\tau^{-1}(Q_0)\cap\Omega'$ is measurable and $\mu(\Omega\setminus\Omega_0)=0$. Substituting $q=\tau(\omega)$ we get $[u'_1(\omega),\dots,u'_N(\omega)]\subset K$ for all $\omega\in\Omega_0$ and

$$\sigma\widehat{\varphi}(\tilde{u}_1,\dots,\tilde{u}_N)(\tau(\omega))=\widehat{\varphi}\big(u'_1(\omega),\dots,u'_N(\omega)\big)\quad(\omega\in\Omega_0),$$

which is equivalent to the required statement. $\triangleright$

4.6. A conic set $C\subset\mathbb{R}^N$ is said to be *multiplicative* if $st:=(s_1t_1,\dots,s_Nt_N)\in C$ for all $s:=(s_1,\dots,s_N)\in C$ and $t:=(t_1,\dots,t_N)\in C$. A function $\varphi:C\to\mathbb{R}$ is called *multiplicative* if $\varphi(st)=\varphi(s)\varphi(t)$ for all $s,t\in C$.

Take a subset $I\subset\{1,\dots,N\}$ and define $\mathbb{R}^N_I$ as the cone in $\mathbb{R}^N$ consisting of 0 and $(s_1,\dots,s_N)\in\mathbb{R}^N_+$ with $s_i>0$ $(i\in I)$. We will write $x_i\gg 0$ $(i\in I)$ if $[x_1,\dots,x_N]\subset\mathbb{R}^N_I$. The general form of a positively homogeneous multiplicative function $\varphi:\mathbb{R}^N_I\to\mathbb{R}$ other that $\varphi\equiv 0$ is given by

$$\begin{gathered}\varphi(t_1,\dots,t_N)=0\quad(t_1\cdot\ldots\cdot t_N=0),\\ \varphi(t_1,\dots,t_N)=\exp(g_1(\ln t_1))\cdot\ldots\cdot\exp(g_N(\ln t_N))\quad(t_1\cdot\ldots\cdot t_N\neq 0),\end{gathered}$$

where $g_1,\dots,g_N$ are some additive functions in $\mathbb{R}$ with $\sum_{i=1}^N g_i=I_{\mathbb{R}}$. If φ is continuous at any interior point of $\mathbb{R}^N_+$ or bounded on any ball contained in $\mathbb{R}^N_I$, then we get a Kobb–Duglas type function and if, in addition, φ is nonnegative, then $\varphi(t_1,\dots,t_N)=t_1^{\alpha_1}\cdot\ldots\cdot t_N^{\alpha_N}$ with $\alpha_1,\dots,\alpha_N\in\mathbb{R}$ and $\sum_{i=1}^N\alpha_i=1$.

By definition $x_i\gg 0$ $(i\in I)$ implies that $\widehat{\varphi}(x_1,\dots,x_N)$ is well defined for every $\varphi\in\mathscr{H}(\mathbb{R}^N_I,[x_1,\dots,x_N])$. Thus, the expression $x_1^{\alpha_1}\cdot\ldots\cdot x_N^{\alpha_N}$ is well defined in E provided that $x_k\gg 0$ for all k with $\alpha_k<0$. At the same time $\varphi\in\mathscr{H}(\mathbb{R}^N_+)$ whenever $I=\varnothing$ and in this case $x_1^{\alpha_1}\cdot\ldots\cdot x_N^{\alpha_N}$ is well defined in E for arbitrary $x_k\geqslant 0$ and $\alpha_k\geqslant 0$ $(k=1,\dots,N)$.

4.7. Proposition. *Let E, F and G be vector lattices with E and F uniformly complete and $b:E\times F\to G$ a lattice bimorphism. Let $\mathfrak{x}:=(x_1,\dots,x_N)\in E^N$, $\mathfrak{y}:=(y_1,\dots,y_N)\in F^N$, and $[\mathfrak{x}]\cup[\mathfrak{y}]\subset K$ for some multiplicative closed conic set $K\subset\mathbb{R}^N$. If $\phi\in\mathscr{H}(C,K)$ is multiplicative on K, then $\widehat{\phi}(b(x_1,y_1),\dots,b(x_N,y_N))$ exists in G and*

$$\widehat{\phi}\,(b(x_1,y_1),\dots,b(x_N,y_N))=b\left(\widehat{\phi}(x_1,\dots,x_N),\widehat{\phi}(y_1,\dots,y_N)\right).$$

$\triangleleft$ Put $u=\widehat{\phi}(x_1,\dots,x_N)$ and $v=\widehat{\phi}(y_1,\dots,y_N)$. Let E_0 and F_0 be the vector sublattices in E and F generated by $\{u,x_1,\dots,x_N\}$ and $\{v,y_1,\dots,y_N\}$, respectively. Let G_0 be the order ideal in G generated by $b(e,f)$ where $e:=|u|+|x_1|+\dots+|x_N|$ and $f:=|v|+|y_1|+\dots+|y_N|$. Observe that $\mathrm{Hom}(G_0)$ separates the points of G_0. By [12, Theorem 3.2] every $\mathbb{R}$-valued lattice bimorphism on $E_0\times F_0$ is of the form $\sigma\otimes\tau:(x,y)\mapsto\sigma(x)\tau(y)$ with $\sigma\in\mathrm{Hom}(E_0)$ and $\tau\in\mathrm{Hom}(F_0)$. Denote by b_0 the restriction of b to $E_0\times F_0$. Given an $\mathbb{R}$-valued lattice homomorphism ω on G_0, we have the representation $\omega\circ b=\sigma\otimes\tau$ for some lattice homomorphisms $\sigma:E_0\to\mathbb{R}$ and $\tau:F_0\to\mathbb{R}$. Since K is multiplicative, we have

$$\begin{aligned}\big(\omega(b(x_1,y_1),\dots,\omega(b(x_N,y_N))\big)&=\big(\sigma(x_1)\tau(y_1),\dots,\sigma(x_N)\tau(y_N)\big)\\ &=(\sigma(x_1),\dots,\sigma(x_N))\cdot(\tau(y_1),\dots,\tau(y_N))\in K,\end{aligned}$$

and thus $[b(x_1,y_1),\dots,b(x_N,y_N)] \subset K$. Now, making use of 3.6 and multiplicativity of ϕ we deduce

$$\begin{aligned}\omega\circ b(u,v) &= \sigma(\widehat{\phi}(x_1,\dots,x_N))\tau(\widehat{\phi}(y_1,\dots,y_N))\\ &= \phi(\sigma(x_1),\dots,\sigma(x_N))\phi(\tau(y_1),\dots,\tau(y_N))\\ &= \phi(\sigma(x_1)\tau(y_1),\dots,\sigma(x_N)\tau(y_N))\\ &= \phi(\omega\circ b(x_1,y_1),\dots,\omega\circ b(x_N,y_N))\\ &= \omega\circ\widehat{\phi}(b(x_1,y_1),\dots,b(x_N,y_N)),\end{aligned}$$

as required by definition 3.2. $\rhd$

4.8. In particular, we can take $G := F\,\bar{\otimes}\,F$, the Fremlin tensor product of E and F [5], or $E^{\odot}$, the square of E [4], and put $b := \otimes$ or $b := \odot$ in 4.7. Thus, under the hypotheses of 4.7 we have

$$\begin{aligned}\widehat{\phi}(x_1\otimes y_1,\dots,x_N\otimes y_N) &= \widehat{\phi}(x_1,\dots,x_N)\otimes\widehat{\phi}(y_1,\dots,y_N),\\ \widehat{\phi}(x_1\odot y_1,\dots,x_N\odot y_N) &= \widehat{\phi}(x_1,\dots,x_N)\odot\widehat{\phi}(y_1,\dots,y_N).\end{aligned}$$

Taking 4.6 into consideration we get the following: If $0\leqslant\alpha_1,\dots,\alpha_N\in\mathbb{R}$, $\alpha_1+\dots+\alpha_N=1$, then $|x_1\otimes y_1|^{\alpha_1}\cdot\ldots\cdot|x_N\otimes y_N|^{\alpha_N}$ exists in $E\,\bar{\otimes}\,F$ for all $x_1,\dots,x_N\in E$ and $y_1,\dots,y_N\in F$ and

$$\prod_{i=1}^{N}|x_i\otimes y_i|^{\alpha_i} = \left(\prod_{i=1}^{N}|x_i|^{\alpha_i}\right)\otimes\left(\prod_{i=1}^{N}|y_i|^{\alpha_i}\right);$$

if, in addition, $E=F$, then we also have

$$\prod_{i=1}^{N}|x_i\odot y_i|^{\alpha_i} = \left(\prod_{i=1}^{N}|x_i|^{\alpha_i}\right)\odot\left(\prod_{i=1}^{N}|y_i|^{\alpha_i}\right).$$

4.10. Proposition. *Let E be a uniformly complete vector lattice, $\mathfrak{x} := (x_1,\dots,x_N)\in E^N$, $\mathfrak{p} := (\pi_1,\dots,\pi_N)\in \mathrm{Orth}(E)^N$, and $[\mathfrak{x}]\cup[\mathfrak{p}]\subset K$ for some multiplicative closed conic set $K\subset C\subset\mathbb{R}^N$. If $\phi\in\mathscr{H}(C,[\mathfrak{x}])\cap\mathscr{H}(C,[\mathfrak{p}])$ is multiplicative on K, then $\widehat{\phi}(\pi_1x_1,\dots,\pi_Nx_N))$ exists in E and*

$$\widehat{\phi}(\pi_1x_1,\dots,\pi_Nx_N) = \widehat{\phi}(\pi_1,\dots,\pi_N)\big(\widehat{\phi}(x_1,\dots,x_N)\big).$$

$\lhd$ The bilinear operator b from $E\times\mathrm{Orth}(E)$ to E defined by $b(x,\pi) := \pi(x)$ is a lattice bimorphism and all we need is to apply Proposition 4.7. $\rhd$

5. Minkowski duality

The *Minkowski duality* is the mapping that assigns to a sublinear function its support set or, in other words, its subdifferential (at zero). For any Hausdorff locally convex space X the Minkowski duality is a bijection between the collections of all lower semicontinuous sublinear functions on X and all closed convex subsets of the conjugate space X', see [13, 19]. The extended functional calculus (Theorems 2.3, 3.3, and 3.7) allows to transplant the Minkowski duality to vector lattice setting.

5.1. A function $\varphi:\mathbb{R}^N\to\mathbb{R}\cup\{+\infty\}$ is called *sublinear* if it is *positively homogeneous*, i. e. $\varphi(0)=0$ and $\varphi(\lambda t)=\lambda\varphi(t)$ for all $\lambda>0$ and $t\in\mathbb{R}^N$, and *subadditive*, i. e. $\varphi(s+t)\leqslant\varphi(s)+\varphi(t)$ for all s, $t\in\mathbb{R}^N$. A function $\psi:\mathbb{R}^N\to\mathbb{R}\cup\{-\infty\}$ is called *superlinear* if $-\psi$ is sublinear. We say that φ is *lower semicontinuous* (ψ is *upper semicontinuous*) if the *epigraph* $\mathrm{epi}(\varphi) := \{(t,\alpha)\in\mathbb{R}^N\times\mathbb{R} : \varphi(t)\leqslant\alpha\}$ (the *hypograph* $\mathrm{hypo}(\varphi) := \{(t,\alpha)\in\mathbb{R}^N\times\mathbb{R} : \varphi(t)\geqslant$

$\alpha\}\big)$ is a closed subset of $\mathbb{R}^N \times \mathbb{R}$. The *effective domain* of a sublinear φ (superlinear ψ) is $\mathrm{dom}(\varphi) := \{t \in \mathbb{R}^N : \varphi(t) < +\infty\}$ $\big(\mathrm{dom}(\psi) := \{t \in \mathbb{R}^N : \psi(t) > -\infty\}\big)$. The *subdifferential* $\underline{\partial}\varphi$ of a sublinear function φ and the *superdifferential* $\overline{\partial}\psi$ of a superlinear function ψ are defined by

$$\underline{\partial}\varphi := \{t \in \mathbb{R}^N : \langle s,t\rangle \leqslant \varphi(s)\, (s \in \mathbb{R}^N)\},$$
$$\overline{\partial}\psi := \{t \in \mathbb{R}^N : \langle s,t\rangle \geqslant \psi(s)\, (s \in \mathbb{R}^N)\},$$

where $s = (s_1 \dots, s_N)$, $t = (t_1 \dots, t_N)$, $\langle s,t\rangle := \sum_{k=1}^N s_k t_k$, see [11, 19]. Denote by $\mathscr{H}_\vee(\mathbb{R}^N, K)$ and $\mathscr{H}_\wedge(\mathbb{R}^N, K)$ respectively the sets of all lower semicontinuous sublinear functions $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and upper semicontinuous superlinear functions $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ which are finite and continuous on a fixed cone $K \subset \mathbb{R}^N$. Put $\mathscr{H}_\vee(\mathbb{R}^N) := \mathscr{H}_\vee(\mathbb{R}^N, \{0\})$ and $\mathscr{H}_\wedge(\mathbb{R}^N) := \mathscr{H}_\wedge(\mathbb{R}^N, \{0\})$. We shall consider $\mathscr{H}_\vee(\mathbb{R}^N)$ and $\mathscr{H}_\wedge(\mathbb{R}^N)$ as subcones of the vector lattice of Borel measurable functions $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ with the convention that all infinite values are replaced by zero value.

5.2. Theorem. *Let $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$ and $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$. Then there exist countable subsets $A \subset \underline{\partial}\varphi$ and $B \subset \overline{\partial}\psi$ such that the representations hold:*

$$\varphi(s) = \sup\{\langle s,t\rangle : \ t \in A\} \quad (s \in \mathbb{R}^N),$$
$$\psi(s) = \inf\{\langle s,t\rangle : \ t \in B\} \quad (s \in \mathbb{R}^N).$$

$\lhd$ The claim is true for $A = \underline{\partial}\varphi$ and $B = \overline{\partial}\psi$ in any locally convex space X. The sets $\underline{\partial}\varphi$ and $\overline{\partial}\psi$ may be replaced by their countable subsets A and B provided that X is a separable Banach space, say $X = \mathbb{R}^N$ (see [8, Proposition A.1]). $\rhd$

5.3. Remark. For this *abstract convexity* see S. S. Kutateladze.

In this section we deal with the description of H-convex elements in E in the event that H is the linear hull of a finite collection $\{x_1, \dots, x_N\} \subset E$. The following two theorems say that under some conditions an element in E is H-convex if and only if it is of the form $\widehat{\mathfrak{x}}(\varphi)$ for some lower semicontinuous sublinear functions φ.

For $A \subset \mathbb{R}^N$ denote by $\langle A, \mathfrak{x}\rangle$ the set of all linear combinations $\sum_{k=1}^N \lambda_k x_k$ in E with $(\lambda_1, \dots, \lambda_N) \in A$, so that

$$\sup\langle A, \mathfrak{x}\rangle := \sup\left\{\sum_{k=1}^N \lambda_k x_k : \ (\lambda_1, \dots, \lambda_N) \in A\right\}.$$

5.4. Theorem. *Let E be a σ-complete vector lattice with an order unit, $x_1, \dots, x_N \in E$, and $\mathfrak{x} := (x_1, \dots, x_N)$. Assume that $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$, and $[\mathfrak{x}] \subset \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$. Then $\mathfrak{x}(\varphi)$ exists in E if and only if $\langle \underline{\partial}\varphi, \mathfrak{x}\rangle$ is order bounded above, $\mathfrak{x}(\psi)$ exists in E if and only if $\langle \underline{\partial}\psi, \mathfrak{x}\rangle$ is order bounded below, and the representations hold:*

$$\widehat{\mathfrak{x}}(\varphi) = \sup\langle \underline{\partial}\varphi, \mathfrak{x}\rangle, \quad \widehat{\mathfrak{x}}(\psi) = \inf\left\langle \overline{\partial}\psi, \mathfrak{x}\right\rangle.$$

Moreover, $\widehat{\varphi}(x_1, \dots, x_N)$ $\big(\widehat{\psi}(x_1, \dots, x_N)\big)$ is an order limit of an increasing (decreasing) sequence which is comprised of the finite suprema (infima) of linear combinations of the form $\sum_{i=1}^N \lambda_i x_i$ with $(\lambda_1, \dots \lambda_N) \in \underline{\partial}\varphi$ $\big((\lambda_1, \dots \lambda_N) \in \overline{\partial}\psi\big)$.

$\lhd$ Assume that $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$ and $[x_1, \dots, x_N] \subset \mathrm{dom}(\varphi)$. Let E_0 denotes the band in E generated by $\mathbb{1} := |x_1| + \dots + |x_N|$ and by $\mathbb{1}$ and $E_0^{u\sigma}$ stands for the universally σ-completion E_0. By Theorem 2.3 $\widehat{\mathfrak{x}}(\varphi)$ always exists in E_0 and the required representation holds true in

$E_0^{u\sigma}$, since φ is Borel. In more details, let φ_0 vanishes on $\mathbb{R}^N \setminus \mathrm{dom}(\varphi)$ and coincides with φ on $\mathrm{dom}(\varphi)$. Then φ_0 is a Borel function on $\mathbb{R}^N$ and according to 5.2 we may choose an increasing sequence (φ_n) of Borel functions such that φ_n coincides with the finite supremum of linear combinations of the form $\sum_{i=1}^N \lambda_i t_i$ on $\mathrm{dom}(\varphi)$ and (φ_n) converges point-wise to φ_0. By Theorem 2.3 the sequence $(\widehat{\mathfrak{x}}(\varphi_n))$ is increasing and order convergent to $\widehat{\mathfrak{x}}(\varphi_0) = \widehat{\mathfrak{x}}(\varphi)$. Now it is clear that $\langle\, \underline{\partial}\varphi, \mathfrak{x}\,\rangle$ is order bounded above in E if and only if $\mathfrak{x}(\varphi) \in E_0$. $\rhd$

5.5. Theorem. *Let E be a relatively uniformly complete vector lattice, $x_1, \dots, x_N \in E$, and $\mathfrak{x} := (x_1, \dots, x_N)$. If $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N; [\,\mathfrak{x}\,])$ and $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N; [\,\mathfrak{x}\,])$, then*

$$\widehat{\mathfrak{x}}(\varphi) = \sup \langle\, \underline{\partial}\varphi, \mathfrak{x}\,\rangle,$$
$$\widehat{\mathfrak{x}}(\psi) = \inf \left\langle\, \overline{\partial}\psi, \mathfrak{x}\,\right\rangle.$$

Moreover, $\widehat{\varphi}(x_1, \dots, x_N)$ $\big(\widehat{\psi}(x_1, \dots, x_N)\big)$ is a relatively uniform limit of an increasing (decreasing) sequence which is comprised of the finite suprema (infima) of linear combinations of the form $\sum_{i=1}^N \lambda_i x_i$ with $\lambda = (\lambda_1, \dots \lambda_N) \in \underline{\partial}\varphi$ ($\lambda \in \overline{\partial}\psi$).

$\lhd$ Consider $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N; [x_1, \dots, x_N])$ and denote $y = \widehat{\varphi}(x_1, \dots, x_N)$. By Theorem 3.3

$$v_\lambda := \lambda_1 x_1 + \ldots + \lambda_N x_N \leqslant y$$

for an arbitrary $\lambda := (\lambda_1, \dots, \lambda_N) \in \underline{\partial}\varphi$. Assume that $v \in E$ is such that $v \geqslant v_\lambda$ for all $\lambda \in \underline{\partial}\varphi$. By the Kreĭns–Kakutani Representation Theorem there is a lattice isomorphism $x \mapsto \tilde{x}$ of the principal ideal E_u generated by $u = |x_1| + \ldots + |x_N| + |v|$ onto $C(Q)$ for some compact Hausdorff space Q. Then v, $x_1, \dots, x_N$, v_λ, and y lie in E_u and for any $\lambda \in \underline{\partial}\varphi$ the point-wise inequality $\widetilde{v}(q) \geqslant \widetilde{v_\lambda}(q)$ $(q \in Q)$ is true. By 4.1 and 3.6 we conclude that

$$\widetilde{y}(q) = \varphi(\widetilde{x_1}(q), \dots, \widetilde{x_N}(q)) = \sup\{\widetilde{v_\lambda}(q) : \ \lambda \in \underline{\partial}\varphi\} \leqslant \widetilde{v}(q).$$

Thus we have $y \leqslant v$ and thereby $y = \sup\{v_\lambda : \ \lambda \in \underline{\partial}\varphi\}$.

Put $U := \{v_\lambda : \ \lambda \in \underline{\partial}\varphi\}$ and denote by $U^\vee$ the subset of E consisting of the suprema of the finite subsets of U. Then $U^\vee \subset E_u$ and the set $\widetilde{U^\vee} := \{\widetilde{v} : \ v \in U^\vee\}$ is upward directed in $C(Q)$ and its point-wise supremum equals to $\widetilde{y}$. By Dini Theorem $\widetilde{U^\vee}$ converges to $\widetilde{y}$ uniformly and thus $U^\vee$ is norm convergent to y in E_u. The superlinear case $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N; [x_1, \dots, x_N])$ is considered in a similar way. $\rhd$

5.6. In some situation it is important to know wether the function is the upper or lower envelope of a family of increasing linear functionals. Suppose that $\mathbb{R}^N$ is preordered by a cone $K \subset \mathbb{R}^N$, i. e. $s \geqslant t$ means that $s - t \in K$. The dual cone of positive linear functionals is denoted by K^*. A function $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ is called *increasing* (with respect to K) if $s \geqslant t$ implies $\phi(s) \geqslant \phi(t)$. A lower semicontinuous sublinear (an upper semicontinuous superlinear) ϕ is increasing if and only if $\underline{\partial}\phi \subset K^*$ ($\overline{\partial}\phi \subset K^*$) and thus ϕ is an upper envelope of a family of increasing linear functionals (is a lower envelope of a family of increasing linear functionals). If ϕ is increasing only on $\mathrm{dom}(\phi)$, then this claim is no longer true but under some mild conditions it is still valid for the restriction of ϕ onto $\mathrm{dom}(\phi)$, see [13, 20].

Proposition. *Let $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be the same as in Theorem 5.2. Suppose that, in addition, $\mathrm{dom}(\varphi) - K = K - \mathrm{dom}(\varphi)$ and $\mathrm{dom}(\psi) - K = K - \mathrm{dom}(\psi)$. Then the following assertions hold:*

(1) *φ is increasing on $\mathrm{dom}(\varphi)$ if and only if*
$$\varphi(s) = \sup\{\langle s, t\rangle : \ t \in (\underline{\partial}\varphi) \cap K^*\} \quad (s \in \mathrm{dom}(\varphi));$$

(2) *ψ is increasing on $\mathrm{dom}(\psi)$ if and only if*
$$\psi(s) = \inf\{\langle s, t\rangle : \ t \in (\overline{\partial}\psi) \cap K^*\} \quad (s \in \mathrm{dom}(\psi)).$$

$\lhd$ Indeed, we may assume $\mathbb{R}^N = \mathrm{dom}(\varphi) - K$ and then the function $\varphi^* : \mathbb{R}^N \to \mathbb{R}$ defined by $\varphi^*(s) = \inf\{\varphi(t) : t \in \mathrm{dom}(\varphi),\, t \geqslant s\}$ $(s \in \mathbb{R}^N)$ is increasing and sublinear and coincides with φ on $\mathrm{dom}(\varphi)$; moreover, $\underline{\partial}\varphi^* = (\underline{\partial}\varphi) \cap K^*$. Similarly, assuming $\mathbb{R}^N = \mathrm{dom}(\psi) - K$, we deduce that the function $\psi_* : \mathbb{R}^N \to \mathbb{R}$ defined by $\psi_*(s) = \sup\{\psi(t) : t \in \mathrm{dom}(\psi),\, t \leqslant s\}$ $(s \in \mathbb{R}^N)$ is increasing and superlinear and agrees with ψ on $\mathrm{dom}(\psi)$; moreover, $\overline{\partial}\psi_* = (\overline{\partial}\psi) \cap K^*$. It remains to observe that φ and ψ are increasing if and only if $\varphi = \varphi^*$ and $\psi = \psi_*$. $\rhd$

5.7. Corollary. *Assume that φ is increasing on* $\mathrm{dom}(\varphi)$, *ψ is increasing on* $\mathrm{dom}(\psi)$, $\mathrm{dom}(\varphi) - K = K - \mathrm{dom}(\varphi)$, *and* $\mathrm{dom}(\psi) - K = K - \mathrm{dom}(\psi)$. *If, in addition, the assumptions of either* 4.4 *or* 4.5 *are fulfilled, then in* 4.4 *and* 4.5 *the sets $\underline{\partial}\,\varphi$ and $\overline{\partial}\varphi$ may be replaced by* $(\underline{\partial}\,\varphi) \cap K^*$ *and* $(\overline{\partial}\psi) \cap K^*$.

5.8. A *gauge* is a sublinear function $\varphi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$. A *co-gauge* is a superlinear function $\psi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{-\infty\}$. The *lower polar* function φ° of a gauge φ and the *upper polar* function $\psi_\circ$ of a co-gauge ψ are defined by

$$\varphi^\circ(t) := \inf\{\lambda \geqslant 0 :\ (\forall\, s \in \mathbb{R}^N)\, \langle s, t\rangle \leqslant \lambda\varphi(s)\} \quad (t \in \mathbb{R}^N),$$
$$\psi_\circ(t) := \sup\{\lambda \geqslant 0 :\ (\forall\, s \in \mathbb{R}^N)\, \langle s, t\rangle \geqslant \lambda\psi(s)\} \quad (t \in \mathbb{R}^N)$$

(with the conventions $\sup\varnothing = -\infty$, $\inf\varnothing = +\infty$, and $0(+\infty) = 0(-\infty) = 0$). Thus, φ° is a gauge and $\psi_\circ$ is a co-gauge. Observe also that the inequalities hold:

$$\langle s, t\rangle \leqslant \varphi(s)\varphi^\circ(t) \quad (s \in \mathrm{dom}(\varphi),\ t \in \mathrm{dom}(\varphi^\circ)),$$
$$\langle s, t\rangle \geqslant \psi(s)\psi_\circ(t) \quad (s \in \mathrm{dom}(\psi),\ t \in \mathrm{dom}(\psi_\circ)).$$

Denote $\varphi^{\circ\circ} := (\varphi^\circ)^\circ$ and $\psi_{\circ\circ} := (\psi_\circ)_\circ$.

5.9. Bipolar Theorem. *Let φ be a gauge and ψ be a co-gauge. Then $\varphi^{\circ\circ} = \varphi$ if and only if φ is lower semicontinuous and $\psi_{\circ\circ} = \psi$ if and only if ψ is upper semicontinuous.*

$\lhd$ See [19]. $\rhd$

5.10. The lower polar function φ° and the upper polar function ψ_0 can be also calculate by

$$\varphi^\circ(t) = \sup_{s \in \mathbb{R}^N} \frac{\langle s, t\rangle}{\varphi(s)} = \sup\{\langle s, t\rangle :\ s \in \mathbb{R}^N,\ \varphi(s) \leqslant 1\} \quad (t \in \mathbb{R}^N)$$

(with the conventions $\alpha/0 = +\infty$ for $\alpha > 0$ and $\alpha/0 = 0$ for $\alpha \leqslant 0$) and

$$\psi_\circ(t) = \inf_{s \in \mathbb{R}^N} \frac{\langle s, t\rangle}{\psi(s)} = \inf\big\{\langle s, t\rangle :\ s \in \mathbb{R}^N,\ \psi(s) \geqslant 1 \text{ or } \psi(s) = 0\big\} \quad (t \in \mathbb{R}^N)$$

(with the conventions $\alpha/0 = +\infty$ for $\alpha \geqslant 0$ and $\alpha/0 = -\infty$ for $\alpha < 0$).

Denote by $\mathscr{G}_\vee(\mathbb{R}^N, K)$ and $\mathscr{G}_\wedge(\mathbb{R}^N, K)$ respectively the sets of all lower semicontinuous gauges $\varphi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ and upper semicontinuous co-gauges $\psi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{-\infty\}$ which are finite and continuous on a fixed cone $K \subset \mathbb{R}^N$. Put $\mathscr{G}_\vee(\mathbb{R}^N) := \mathscr{G}_\vee(\mathbb{R}^N, \{0\})$ and $\mathscr{G}_\wedge(\mathbb{R}^N) := \mathscr{G}_\wedge(\mathbb{R}^N, \{0\})$. Observe that $\mathscr{G}_\vee(\mathbb{R}^N) \subset \mathscr{H}_\vee(\mathbb{R}^N)$ and $\mathscr{G}_\wedge(\mathbb{R}^N) \subset \mathscr{H}_\wedge(\mathbb{R}^N)$.

5.11. Corollary. *Assume that either the assumptions of* 5.4 *are fulfilled and, in addition,* $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N)$ *and* $\psi \in \mathscr{G}_\wedge(\mathbb{R}^N)$, *or the assumptions of* 5.5 *are fulfilled and additionally* $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N; [\mathfrak{r}])$ *and* $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N; [\mathfrak{r}])$. *Then in* 5.4 *and* 5.5 *the sets $\underline{\partial}\,\varphi$ and $\overline{\partial}\varphi$ may be replaced by* $\{t \in \mathbb{R}^N : \varphi^\circ(t) \leqslant 1\}$ *and* $\{t \in \mathbb{R}^N : \psi_\circ(t) \geqslant 1\}$, *respectively.*

$\lhd$ It is immediate from the Bipolar Theorem and the above definitions, since obviously $\underline{\partial}\,\varphi = \{t \in \mathbb{R}^N : \varphi^\circ(t) \leqslant 1\}$ and, $\overline{\partial}\,\psi = \{t \in \mathbb{R}^N : \psi^\circ(t) \geqslant 1\}$. $\rhd$

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Received April 12, 2009.

Anatoly G. Kusraev
South Mathematical Institute
Vladikavkaz Science Center of the RAS, *Director*
Russia, 362027, Vladikavkaz, Markus street, 22
E-mail: `kusraev@smath.ru`