Владикавказский математический журнал
2008, Том 10, Выпуск 3, С. 29–33

УДК **517.98**

ON SOME PROPERTIES
OF ORTHOSYMMETRIC BILINEAR OPERATORS[1]

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This note contains some properties of positive orthosymmetric bilinear operators on vector lattices which are well known for almost f-algebra multiplication but despite of their simplicity does not seem appeared in the literature.

Mathematics Subject Classification (2000): 46A40, 47A65.

Key words: vector lattice, square of a vector lattice, bilinear operator, orthosymmetry, lattice bimorphism, f-algebra multiplication.

The aim of this note is to present some properties of orthosymmetric bilinear operators which are well known for f-algebra multiplication but despite of their simplicity does not seem appeared in the literature. All unexplained terms can be found in [1] or [13]. All vector lattices under consideration are assumed to be Archimedean.

1. A bilinear operator $b : E \times E \to G$ is called *orthosymmetric* if $x \wedge y = 0$ implies $b(x, y) = 0$ for all $x, y \in E$. This definition was introduced in [8]. Recall also that b is said to be *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in E$ and *positively semidefinite* if $b(x, x) \geqslant 0$ for every $x \in E$. In the special case that b is the multiplication of a commutative almost f-algebra the following proposition is presented in [2, Proposition 1.13].

Proposition 1. *Let F and G be vector lattices. A positive bilinear operator b from $E \times E$ to G is orthosymmetric if and only if $b(x, y) = b(x \vee y, x \wedge y)$ for all $x, y \in E$.*

$\lhd$ Orthosymmetry implies $b(x - x \wedge y, y - x \wedge y) = 0$. Since a positive orthosymmetric bilinear operastor is symmetric (see [8]), we deduce

$$\begin{aligned} b(x, y) &= b(x, x \wedge y) + b(x \wedge y, y) - b(x \wedge y, x \wedge y) \\ &= b(x + y - x \wedge y, x \wedge y) = b(x \vee y, x \wedge y). \end{aligned}$$

Conversely, if $x \wedge y = 0$, then $b(x, y) = b(x \vee y, 0) = 0$. $\rhd$

A bilinear operator $b : E \times F \to G$ is said to be *lattice bimorphism* if the mappings $y \mapsto b(e, y)$ $(y \in F)$ and $x \mapsto b(x, f)$ $(x \in E)$ are lattice homomorphisms for all $0 \leqslant e \in E$ and $0 \leqslant f \in F$, see [11]. Evidently, every lattice bimorphism is positive. The following characterization of lattice bimorphism was given in [15].

Proposition 2. *For a positive bilinear operator $b : E \times E \to G$ the following assertions are equivalent:*

(1) *b is a lattice bimorphism;*

(2) *$|b(x, y)| = b(|x|, |y|)$ for all $x \in E$ and $y \in F$;*

[1]Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований, проект № 06-01-00622.

(3) *if* $0 \leqslant x, u \in E$ *and* $0 \leqslant y, v \in F$ *satisfy* $x \wedge u = 0$ *and* $y \wedge v = 0$, *then* $b(x,y) \wedge b(u,v) = 0$.

2. It was mentioned in [12] that an orthosymmetric positive bilinear operator is positively semidefinite. The converse is also true for lattice bimorphisms as was observed in [7, Proposition 1.7]. The following characterization of symmetric lattice bimorphisms is well known at least for d-algebra multiplication (see, for example, [2, Theorems 4.3, 4.4, 4.5] and [4, Proposition 3.6]).

Theorem 1. *Let E and F be vector lattices and let $b : E \times E \to F$ be a lattice bimorphism. Then the following assertions are equivalent:*

(1) *b is symmetric;*

(2) $b(x,x) - b(y,y) = b(x-y, x+y)$ *for all* $x, y \in E$;

(3) $b(x,x) \wedge b(y,y) \leqslant b(x,y) \leqslant b(x,x) \vee b(y,y)$ *for all* $x, y \in E_+$;

(4) $b(x \wedge y, x \wedge y) = b(x,x) \wedge b(y,y)$ *and* $b(x \vee y, x \vee y) = b(x,x) \vee b(y,y)$ *for all* $x, y \in E_+$;

(5) $x \wedge y = 0$ *implies* $b(x,y) = b(y,x)$ *for all* $x, y \in E$;

(6) $b(x, |x|) = b(x^+, x^+) - b(x^-, x^-)$ *for all* $x \in E$;

(7) *b is orthosymmetric;*

(8) *b is positively semidefinite.*

$\lhd$ (1) $\Leftrightarrow$ (2): It is obviously true for every bilinear operator b.

(2) $\Rightarrow$ (3): For any $x, y \in E_+$ we deduce making use of (2):

$$
\begin{aligned}
b(x,x) \wedge b(y,y) - b(x,y) &\leqslant b(x,x) \wedge b(y,y) - b(x \wedge y, x \wedge y) \\
&= [b(x,x) - b(x \wedge y, x \wedge y)] \wedge [b(y,y) - b(x \wedge y, x \wedge y)] \\
&= b(x - x \wedge y, x + x \wedge y) \wedge b(y - x \wedge y, y + x \wedge y) \\
&\leqslant b(x - x \wedge y, x + y) \wedge b(y - x \wedge y, x + y) \\
&= b((x - x \wedge y) \wedge (y - x \wedge y), x + y) = 0.
\end{aligned}
$$

The second inequality is deduced likewise.

(3) $\Rightarrow$ (4): Using the first inequality in (3) we can write the following chain of equalities:

$$
\begin{aligned}
b(x,x) \wedge b(y,y) = [b(x,x) \wedge b(x,y)] \wedge [b(y,x) \wedge b(y,y)] \\
= b(x, x \wedge y) \wedge b(y, x \wedge y) = b(x \wedge y, x \wedge y).
\end{aligned}
$$

The second equality is deduced likewise.

(4) $\Rightarrow$ (5): Take $x, y \in E$ with $x \wedge y = 0$. By the first equality of (3) $b(x,x)$ and $b(y,y)$ are disjoint. Using the second equality we have $b(x,x) + b(y,y) = b(x \vee y, x \vee y) = b(x+y, x+y) = b(x,x) + b(x,y) + b(y,x) + b(y,y)$, so that $b(x,y) = b(y,x) = 0$.

(5) $\Rightarrow$ (6): It is sufficient to observe that $b(x, |x|) - b(x^+, x^+) + b(x^-, x^-) = b(x^+, x^-) - b(x^-, x^+)$.

(6) $\Rightarrow$ (7): If b obey (6), then $b(x^+, x^-)$ and $b(x^-, x^+)$ coincide, see (5) $\Rightarrow$ (6). At the same time these elements are disjoint, since $b(x^+, x^-) \leqslant b(x^+, |x|)$, $b(x^-, x^+) \leqslant b(x^-, |x|)$ and $b(x^+, |x|) \wedge b(x^-, |x|) = 0$. Thus, $b(x^+, x^-) = b(x^-, x^+) = 0$, from which (7) follows

(7) $\Rightarrow$ (1): Follows from [8, Corollary 2].

(7) $\Rightarrow$ (8): If b is ortosymmetric, then $b(x,x) = b(x^+, x^+) - b(x^+, x^-) - b(x^-, x^+) + b(x^-, x^-) = b(x^+, x^+) + b(x^-, x^-) \geqslant 0$, see [12].

(8) $\Leftrightarrow$ (7): Let b be a positively semidefinite lattice bimorphism. Take $x, y \in E$ and put $\alpha := b(x,x)$, $\beta := b(y,y)$, $\gamma := b(x,y) + b(y,x)$. Then $\alpha + \beta - \gamma = b(x-y, x-y) \geqslant 0$. If $x \wedge y = 0$, then $b(x,y) \geqslant b(x,y) \wedge b(y,y) = b(x \wedge y, y) = 0$ and, since $b(x, \cdot)$ and $b(\cdot, x)$ are lattice homomorphisms, we have $\alpha \wedge b(x,y) = b(x, x \wedge y) = 0$ and $\alpha \wedge b(y,x) = b(x \wedge y, x) = 0$.

Thus, $\alpha \perp \gamma$ and analogously $\beta \perp \gamma$. Therefore, $(\alpha+\beta) \perp \gamma$, and taking into account the inequality $\alpha+\beta-\gamma \geqslant 0$ we derive $\gamma = 0$, i.e. $b(x,y) = b(y,x) = 0$. $\rhd$

3. Let E be a vector lattice. A pair $(E^{\odot}, \odot)$ is said to be a *square* of E if the following two conditions are fulfilled:

(1) $E^{\odot}$ is a vector lattice and $\odot$ is a symmetric lattice bimorphism from $E \times E$ to $E^{\odot}$,

(2) if b is a symmetric lattice bimorphism from $E \times E$ to some vector lattice F, then there exists a unique lattice homomorphism $\Phi_b : E^{\odot} \to F$ with $b = \Phi_b \odot$.

For an arbitrary vector lattice E there exists the square $(E^{\odot}, \odot)$ which is essentially unique, i. e. if some pair $(E^{\circledcirc}, \circledcirc)$ obeys (1) and (2) above, then there exists a lattice isomorphism i from $E^{\odot}$ onto $E^{\circledcirc}$ such that $i\odot = \circledcirc$ (and, of course, $i^{-1}\circledcirc = \odot$), see [10]. Moreover (see [10] and [7, Theorem 3.1]), for every positive bilinear orthoregular operator $b : E \times E \to G$ there exists a unique linear regular operator $\Phi_b : E^{\odot} \to G$ such that

$$b(x,y) = \Phi_b(x \odot y) \quad (x, y \in E).$$

The symmetric lattice bimorphism $\odot : E \times E \to E^{\odot}$ is called the *canonical bimorphism* of the square. The operator Φ_b is called the *linearization* of b *via square.* If E is a sublattice of a semiprime f-algebra A, then the canonical bimorphism $\odot$ can be expressed in terms of the algebra multiplication, see [7, Proposition 2.5].

Proposition 3. *Let A be a semiprime f-algebra with a multiplication $\bullet$ and E be a sublattice of A. Then there exists a sublattice $F \subset A$ and an isomorphism ι from $E^{\odot}$ onto F such that $\iota(x \odot y) = x \bullet y$ for all $x, y \in E$. In other words, the pair $(F, \bullet)$ is a square of E.*

4. A vector lattice E is called *square-mean closed* if the set $\{(\cos\theta)x + (\sin\theta)y : 0 \leqslant \theta < 2\pi\}$ has a supremum $\mathfrak{s}(x,y)$ in E for all $x, y \in E$. A vector lattice E is called *geometric-mean closed* if the set $\{(t/2)x + (1/2t)y : 0 < t < +\infty\}$ has an infimum $\mathfrak{g}(x,y)$ in E for all $x, y \in E_+$. The following result see in [5, Theorems 3.1 and 3.4].

Proposition 4. *If A is a square-mean closed Archimedean f-algebra, then*

$$\mathfrak{s}(x,y)^2 = x^2 + y^2 \quad (x, y \in A).$$

If A is a geometric-mean closed Archimedean f-algebra, then

$$\mathfrak{g}(x,y)^2 = xy \quad (x, y \in A_+).$$

Every relatively uniformly complete vector lattice is square-mean closed and geometric-mean closed [5, Theorems 3.3]. However, neither a square-mean closed nor a geometric-mean closed Archimedean vector lattice need not be uniformly complete. But a geometric-mean closed Archimedean f-algebra is square-mean closed [5, Theorem 3.6]. The following result is a generalization of Proposition 4.

Theorem 2. *Let E and F be vector lattices and $b : E \times E \to F$ a positive orthosymmetric bilinear operator. If E is square-mean closed, then*

$$\mathfrak{s}(x,y) \odot \mathfrak{s}(x,y) = x \odot x + y \odot y,$$
$$b(\mathfrak{s}(x,y), \mathfrak{s}(x,y)) = b(x,x) + b(y,y)$$

for all $x, y \in E$. If E is geometric-mean closed, then for all $x, y \in E_+$ we have

$$\mathfrak{g}(x,y) \odot \mathfrak{g}(x,y) = x \odot y,$$
$$b(\mathfrak{g}(x,y), \mathfrak{g}(x,y)) = b(x,y).$$

$\lhd$ In each of two cases under consideration the second equality follows from the first one by applying Φ_b, the linearization via square of b. Let A denotes the universal completion of E endowed with a semiprime f-algebra multiplication. Then by Proposition 3 there is a lattice isomorphism ι of $E^{\odot}$ onto a sublattice $F \subset A$. At the same time, according to Proposition 4, the following equalities are true in A:

$$\mathfrak{s}(x,y) \bullet \mathfrak{s}(x,y) = x \bullet x + y \bullet y \quad (x, y \in E),$$
$$\mathfrak{g}(x,y) \bullet \mathfrak{g}(x,y) = x \bullet y \quad (x, y \in E_+).$$

Now, the first equalities are immediate by applying ι^{-1}, since $\mathfrak{s}(x,y) \in E$ and $\mathfrak{g}(x,y) \in E$ under the stated hypotheses and $\iota^{-1}(x \bullet y) = x \odot y$. $\rhd$

5. In conclusion we present some corollaries to Theorem 2.

Corollary 1. *Let E and F be vector lattices with E square-mean closed and $b: E\times E \to F$ be a positive orthosymmetric bilinear operator. Then $E_+^{(b)} := \{b(x,x) : x \in E\}$ is a convex pointed cone and $E^{(b)} := b(E \times E)$ is a vector subspace of F ordered by a positive cone $E_+^{(b)}$ such that $E^{(b)} = E_+^{(b)} - E_+^{(b)}$. If, in addition, b is a lattice bimorphism, then $E^{(b)}$ is a vector sublattice of F.*

$\lhd$ The first part of Theorem 2 implies that $E_+^{(b)} \subset F_+$ is a pointed cone. The equalities $b(x,y) = (1/4)[b(x+y, x+y) - b(x-y, x-y)])$ and $b(x,x) - b(y,y) = b(x+y, x-y)$ show that $E^{(b)} = E_+^{(b)} - E_+^{(b)}$. Thus, $(E^{(b)}, E_+^{(b)})$ is an ordered vector space. If b is a lattice bimorphism, then $E_+^{(b)}$ is a sublattice of F_+ in virtue of Theorem 1 (2). $\rhd$

For an almost f-algebra multiplication this result was obtained in [4, Prposition 3.3, Corollary 3.7]. The first statement of the following corollary was proved in [9, Lemma 8] in case of uniformly complete E.

Corollary 2. *Let E be a square-mean closed vector lattice. The the assertions hold:*

(1) $E^{\odot} = \{x \odot y : x, y \in E\}$ and $E_+^{\odot} = \{x \odot x : x \in E\}$;

(2) If $F = h(E)$, then $F^{\odot} = h^{\odot}(E^{\odot})$ for any vector lattice F and lattice homomorphism $h: E \to F$;

(3) If J is a uniformly closed order ideal of E, then $J^{\diamond} := \{x \odot y : |x| \wedge |y| \in J\}$ is a uniformly closed order ideal of $E^{\odot}$ and the map $x \odot y + J^{\diamond} \mapsto (x + J) \odot (y + J)$ implements a lattice isomorphism of $E^{\odot}/J^{\diamond}$ onto $(E/J)^{\odot}$.

$\lhd$ (1) Put $b := \odot$ in Corollary 1 and observe that $E^{\odot} = E^{(b)}$, since $E^{\odot}$ coincides with the sublattice generated by $b(E \times E) = \{x \odot y : x, y \in E\}$.

(2) If $h : F \to E$ is a lattice homomorphism then by [7, Proposition 2.4] there exists a lattice homomorphism $h^{\odot} : F^{\odot} \to E^{\odot}$ such that $h^{\odot}(x \odot y) = h(x) \odot h(y)$ $(x, y \in F)$. Assume that $T(E) = F$. Then making use of by (1) we deduce

$$E^{\odot} = \{h(x) \odot h(y) : x, y \in F\} = \{h^{\odot}(x \odot y) : x, y \in F) \subset h^{\odot}(F^{\odot}) \subset E^{\odot}.$$

(3): If $\phi : E \to E/J$ is a quotient homomorphism, then $\phi^{\odot}$ is a surjective map from $E^{\odot}$ to $(E/J)^{\odot}$ by (2). According to (1) any $u \in E^{\odot}$ have the representation $u = x \odot y$ for some $x, y \in E$ and $0 = \phi^{\odot}(u) = \phi(x) \odot \phi(y)$ implies $\phi(x) \perp \phi(y)$ by [7, Theorem 2.1 (3)]. But the latter is equivalent to $|x| \wedge |y| \in J$, since ϕ is a lattice homomorphism. Thus, $J^{\diamond} = \ker(\phi^{\odot})$ and the proof is complete. $\rhd$

Corollary 3. *Let E and F be vector lattices with E square-mean closed and let $b : E \times E \to F$ be an order bounded orthosymmetric bilinear operator. Then for any finite collections $x_1, y_1, \dots, x_N, y_N \in E$ there exist $u, v \in E$ such that $\sum_{k=1}^{N} b(x_k, y_k) = b(u, v)$.*

$\lhd$ According to Corollary 1 (1) there exist $u, v \in E$ such that $u \odot v = \sum_{k=1}^{N} x_k \odot y_k$. Now, if $b = \Phi_b \odot$ for a linear operator Φ_b from $E^{\odot}$ to F, then

$$b(u, v) = \Phi_b(u \odot v) = \Phi_b \left(\sum_{k=1}^{N} x_k \odot y_k \right) = \sum_{k=1}^{N} b(x_k, y_k)$$

which is the desired representation. $\rhd$

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Received June 5, 2008.

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