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REPRESENTATION AND EXTENSION OF ORTHOREGULAR BILINEAR OPERATORS

Buskes G., Kusraev A. G.[1]

In this paper we study some important structural properties of orthosymmetric bilinear operators using the concept of the square of an Archimedean vector lattice. Some new results on extension and analytical representation of such operators are presented.

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Introduction

Recently the class of orthosymmetric bilinear operators in vector lattices, introduced in [12], has aroused considerable interest. This is due to the study of lattice ordered algebras [4, 11, 12, 29], Hilbert lattices [17], various structural properties of positive bilinear [8, 14, 24, 25, 26] and multilinear [5, 6] operators, etc. A number of important properties of such operators was revealed. For example, a positive orthosymmetric bilinear operator is symmetric [12] and every positive orthosymmetric bilinear operator defined on a sublattice of an f-algebra can be factored through a positive linear operator and the algebra multiplication [11, 12, 29]. These results gave rise to the concept of the square of a vector lattice, developed in [13].

This paper is a continuation of [6, 11, 12, 13, 24, 25, 28][2]. A general idea behind the paper can be stated as follows: In the theory of orthosymmetric bilinear operators, the role played by the square of an Archimedean vector lattice is as important as that of Fremlin's tensor product of Archimedean vector lattices in the theory of bilinear operators, see [25]. In particular, the square of a vector lattice possesses the following universal property: On every Archimedean vector lattice there exists a unique symmetric lattice bimorphism with values in the square of the initial vector lattice, similar to f-algebra multiplication, such that an arbitrary regular orthosymmetric bilinear operator defined on this vector lattice and with values in a uniformly complete vector lattice is representable as a composition of the said bimorphism and some regular linear operator uniquely defined on the square [24]. This approach allows us to improve and systematize some known results as well as to obtain several new facts on extension and analytical representation of orthosymmetric bilinear operators.

For the theory of vector lattices and positive operators we refer to the books [2] and [23].

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[2]The second named author regrets neglecting mention of [13] and [7] in [25] and [28], respectively.

1. Prerequisites

In this section we fix the notation and terminology and outline some results needed in the sequel. Throughout this paper a vector lattice means an Archimedean vector lattice over the field of real numbers.

1.1. Let E, F, and G be vector lattices. A bilinear operator $b : E \times F \to G$ is called *positive* if $b(x, y) \geqslant 0$ for all $0 \leqslant x \in E$ and $0 \leqslant y \in F$, and *regular* if it can be represented as a difference of two positive bilinear operators. The set $BL_r(E, F; G)$ of all regular bilinear operators from $E \times F$ to G serves as an ordered vector space if an order relation is defined by the cone of positive bilinear operators $BL_+(E, F; G)$. This space is an order complete vector lattice provided that G is order complete, see [14, 22, 26].

A bilinear operator b is said to be *lattice bimorphism* if the mappings $b_x : y' \mapsto b(x, y')$ ($y' \in F$) and $b_y : x' \mapsto b(x', y)$ ($x' \in E$) are lattice homomorphisms for all $0 \leqslant x \in E$ and $0 \leqslant y \in F$, see [16].

The following fundamental result was established by D. Fremlin in [16].

1.2. Theorem. *Let E and F be vector lattices. Then there is a unique up to isomorphism vector lattice $E \mathbin{\bar{\otimes}} F$ and a vector bimorphism $\phi : E \times F \to E \mathbin{\bar{\otimes}} F$ such that:*

(1) *whenever G is a vector lattice and $\psi : E \times F \to G$ is a lattice bimorphism, there is a unique lattice homomorphism $T : E \mathbin{\bar{\otimes}} F \to G$ with $T \circ \phi = \psi$;*

(2) *ϕ induces an embedding of the algebraic tensor product $E \otimes F$ into $E \mathbin{\bar{\otimes}} F$;*

(3) *$E \otimes F$ is dense in $E \mathbin{\bar{\otimes}} F$ in the sense that for every $v \in E \mathbin{\bar{\otimes}} F$ there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\varepsilon > 0$ there is an element $u \in E \otimes F$ with $|v - u| \leqslant \varepsilon x_0 \otimes y_0$;*

(4) *if $0 < v \in E \mathbin{\bar{\otimes}} F$, then here exist $x \in E_+$ and $y \in F_+$ with $0 < x \otimes y \leqslant v$.*

The lattice bimorphism ϕ from the theorem is conventionally denoted by $\otimes$ and the algebraic tensor product $E \otimes F$ is regarded as actually embedded in $E \mathbin{\bar{\otimes}} F$.

1.3. Let ψ and T be the same as in the statement of Theorem 1.2 (1). Suppose that for any $x \in E_+$ and $y \in E_+$ the equality $\psi(x, y) = 0$ implies $x = 0$ or $y = 0$. In this case T is injective and thus maps $E \mathbin{\bar{\otimes}} F$ onto a vector sublattice of G generated by $\operatorname{im} \psi := \psi(E \times F)$. In particular, if E_0 and F_0 are vector sublattices in E and F, respectively, then the tensor product $E_0 \mathbin{\bar{\otimes}} F_0$ is isomorphic to the vector sublattice in $E \mathbin{\bar{\otimes}} F$ generated by $E_0 \otimes F_0$, see [16]. Therefore, $E_0 \mathbin{\bar{\otimes}} F_0$ is regarded as a vector sublattice of $E \mathbin{\bar{\otimes}} F$. Moreover, as it is seen from 1.2 (3), $E_0 \mathbin{\bar{\otimes}} F_0$ is a majorizing sublattice provided that E_0 and F_0 are majorizing sublattices.

D. Fremlin [16] proved also the following important universal property of the tensor product of vector lattices.

1.4. Theorem. *Let E, F, and G be vector lattices with G relatively uniformly complete. Then for every positive bilinear operator $b : E \times F \to G$ there exists a unique positive linear operator $T : E \mathbin{\bar{\otimes}} F \to G$ such that $T\otimes = b$.*

It follows, in particular, that for the same E, F, and G the mapping $T \mapsto T\otimes$ is an isomorphism of ordered vector spaces $L_r(E \mathbin{\bar{\otimes}} F, G)$ and $BL_r(E, F; G)$, where $L_r(H, G)$ stands for the space of all linear regular operators from H to G. More generally, the same relationship holds between the spaces of order bounded linear operators from $E \mathbin{\bar{\otimes}} F$ to G and bilinear operators of order bounded variation from $E \times F$ to G, see [14]. (The definition of bilinear operator of order bounded variation see below in 3.3).

Thus, the Fremlin's tensor product lends itself to a transfer of known results on regular linear operators to regular bilinear operators as well as on order bounded linear operators to bilinear operators of order bounded variation. This and certain other aspects of bilinear operators on products of vector lattices are presented in the forthcoming survey paper [9].

1.5. A bilinear operator $b : E \times E \to G$ is said to be *orthosymmetric* if $x \perp y$ implies $b(x, y) = 0$ for arbitrary $x, y \in E$, see [12]. Recall that $x \perp y$ means $|x| \wedge |y| = 0$. The difference of two positive orthosymmetric bilinear operators is called *orthoregular*. Denote by $BL_{or}(E, G)$ the space of all orthoregular bilinear operators from $E \times E$ to G ordered by the cone of positive orthosymmetric operators.

Let X be a vector space. A bilinear operator $b : X \times X \to G$ is said to be *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in X$, *positively semidefinite* if $b(x, x) \geqslant 0$ for every $x \in X$, and *positively definite* if it is positively semidefinite and $b(x, x) = 0$ implies $x = 0$. It can easily be seen that an orthosymmetric positive bilinear operator is positively semidefinite [17]. More subtle is the fact that any orthosymmetric positive bilinear operator is symmetric, see [12, Corollary 2]. The following lemma, obtained in [24], is crucial.

1.6. *Assume that a positive linear operator $T : E \bar{\otimes} E \to G$ is such that the bilinear operator $T\otimes : E \times E \to G$ is orthosymmetric. If $Tu > 0$ for some $0 \leqslant u \in E \bar{\otimes} E$ then there is an element $e \in E$ with $0 < e \otimes e \leqslant u$.*

$\lhd$ We briefly sketch the proof. By virtue of 1.2 (3) there exists $e_0 \in E_+$ with $u \leqslant e_0 \otimes e_0$. Let E_0 be the order ideal in E generated by e_0. Then $E_0 \bar{\otimes} E_0$ is the sublattice of $E \bar{\otimes} E$ generated by $E_0 \otimes E_0$ (see 1.3). Now, if T_0 is the restriction of T to $E_0 \otimes E_0$ then the bilinear operator $b_0 := T_0\otimes : E_0 \times E_0 \to G$ is positive and orthosymmetric. One can consider E_0 as a uniformly closed sublattice of $C(Q)$ containing constants and separating points for some Hausdorff compact spaces Q. As was shown in [27, Proposition 1.7], there is a countable additive positive quasiregular Borel measure μ on Q with values in the Dedekind completion $\widehat{G}$ such that

$$b_0(x, y) = \int\limits_{Q \times Q} x(s)y(t)\, d\mu(s, t) \quad (x, y \in E_0).$$

Moreover, $\mu(Q \times Q \setminus \Delta) = 0$ where $\Delta := \{(q, q) : q \in Q\}$. It follows that if a function $u \in E_0 \bar{\otimes} E_0$ vanishes on Δ then $Tu = 0$, which contradicts to the hypotheses $Tu > 0$. Thus, $u(q, q) > 0$ for some $q \in Q$. Employing the Uryson Lemma one can choose a continuous function $x : Q \to [0, 1]$ and a real $0 < \varepsilon$ with $\varepsilon x \otimes x \leqslant u$. Since E_0 is uniformly dense in $C(Q)$, there is a function $e \in E_0$ such that $0 < e \leqslant \varepsilon x$, from which $0 < e \otimes e \leqslant u$. $\rhd$

1.7. *For any lattice bimorphism $b : E \times E \to F$ the following are equivalent:*

(1) *b is symmetric;*

(2) *b is orthosymmetric;*

(3) *b is positively semidefinite.*

$\lhd$ If b is symmetric then $b(x^+, x^-)$ and $b(x^-, x^+)$ coincide. At the same time these elements are disjoint, since $b(x^+, x^-) \leqslant b(x^+, |x|)$, $b(x^-, x^+) \leqslant b(x^-, |x|)$ and $b(x^+, |x|) \wedge b(x^-, |x|) = 0$. Thus, $b(x^+, x^-) = b(x^-, x^+) = 0$, from which the implication (1) $\to$ (3) follows. In addition, (3) $\to$ (2) was observed in [24, Lemma 1] and (2) $\to$ (1) follows from [12, Corollary 2]. $\rhd$

1.8. *Let A be an f-algebra with a multiplication $\cdot$ and E be a sublattice of A. Denote by $E^{(2)}$ the linear hull of $E \cdot E := \{x \cdot y : x, y \in E\}$. If E is relatively uniformly complete then $E^{(2)}$ is a vector lattice and $E^{(2)} = E \cdot E$.*

$\lhd$ This fact was proved in [11, Lemma 8]. $\rhd$

2. Canonical bimorphism

This section deals with the existence and some nice properties of the square of a vector lattice. For completeness sake we reproduce some proofs from [13].

2.1. Theorem. *For an arbitrary vector lattice E there exists a vector lattice $E^{\odot}$ (unique up to isomorphism) and a lattice bimorphism $\odot : (x, y) \mapsto x \odot y$ from $E \times E$ to $E^{\odot}$ such that the following hold:*

(1) *if b is a symmetric lattice bimorphism from $E \times E$ to some vector lattice F then there is a unique lattice homomorphism $\Phi_b : E^{\odot} \to F$ with $b = \Phi_b \odot$;*

(2) *given an arbitrary $u \in E^{\odot}$, there is $e_0 \in E_+$ such that, for every $\varepsilon > 0$, one can choose $x_1, \dots, x_n, y_1, \dots, y_n \in E$ with*

$$\left| u - \sum_{i=1}^{n} x_i \odot y_i \right| \leqslant \varepsilon e_0 \odot e_0;$$

(3) *for any $x, y \in E$ we have $x \odot y = 0$ if and only if $x \perp y$;*

(4) *given an element $0 < u \in E^{\odot}$, there exists $e \in E_+$ with $0 < e \odot e \leqslant u$.*

$\lhd$ Denote by J the smallest relatively uniformly closed order ideal in the tensor product $E \bar{\otimes} E$ containing the set $\{x \otimes y : x, y \in E, x \perp y\}$. Define $E^{\odot} := E \bar{\otimes} E / J$ just as in [13]. Let $\phi : E \bar{\otimes} E \to E^{\odot}$ be the quotient homomorphism and denote $\odot := \phi \otimes$. Then $E^{\odot}$ is an Archimedean vector lattice and $\odot$ is a lattice bimorphism. Observe that $\odot$ is orthosymmetric. Indeed, if $x \perp y$ then $x \otimes y \in J = \ker(\phi)$, thus $x \odot y = \phi(x \otimes y) = 0$. We now demonstrate the assertions (1–4).

(1): We repeat the arguments from [13, Theorem 4]. Let b be a symmetric bimorphism from $E \times E$ to a vector lattice F. Then b is orthosymmetric by virtue of 1.7. According to 1.2 (1) there is a unique lattice homomorphism $S : E \bar{\otimes} E \to F$ with $b = S \otimes$. The set $\ker(S)$ contains all elements of the form $x \otimes y$ with $x \perp y$ as the operator b is orthosymmetric. Since $\ker(S)$ is a relatively uniformly closed order ideal, we have $J \subset \ker(S)$. The existence of a linear operator $\Phi_b : E^{\odot} \to F$ with $S = \Phi_b \phi$ follows. Given $v = \phi(u)$, we have

$$|\Phi_b(v)| = |S(u)| = S(|u|) = \Phi_b(\phi(|u|)) = \Phi_b(|v|),$$

and the operator Φ_b is thus seen to be a lattice homomorphism. The uniqueness of Φ_b follows from the surjectivity of ϕ.

(2): If $u \in E^{\odot}$ then $u = \phi(v)$ for some $v \in E \bar{\otimes} E$. According to 1.2 (3) we can find $x_0, y_0 \in E$ such that, for every $\varepsilon > 0$, there are $x'_1, \dots, x'_n, y'_1, \dots, y'_n \in E$ with

$$\left| v - \sum_{i=1}^{n} x'_i \otimes y'_i \right| \leqslant \varepsilon x_0 \odot y_0.$$

Now, we only need to apply the lattice homomorphism ϕ to the last inequality and to put $e_0 = \phi(x_0) \vee \phi(y_0)$, $x_i = \phi(x'_i)$, $y_i = \phi(y'_i)$ $(i = 1, \dots, n)$.

(3): Take a universally complete vector lattice G whose base is isomorphic to the base of E. (Recall that a vector lattice is said to be *universally complete* if it is Dedekind complete and every nonempty set of pairwise disjoint positive elements has a supremum.) Then there is a lattice isomorphism ι from E onto an order dense sublattice $\iota(E)$ in G. Fix a multiplicative structure in G that is uniquely determined by a choice of a weak order unit in G and providing G with the structure of a semiprime f-algebra. Define the symmetric lattice bimorphism $b : E \times E \to G$ by $b(x, y) := \iota(x)\iota(y)$. According to (1) there exists a lattice homomorphism $S : E^{\odot} \to G$ for which $b = S \odot$. If $x \odot y = 0$ then $0 = b(x, y) = \iota(x)\iota(y)$, thus $x \perp y$. The converse was mentioned above.

(4): For $0 < u \in E^{\odot}$ one can choose $0 < v \in E \bar{\otimes} E$ with $u = \phi(v)$. By 1.6 there exists $e_0 \in E$ such that $0 < e_0 \otimes e_0 \leqslant v$. If $e := \phi(e_0)$ then $0 \leqslant e \otimes e \leqslant u$. In addition, $0 < e \odot e$, since assuming $0 = e \odot e$ we arrive at the contradictory equality $e = 0$ by virtue of (3).

The uniqueness (up to isomorphism) was demonstrated in [13, Theorem 4] as follows. Assume that for some vector lattice $E^{\circledcirc}$ and lattice bimorphism $\circledcirc : E \times E \to E^{\circledcirc}$ the pair $(E^{\circledcirc}, \circledcirc)$ obeys the conditions (1)–(2). Then there are lattice homomorphisms $\Phi : E^{\odot} \to E^{\circledcirc}$ and $\Psi : E^{\circledcirc} \to E^{\odot}$ such that $\Phi\odot = \circledcirc$ and $\Psi\circledcirc = \odot$. It follows that $\Psi \circ \Phi(x \odot y) = x \odot y$ and thus the set $\{x \odot y : x, y \in E\}$ is contained in $U := \{u \in E^{\odot} : \Psi \circ \Phi(u) = u\}$. At the same time U serves as a relatively uniformly closed sublattice in $E^{\odot}$. Therefore, by virtue of 2.1 (2), $U = E^{\odot}$ and $\Psi \circ \Phi = I_{E^{\odot}}$. In a similar way $\Phi \circ \Psi = I_{E^{\circledcirc}}$. Thereby Φ and Ψ are reciprocal lattice homomorphisms implementing an isomorphism of vector lattices $E^{\odot}$ and $E^{\circledcirc}$. $\rhd$

2.2. The vector lattice $E^{\odot}$ uniquely (up to lattice isomorphism) determined by an arbitrary vector lattice E is called the *square* of E. The symmetric lattice bimorphism $\odot : E \times E \to E^{\odot}$ is called the *canonical bimorphism*. By definition the vector lattice $E^{\odot}$ serves as a homomorphic image of $E \bar{\otimes} E$. As was mentioned above the construction of $E^{\odot}$ was first introduced in [13] where two more equivalent approaches to the notion of the square of a vector lattice are presented. Noteworthy is also the forthcoming paper [3] providing one more way to gain insight into the square of a vector lattice.

2.3. *Let $b : E \times E \to F$ and $\Phi : E^{\odot} \to F$ be respectively a symmetric bimorphism and a lattice homomorphism with $b = \Phi\odot$. Then Φ is injective if and only if b is positively definite.*

$\lhd$ Assume that b is positively definite and $\Phi(u) = 0$ for some $0 < u \in E^{\odot}$. Using 2.1 (4) choose $0 < e \in E_+$ with $e \odot e \leqslant u$. Then we arrive at the relation $0 \leqslant b(e, e) = \Phi(e \odot e) \leqslant \Phi(u) = 0$, which contradicts to the positive definiteness of b. Conversely, if Φ is injective and $b(e, e) = 0$ then $\Phi(e \odot e) = 0$, whence $e \odot e = 0$ and, by virtue of 2.1 (3), we obtain $e = 0$. $\rhd$

2.4. *If $T : E \to F$ is a lattice homomorphism then there is a unique lattice homomorphism $T^{\odot} : E^{\odot} \to F^{\odot}$ such that $T^{\odot}(x \odot y) = Tx \odot Ty$ $(x, y \in E)$.*

$\lhd$ Given a lattice homomorphism $T : E \to F$, put $b(x, y) := Tx \odot Ty$ $(x, y \in E)$. Then b is a symmetric lattice bimorphism from $E \times E$ into $F^{\odot}$. By 2.1 (1) there is a lattice homomorphism $S : E^{\odot} \to F^{\odot}$ with $b = S\odot$. Clearly, $T^{\odot} := S$ is the desired operator. $\rhd$

2.5. If E is a sublattice of a lattice ordered algebra A then it is natural to expect that the canonical bimorphism $\odot$ can be expressed in terms of the algebra multiplication. This can be done in case of a semiprime f-algebra A, see [13, Theorem 8].

Let A be a semiprime f-algebra with a multiplication $\cdot$ and E be a sublattice of A. Then there exists sublattice $F \subset A$ and an isomorphism ι from $E^{\odot}$ onto F such that $\iota(x \odot y) = x \cdot y$ for all $x, y \in E$. If, in addition, E is relatively uniformly complete then $F = E \cdot E$ and thus $E \cdot E$ is a vector lattice isomorphic to $E^{\odot}$.

$\lhd$ The first part is immediate from 2.1 (1) and 2.3, since the multiplication in a semiprime f-algebra is a positively definite symmetric lattice bimorphism. The second part follows from 1.8. $\rhd$

2.6. *If a vector lattice E is relatively uniformly complete then $E^{\odot} = E \odot E := \{x \odot y : x, y \in E\}$.*

$\lhd$ In the same way as in 2.1 (3) E can be taken as being a sublattice of the semiprime f-algebra $A := C_\infty(Q)$ for a suitable extremally disconnected compact space Q. Then by Proposition 2.5 $E^{\odot} = \iota^{-1}(E \cdot E) = E \odot E$. $\rhd$

Note that though we have for convenience employed the universal completion in this section, the results are constructively valid in ZF set theory. However, our results in the sections 4 and 5 below of course do involve by necessity some form of the Axiom of Choice.

2.7. *If E_0 is a sublattice in E and $\circledcirc$ is the canonical bimorphism of E_0 then there exists an injective lattice homomorphism h from $E_0^{\circledcirc}$ into $E^{\odot}$ such that $h(x \circledcirc y) = x \odot y$ for all*

$x, y \in E_0$. Therefore, $E_0^{\odot}$ and $\circledcirc$ can be considered respectively as a sublattice in $E^{\odot}$ and the restriction of $\odot$ to $E_0 \times E_0$. If E_0 is a majorizing (an order dense) sublattice in E then $E_0^{\odot}$ is a majorizing (an order dense) sublattice in $E^{\odot}$.

$\lhd$ To prove the first assertion we only need to apply 2.1 (1) and 2.3 to the bimorphism $(x, y) \mapsto \iota(x) \odot \iota(y)$ $(x, y \in E_0)$, where ι is the inclusion map from E_0 into E. The second assertion follows from 2.1 (2, 4). $\rhd$

2.8. *If F serves as a homomorphic image of a vector lattice E and F is relatively uniformly complete then $F^{\odot}$ is a homomorphic image of a vector lattice $E^{\odot}$.*

$\lhd$ Let $T : E \to F$ and $T^{\odot} : E^{\odot} \to F^{\odot}$ be the same as in 2.4 and $T(E) = F$. It suffices to observe that, by virtue 2.6, $F^{\odot} = F \odot F = T^{\odot}(E \odot E) \subset T^{\odot}(E^{\odot})$. $\rhd$

3. Representation of bilinear orthoregular operators

In this section we show that all orthoregular (orthosymmetric order bounded) bilinear operators from $E \times E$ to the relatively uniformly complete vector lattice G can be represented as compositions of regular (order bounded) linear operators from $E^{\odot}$ to G with the canonical bimorphism. This universal property of the square of a vector lattice is similar to that of Fremlin's tensor product. A general form of the classical Cauchy–Bunyakowski–Schwarz inequality is also proved.

3.1. Theorem. *Let E and G be vector lattices with G relatively uniformly complete. Then for every bilinear orthoregular operator $b : E \times E \to G$ there exists a unique linear regular operator $\Phi_b : E^{\odot} \to G$ such that*

$$b(x, y) = \Phi_b(x \odot y) \quad (x, y \in E).$$

The correspondence $b \mapsto \Phi_b$ is an isomorphism of the ordered vector spaces $BL_{or}(E, G)$ and $L_r(E^{\odot}, G)$.

$\lhd$ Take an orthosymmetric positive bilinear operator $b : E \times E \to G$. By virtue of 1.4 there exists a unique linear positive operator $T : E \mathbin{\bar{\otimes}} E \to G$ for which $T\otimes = b$. Having chosen a linear operator $\Phi_b := \Phi : F \to G$ with $T = \Phi \circ \phi$, we observe that the diagram

$$\begin{tikzcd}
E \times E \arrow[dd, "\otimes"'] \arrow[dr, "\odot"] \arrow[drr, "b"] & & \\
& E^{\odot} \arrow[r, dashed, "\Phi"] & G \\
E \mathbin{\bar{\otimes}} E \arrow[ur, "\phi"] \arrow[urr, "T"'] & &
\end{tikzcd}$$

is commutative. In particular, $b = \Phi \circ \odot$. If $f \in F_+$ then $f = \phi(u)$ for some $0 \leqslant u \in E \mathbin{\bar{\otimes}} E$, since ϕ is an onto mapping. Thus $\Phi(f) = Tu \geqslant 0$ and Φ is positive. Uniqueness of Φ follows from the relation $\Phi \circ \phi = T$. Now it is clear that to prove the existence of Φ with required properties it suffices to establish the inclusion $\ker(\phi) \subset \ker(T)$. Since ϕ is a lattice homomorphism, $u \in \ker(\phi)$ if and only if $|u| \in \ker(\phi)$, so that we can restrict ourselves to the case of positive $u \in E \mathbin{\bar{\otimes}} E$. If $0 \leqslant u \notin \ker(T)$ then, in view of 1.6, one can choose $0 < e \in E$ with $e \otimes e \leqslant u$.

It follows $0 = \phi(u) \geqslant \phi(e \otimes e) = e \odot e$, which contradicts to positive definiteness of $\odot$. Thus, $\ker(\phi) \subset \ker(T)$ as required.

Suppose now that b is an orthoregular bilinear operator. By definition $b = b_1 - b_2$ for some positive orthosymmetric bilinear operators $b_1, b_2 : E \times E \to G$. According to what has been just proved there exists a pair of positive linear operators $\Phi_1, \Phi_2 \in L_r(F, G)$ such that $b_k = \Phi_k \odot$ $(k := 1, 2)$. For $\Phi := \Phi_1 - \Phi_2$ we have $\Phi \in L_r(F, G)$ and $b = \Phi\odot$. The uniqueness of Φ is seen from the representation $\Phi \circ \phi = T_1 - T_2$ where $T_1, T_2 \in L_r(E \bar{\otimes} E, G)$ are positive operators uniquely defined by $T_k \otimes = b_k$ $(k := 1, 2)$. $\rhd$

By virtue of 2.4 Theorem 3.1 generalizes Theorem 1 from [12] and Lemma 4 from [11]. Consider some more results that can be derived from the above theorem.

3.2. *If G is an order complete vector lattice then $BL_{or}(E, G)$ is also an order complete vector lattice. Moreover, every regular orthosymmetric bilinear operator b is orthoregular and the following representations hold:*

$$|b| = |\Phi_b|\odot, \quad b^+ = \Phi_b^+\odot, \quad b^- = \Phi_b^-\odot\,.$$

$\lhd$ By virtue of Theorem 3.1, the ordered vector spaces $BL_{or}(E, G)$ and $L_r(E^\odot, G)$ are isomorphic. Therefore, it suffices to apply the Riesz–Kantorovich Theorem and observe that a linear and order isomorphism preserves lattice operations whenever they exist. $\rhd$

3.3. A bilinear operator $b : E \times F \to G$ is said to be *of order bounded variation* if for all $0 \leqslant x \in E$ and $0 \leqslant y \in F$ the set

$$\Big\{ \sum_{k=1}^{n} \sum_{l=1}^{m} b(x_k, y_l) :\ 0 \leqslant x_k \in E\,(1 \leqslant k \leqslant n \in \mathbb{N}),$$
$$0 \leqslant y_l \in E \ \ (1 \leqslant l \leqslant m \in \mathbb{N}),\, x = \sum_{k=1}^{n} x_k,\, y = \sum_{l=1}^{m} y_l \Big\}$$

is order bounded in G. The set of all bilinear operators $b : E \times F \to G$ that are of order bounded variations is denoted by $BL_{bv}(E, F; G)$ and forms an ordered vector space with the positive cone $BL_+(E, F; G)$, since every positive bilinear operator is of order bounded variation. It was proved in [14, Theorem 3.2] that if G is relatively uniformly complete then the correspondence $T \mapsto T\otimes$ is a linear and order isomorphism from the space of order bounded linear operators $L^\sim(E \bar{\otimes} F, G)$ onto $BL_{bv}(E, F; G)$. A similar result is true for orthosymmetric operators. But more can be said in this case: as was observed in [13, Theorem 9] any order bounded orthosymmetric bilinear operator is of order bounded variation; in symbols, $BL_{bv}(E, E; G) \supset BL_o^\sim(E; G)$ where $BL_o^\sim(E; G)$ denotes the set of all order bounded orthosymmetric bilinear operators from $E \times E$ to G.

3.4. Theorem. *Let E and G be vector lattices with G relatively uniformly complete. Then for every order bounded orthosymmetric bilinear operator $b : E \times E \to G$ there exists a unique order bounded linear operator $\Phi_b : E^\odot \to G$ such that*

$$b(x, y) = \Phi_b(x \odot y) \quad (x, y \in E).$$

The correspondence $b \mapsto \Phi_b$ is an isomorphism between ordered vector spaces $BL_o^\sim(E; G)$ and $L^\sim(E^\odot, G)$.

$\lhd$ Our arguments are similar to [14, Theorem 3.2]. Denote by $\widehat{G}$ the Dedekind completion of G. Take any $b \in BL_o^\sim(E, G)$ viewed as an operator with the target space $\widehat{G}$. By 3.1 and 3.3

there is an operator $\Phi_b \in L^{\sim}(E^{\odot}, \widehat{G})$ with $b = \Phi_b\odot$. It follows that $\Phi_b(U)$ is contained in G if U denotes the linear hull of $\{x \odot y : x, y \in E\}$. Thus, $\Phi_b(E^{\odot})$ is also contained in G, since an order bounded operator Φ_b preserves relative uniform convergence, U is relatively uniformly dense in $E^{\odot}$ by 2.1, and G is relatively uniformly closed in $\widehat{G}$. Evidently, the correspondence $b \mapsto \Phi_b$ is an order preserving linear bijection. $\rhd$

3.5. Below we need the following result established in [28]. Let G be a universally complete vector lattice and we fix a structure of a semiprime f-algebra in it that is uniquely determined by a choice of an order unit.

For any lattice bimorphism $b : E \times F \to G$ there exist lattice homomorphisms $S : E \to G$ and $T : F \to G$ such that

$$b(x, y) = S(x)T(y) \quad (x \in E,\ y \in F).$$

Moreover, b is symmetric if and only $S = \alpha T$ for some positive orthomorphism α in G.

3.6. Theorem. *Let E and G be vector lattices and $b : E \times E \to G$ be a symmetric lattice bimorphism. Then there exist a vector lattice F, a lattice homomorphism $S : E \to F$, and an isomorphic embedding $h : F \odot F \to G$ such that $S(E) = F$ and the representation hold:*

$$b(x, y) = h(S(x) \odot S(y)) \quad (x, y \in E).$$

$\lhd$ Immediate from 3.5 and 2.4. $\rhd$

3.7. In [12] the following general form of the classical Cauchy–Bunyakowski–Schwarz inequality was proved: if X is a real vector space and $b : X \times X \to F$ is a positively semidefinite symmetric bilinear operator with values in an almost f-algebra F then

$$b(x, y)b(x, y) \leqslant b(x, x)b(y, y) \quad (x, y \in X).$$

In the case of a semiprime f-algebra F this fact was established earlier in [21] and it was shown in [4] that the semiprimeness assumption can be omitted. Using Theorem 3.1 we point to another improvement announced in [25], replacing the almost f-algebra multiplication by an arbitrary positive orthosymmetric bilinear operator. A review of different generalizations and refinements of the classical Cauchy–Bunyakowski–Schwarz inequality one can find in [15].

3.8. Theorem. *Let X be a real vector space, E be a vector lattice, and $\langle\cdot,\cdot\rangle$ be a positively semidefinite symmetric bilinear operator from $X \times X$ to E. Let F be another vector lattice and $\circ : E \times E \to F$ be a positive orthosymmetric bilinear operator. Then*

$$\langle x, y\rangle \circ \langle x, y\rangle \leqslant \langle x, x\rangle \circ \langle y, y\rangle \quad (x, y \in X).$$

$\lhd$ Fix any two elements $x, y \in X$. Assume that the subspace $X_0 \subset X$ spanned by $\{x, y\}$ is two-dimensional, since otherwise there is nothing to prove. Put $u := \langle x, x\rangle$, $v := \langle y, y\rangle$, $w := \langle x, y\rangle$, and $e := u + v + |w|$. Let E_0 be an order ideal in E generated by e. Without loss of generality, we may assume that E_0 is a uniformly closed sublattice of the f-algebra $C(Q)$ for some compact Hausdorff space Q and e denotes the function on Q which is identically equal to 1. If b denotes the restriction of $\langle\cdot\,,\,\cdot\rangle$ onto $E_0 \times E_0$ then for every $q \in Q$ the relation $b_q(x', y') := b(x', y')(q)$ defines correctly a positively semidefinite symmetric bilinear form on $X_0 \times X_0$. By virtue of the classical Cauchy–Bunyakowski–Schwarz inequality $b_q(x', y')^2 \leqslant b_q(x', x')b_q(y', y')$ $(x', y' \in E_0)$. It follows that $b(x, y)b(x, y) \leqslant b(x, x)b(y, y)$.

Let juxtaposition mean the natural multiplication in the f-algebra $C(Q)$, and denote by d the restriction of $(e_1, e_2) \mapsto e_1 \circ e_2$ onto $E_0 \times E_0$. By Theorem 3.1 $d = S\odot$ for a suitable positive linear operator $S : E_0^{\odot} \to \widehat{F}$ where $\widehat{F}$ is a Dedekind completion of F. By virtue of 2.4 there exist a sublattice $G \subset C(Q)$ and an isomorphism ι from $E^{\odot}$ onto G such that $\iota(e_1 \odot e_2) = e_1 e_2$. As is easily seen, the operator $S' := S \circ \iota^{-1}$ is linear, positive, and $e_1 \circ e_2 = d(e_1, e_2) = S'(e_1 e_2)$. Applying S' to the inequality $b(x, y)b(x, y) \leqslant b(x, x)b(y, y)$ and taking the previous identity into consideration we arrive at $b(x, y) \circ b(x, y) \leqslant b(x, x) \circ b(y, y)$, which is equivalent to the required. $\rhd$

4. Extension of orthoregular bilinear operators

A positive bilinear operator defined on the Cartesian product of majorizing sublattices admits a positive bilinear extension to the Cartesian product of the ambient vector lattices [18, 22, 26]. If the given operator is orthosymmetric then a question arises as to whether there is a positive orthosymmetric bilinear extension. A positive answer can be derived using Theorem 3.1 and Proposition 2.7, see [25].

4.1. Theorem. *Let E_0 be a majorizing sublattice in a vector lattice E and G be an order complete vector lattice. Then every positive orthosymmetric bilinear operator $b_0 : E_0 \times E_0 \to G$ admits a positive orthosymmetric bilinear extension $b : E \times E \to G$.*

$\lhd$ Denote by ι the inclusion map from E_0 into E. According to 2.7 there is an isomorphic embedding $\iota^{\odot}$ of $E_0^{\odot}$ into $E^{\odot}$ such that the canonical bimorphism of E_0 is the restriction of the canonical bimorphism $\odot$ of E, i. e. $\odot(\iota \times \iota) = \iota^{\odot}\odot$. Moreover, $\iota^{\odot}(E_0^{\odot})$ is a majorizing sublettice in $E^{\odot}$. By virtue of Theorem 3.1 there is a linear positive operator $\Phi_0 : E_0^{\odot} \to G$ with $b_0 = \Phi_0\odot$. Choose any positive linear extension $\Phi : E^{\odot} \to G$ of Φ_0 and put $b := \Phi\odot$. Then the operator $b : E \times E \to G$ is bilinear, positive, and orthosymmetric and the diagram

$$\begin{tikzcd}
E_0 \times E_0 \arrow[rr, "\odot"] \arrow[dr, "b_0"] \arrow[dd, "\iota\times\iota"'] & & E_0^{\odot} \arrow[dl, "\Phi_0"'] \arrow[dd, "\iota^{\odot}"] \\
& G & \\
E \times E \arrow[rr, "\odot"'] \arrow[ur, dashed, "b"] & & E^{\odot} \arrow[ul, dashed, "\Phi"']
\end{tikzcd}$$

is commutative. In particular, $b(\iota \times \iota) = \Phi \odot (\iota \times \iota) = \Phi\iota^{\odot}\odot = \Phi_0\odot = b_0$ and b is thus an extension of b_0. $\rhd$

4.2. To prove our next extension theorem we need the following result: If E_0 is a majorizing vector sublattice of a vector lattice E and G is an order complete vector lattice then there exists an order continuous lattice homomorphism ε' from $L_r(E_0, G)$ to $L_r(E, G)$ such that $\varrho' \circ \varepsilon'$ is the identical mapping on $L_r(E_0, G)$, where $\rho' : L_r(E, G) \to L_r(E_0, G)$ is the restriction operator $S \mapsto S|_{E_0}$.

This fact stating the existence of an order continuous "simultaneous extension" of linear regular operators from a majorizing sublattice to the ambient vector lattice was proved in [22, Theorem 3], see also [23, Theorem 3.4.11].

4.3. Theorem. *Let E_0, E, and G be the same as in 4.1. Then there exists a "simultaneous extension" of the orthoregular bilinear operators from E_0 to E, i. e. there exists an order continuous lattice homomorphism $\varepsilon : BL_{or}(E_0, G) \to BL_{or}(E, G)$ such that $\rho \circ \varepsilon$ is the identity mapping on $BL_{or}(E_0, G)$, where $\rho : BL_{or}(E, G) \to BL_{or}(E_0, G)$ denotes the restriction operator $b \mapsto b|_{E_0 \times E_0}$.*

◁ According to 4.2 there exists a "simultaneous extension" $\varepsilon' : L_r(E_0, G) \to L_r(E, G)$. Consider two operators $\psi_0 : S_0 \mapsto S_0\odot$ $(S_0 \in L_r(E_0, G))$ and $\psi : S \mapsto S\odot$ $(S \in L_r(E, G))$. By Theorem 3.1 these operators are linear and lattice isomorphisms. Moreover, the right square of the diagram

$$\begin{tikzcd} L_r(E_0,G) \arrow[r, "\varepsilon'"] & L_r(E,G) \arrow[r, "\rho'"] \arrow[d, "\psi"] & L_r(E_0,G) \arrow[d, "\psi_0"] \\ BL_{or}(E_0,G) \arrow[u, "\psi_0^{-1}"] \arrow[r, dashed, "\varepsilon"'] & BL_{or}(E,G) \arrow[r, "\rho"'] & BL_{or}(E_0,G) \end{tikzcd}$$

is commutative. The operator $\varepsilon := \psi \circ \varepsilon' \circ \psi_0^{-1}$ makes the whole diagram commutative. In particular, $\rho \circ \varepsilon = \rho \circ \psi \circ \varepsilon' \circ \psi_0^{-1} = \psi_0 \circ \rho' \circ \varepsilon' \circ \psi_0^{-1} = \psi_0 \circ id \circ \psi_0^{-1} = Id$, where id and Id are identity mappings on $L_r(E_0, G)$ and $BL_{or}(E_0, G)$, respectively. ▷

4.4. Theorem. *If E_0 is a majorizing sublattice of a vector lattice E and G is an order complete vector lattice then every symmetric lattice bimorphism $b_0 : E_0 \times E_0 \to G$ admits a bilinear extension $b : E \times E \to G$ which is a symmetric lattice bimorphism.*

◁ Let $S_0 : E_0^{\odot} \to G$ be a positive linear operator with $b_0 = S_0\odot$, see Theorem 3.1. Denote by $\varepsilon^+(b_0)$ and $\varepsilon^+(S_0)$ the set of all positive orthosymmetric bilinear extensions of b_0 to $E \times E$ and the set of all positive linear extensions of S_0 to $E^{\odot}$, respectively. Using 2.1 (2) it is easy to verify that the correspondence $S \mapsto S\odot$ represents an affine bijection from $\varepsilon^+(S_0)$ onto $\varepsilon^+(b_0)$. Clearly, this bijection preserves extreme points. But it is known (see [10] and [23, 3.3.9 (3)]) that the extreme points of $\varepsilon^+(S_0)$ serves as the lattice homomorphisms, thus the extreme points of $\varepsilon^+(b_0)$ are symmetric lattice bimorphisms. ▷

5. Disjointness preserving operators

In this section we prove that an order bounded disjointness preserving orthosymmetric bilinear operator can be decomposed into a strongly disjoint sum of operators admitting weight-shift-weight representation.

5.1. Let E, F, and G be vector lattices. A bilinear operator $b : E \times F \to G$ is said to be *disjointness preserving* if

$$x_1 \perp x_2 \implies b(x_1, y) \perp b(x_2, y),$$
$$y_1 \perp y_2 \implies b(x, y_1) \perp b(x, y_2)$$

for arbitrary $x \in E$ and $y \in F$. If $E = F$ and b is orthosymmetric then b is symmetric (see 1.5) and any of these conditions implies the other.

Suppose that a bilinear operator $b : E \times F \to G$ is order bounded and disjointness preserving. Then b has the positive part b^+, the negative part b^-, and the modulus $|b|$, which are lattice bimorphisms. Moreover,

$$b^+(x, y) = b(x, y)^+, \quad b^-(x, y) = b(x, y)^- \quad (0 \leqslant x \in E,\ 0 \leqslant y \in F);$$

$$|b|(x, y) = |b(|x|, |y|)| \quad (x \in E,\ y \in F).$$

In particular, b is regular. Thus, any disjointness preserving bilinear operator is orthoregular if and only if it is order bounded and orthosymmetric, see [27, Theorem 3.4] and [7, Theorems 4 and 5].

5.2. *Let $b : E \times E \to G$ be an order bounded disjointness preserving orthosymmetric bilinear operator. If a regular linear operator $\Phi : E^{\odot} \to G$ satisfies the condition $b = \Phi\odot$, then Φ is disjointness preserving. Thus, the correspondence $\Phi \mapsto \Phi\odot$ is a bijection between the set of all order bounded disjointness preserving bilinear operators from $E \times E$ to G and the set of all order bounded disjointness preserving linear operators from $E^{\odot}$ to G.*

$\lhd$ Indeed, according to 5.1, b^+ and b^- are lattice bimorphisms, and by Theorem 3.1 they admit the representations $b^+ = S_1\odot$ and $b^- = S_2\odot$ for some lattice homomorphisms $S_1, S_2 : E^{\odot} \to G$. If

$$u := \sum_{k=1}^{n} x_k \odot y_k, \quad e := \sum_{k=1}^{n} |x_k| + \sum_{k=1}^{n} |y_k|,$$

then $|u| \leqslant e \odot e$; therefore, $|S_1(u)| = S_1(|u|) \leqslant S_1(e \odot e) = b^+(e, e) = (b(e, e))^+$. Analogously, $|S_2(u)| \leqslant b^-(e, e) \leqslant (b(e, e))^-$. Thus, $S_1(u) \perp S_2(u)$ for all $u \in U$ where U is the linear hull of the set $\{x \odot y : x, y \in E\}$. Using 2.1 (2), one can easily observe that the last statement is true for all $u \in E^{\odot}$, i. e. the operator S_1 and S_2 are strongly disjoint. Put $\Phi := S_1 - S_2$ and observe that Φ preserves disjointness by virtue of strong disjointness of S_1 and S_2. It is also obvious that $b = \Phi\odot$, $\Phi^+ = S_1$, $\Phi^- = S_2$ and $|b| = (S_1 + S_2)\odot$. $\rhd$

5.3. Bilinear operators b and d from $E\times F$ to G are called *strongly disjoint* if $\mathrm{im}(b) \perp \mathrm{im}(d)$. It can be easily shown that bilinear operators b and d are strongly disjoint if and only if $b(x, y) \perp d(x, y)$ for all $x \in E$ and $y \in F$.

Let $(b_\xi)_{\xi\in\Xi}$ be a family of bilinear operators from $E \times F$ to G. Say that an operator $b : E \times F \to G$ decomposes into the *strongly disjoint sum* of operators b_ξ $\big($and write $b = \bigoplus_{\xi\in\Xi} b_\xi\big)$, whenever the operators b_ξ are pairwise strongly disjoint and

$$b(x, y) = o\text{-}\sum_{\xi\in\Xi} b_\xi(x, y) \quad (x \in E,\ y \in F).$$

It is easy to observe that the strongly disjoint sum of (b_ξ) is disjointness preserving if and only if each its summand b_ξ is.

5.4. Order bounded disjointness preserving bilinear operators admit a nice analytical representation, see [28]. Now we are going to specialize to the corresponding representation results for orthosymmetric operators.

Let E and G be order-dense ideals of some universally complete vector lattices $\mathscr{E}$ and $\mathscr{G}$. In $\mathscr{E}$ and $\mathscr{G}$ we fix order-unities $1_{\mathscr{E}}$ and $1_{\mathscr{G}}$ and consider multiplications that make these spaces f-algebras with unities $1_{\mathscr{E}}$ and $1_{\mathscr{G}}$, respectively. We recall that orthomorphisms in $\mathscr{E}$ and $\mathscr{G}$ are multiplication operators and we identify them with the corresponding multipliers. For every $f \in \mathscr{E}$, there exists a unique element $g \in \mathscr{E}$ such that $fg = [f]1_{\mathscr{E}}$ and $[f] = [g]$, where $[f]$ stands for the band projection onto $f^{\perp\perp}$. We denote such an element g by $1_{\mathscr{E}}/f$ and put $g/f = g(1_{\mathscr{E}}/f)$. The orthomorphism $g \mapsto g/f$ is also denoted by $1_{\mathscr{E}}/f$.

Consider order dense ideals $E' \subset \mathscr{E}$, $G' \subset \mathscr{G}$, and $G'' \subset \mathscr{G}$. Denote by $G' \cdot G''$ the vector sublattice in $\mathscr{G}$ generated by the set $\{g'g'' : g' \in G', g'' \in G''\}$ (and actually coinciding with this set by 1.8).

If $w : E \to E'$ and $S : E' \to G'$ are linear operators then $w \times w : E \times E \to E' \times E'$ and $S \bullet S : E' \times E' \to G' \cdot G'$ denote the linear and bilinear operators defined by $(x, y) \mapsto (wx, wy)$ and $(x, y) \mapsto S(x)S(y)$, respectively. We say that a bilinear orthosymmetric operator $b : E \times E \to G$ admits a *weight-shift-weight representation* (*WSW*-representation for short) if there exist order-dense ideals $E' \subset \mathscr{E}$ and $G' \subset \mathscr{G}$, orthomorphisms $w : E \to E'$ and

$W : G' \cdot G' \to G$, and a shift operator $S : E' \to G'$ such that $b = W \circ (S \bullet S) \circ (w \times w)$, i. e. the diagram

$$\begin{tikzcd}
E \times E \arrow[r, "b"] \arrow[d, "w\times w"'] & G \\
E' \times E' \arrow[r, "S\bullet S"'] & G' \cdot G' \arrow[u, "W"']
\end{tikzcd}$$

is commutative.

A *shift operator* from E' to G' is a restriction to E' of a positive linear map $\hat{S} : \hat{E} \to \mathscr{G}$ satisfying the properties: 1) $\hat{E}$ is an order dense ideal in $\mathscr{E}$ containing E' and $1_{\mathscr{E}}$; 2) $\hat{S}$ sends any component of $1_{\mathscr{E}}$ into a component of $1_{\mathscr{G}}$; 3) $\hat{S}$ is disjointness preserving; 4) $\hat{S}(1_{\mathscr{E}})^{\perp\perp} = \mathscr{G}$. The operators W, S, and w are respectively called the *outer weight*, *shift*, and *inner weight* of the representation $W \circ (S \bullet S) \circ (w \times w)$.

Now we can deduce our representation result for order bounded disjointness preserving bilinear operators using the corresponding results for linear operators due to A. E. Gutman, see [19, 20, 23].

5.5. Theorem. *Let $b : E \times E \to G$ be a disjointness preserving orthoregular bilinear operator. Then there exist a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the Boolean algebra $\mathfrak{P}(F)$ and a family $(e_\xi)_{\xi\in\Xi}$ in E^+ such that for each $\xi \in \Xi$ the composite $\rho_\xi \circ b$ admits a WSW-representation (with an inner weight $1_{\mathscr{E}}/e_\xi$)*

$$\rho_\xi \circ b = \rho_\xi W \circ (S \bullet S) \circ (1_{\mathscr{E}}/e_\xi \times 1_{\mathscr{E}}/e_\xi) \quad (\xi \in \Xi),$$

where S is a shift operator and the outer weight $W : \mathscr{G} \to \mathscr{G}$ is the orthomorphism of multiplication by o-$\sum_{\xi\in\Xi} \rho_\xi b(e_\xi, e_\xi)$. Thus, the operator b decomposes into the strongly disjoint sum of the operators $\rho_\xi \circ b$ admitting WSW-representations with the same shift operator.

$\triangleleft$ Let $b : E \times E \to G$ be a disjointness preserving orthoregular bilinear operator. Then $|b|$ is a lattice bimorphism. By virtue of 3.5 $|b|(x, y) = T(x)T(y)$, where $T : E \to \mathscr{F}$ is a lattice homomorphism. According to Theorem 5.1 there is a band projection π in $\mathscr{F}$ such that $\pi|b| = b^+$ and $\pi^\perp|b| = b^-$. Apply the representation result from [23, Theorem 5.4.5] which implies the existence of a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathfrak{P}(G)$ and a family e_ξ of positive elements in E such that, for each $\xi \in \Xi$, the composite $\rho_\xi \circ T$ admits a WSW-representation $\rho_\xi \circ T = W_0 \circ \rho_\xi S \circ (1_{\mathscr{E}}/e_\xi)$, where $S : E' \to G'$ is the shift of T and $W : \mathscr{F} \to \mathscr{F}$ is the orthomorphism of multiplication by o-$\sum_{\xi\in\Xi} \rho_\xi T e_\xi$. Taking into consideration disjointness of ρ_ξ and ρ_η with $\xi \neq \eta$, we deduce

$$\rho_\xi|b|(x, y) = \rho_\xi T(x)\rho_\xi T(y) = W_0^2 \circ \rho_\xi S(x/e_\xi)S(y/e_\xi) \quad (x, y \in E).$$

Denote $W := o$-$\sum_\xi \pi\rho_\xi W_0^2 - \pi^\perp\rho_\xi W_0^2$ and observe that $\rho_\xi b = \rho_\xi W \circ (S \bullet S) \circ (1_{\mathscr{E}}/e_\xi \times 1_{\mathscr{E}}/e_\xi)$, from which the desired assertion follows. $\triangleright$

5.6. Let P and Q be extremally disconnected Hausdorff compact topological spaces, E and G be order-dense ideals in the universally complete vector lattices $\mathscr{E} := C_\infty(P)$ and $\mathscr{G} := C_\infty(Q)$, respectively. Denote by $C_0(Q, P)$ the totality of all continuous functions $\sigma : Q_0 := \mathrm{dom}(\sigma) \to P$ defined on various clopen subsets $Q_0 \subset Q$. Given arbitrary $\sigma \in C_0(Q, P)$

and $e \in C_\infty(P)$, the function $\sigma^*e : Q \to \overline{\mathbb{R}}$ is defined as follows:

$$(\sigma^*e)(q) := \begin{cases} e\big(\sigma(q)\big), & \text{if } q \in \mathrm{dom}(\sigma), \\ 0, & \text{if } q \in Q \setminus \mathrm{dom}(\sigma). \end{cases}$$

Obviously, the function σ^*e is continuous but, in general, is not an element of $C_\infty(Q)$, since it can assume infinite values on a set with non-empty interior Q_∞. If some $W \in C_\infty(Q)$ vanishes on Q_∞ then we assume that $W\sigma^*x \in C_\infty(Q)$.

5.7. Theorem. *Let E and G be order dense ideals in $C_\infty(P)$ and $C_\infty(Q)$, respectively, and let $b : E \times E \to G$ be a disjointness preserving orthoregular bilinear operator. Then there exists a continuous mapping $\sigma \in C_0(Q, P)$, a family of positive functions $(w_\xi)_{\xi\in\Xi}$ in $C_\infty(P)$, and a family of pairwise disjoint functions $(W_\xi)_{\xi\in\Xi}$ in $C_\infty(Q)$ such that $1/w_\xi \in E$ for all $\xi \in \Xi$ and*

$$b(x, y) = o\text{-}\sum_{\xi\in\Xi} W_\xi \sigma^*(w_\xi x)\sigma^*(w_\xi y) \quad (x, y \in E).$$

$\lhd$ We need only to apply 5.6 and to use the following result from [23, Theorem 5.4.5]: A linear operator $T : E \to G$ admits a WSW-representation if and only if there exist functions $\sigma \in C_0(Q, P)$, $w \in C_\infty(P)$, and $W \in C_\infty(Q)$ such that $\sigma^*(we) \in C_\infty(Q)$ and $Te = W\sigma^*(we)$ for all $e \in E$. $\rhd$

5.8. Theorem 5.7 gives an analytical representations of order bounded disjointness preserving bilinear operators with the help of the operations of continuous change of variable and pointwise multiplication by a real-valued function. This type of analytical representation, often called *multiplicative representation*, stems from the work [1] by Yu. A. Abramovich.

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Gerard Buskes
Department of Mathematics
University of Mississippi
University, MS 38677, USA
E-mail: `mmbuskes@olemiss.edu`

Anatoly Kusraev
Institute of Applied Mathematics and Informatics
Vladikavkaz Science Center of the RAS
Vladikavkaz, 362040, RUSSIA
E-mail: `kusraev@alanianet.ru`