Владикавказский математический журнал
Апрель–июнь, 2006, Том 8, Выпуск 2

УДК **512.541**

ON THE BALANCED SUBGROUPS OF MODULAR GROUP RINGS

P. V. Danchev

The balanced property of certain subgroups of the group of all normalized p-torsion invertible elements in a modular group ring of characteristic p is explored.

Introduction

Let $S(RG)$ be the normed p-unit group in a group ring RG, formed by an abelian group G and a commutative ring R with identity of prime characteristic p. All unexplained symbols and letters as well as the terminology and definitions from the abelian group theory (including the topological ones) can be found in the classical book monographs [7]. For a background material in that direction, we refer the reader also to [1]–[6].

The major goal motivating the present paper is to find some special nice and isotype subgroups of $S(RG)$, a problem that arises naturally in the examination of the total projectivity both in modular and semi-simple aspects (cf. [1] and [6]). Thus the property of subgroups being balanced in modular group rings is crucial for the investigation of nice composition series and nice bases in such rings (see, for instance, [9] or [5]).

Moreover, the balanced subgroups play an important role for the quasi-completeness (e. g. [2, 3]) and torsion-completeness (e. g. [4]) in group algebras by using either an algebraical or topological technique in terms of bounded convergent Cauchy sequences.

The query for the balanced property of $S(KH)$ in $S(KG)$ when KG is semisimple, such that G is p-primary and K is either a field having arbitrary characteristic or is a special ring of zero characteristic, is considered and settled in some way by us in [4].

In [9] and [8], May and Hill-Ullery studied the case when R is a field, whereas we here investigate the general situation which cannot be treated by similar reasons.

The main result

We start with a single key assertion needed for future applications. It discovers the balanced property in $S(RG)$ of subgroups of the type $S(RH)$, whenever $H \leqslant G$; for certain other balanced subgroups the readers can see [6].

Proposition. *Let H be a p-balanced (that is p-nice and p-isotype) subgroup of G. Then $S(PH)$ is balanced in $S(RG)$, provided P is a perfect subring of R with the same unity.*

$\lhd$ **«p-nice»**. Bearing in mind [7], it is enough to calculate that $\bigcap_{\alpha<\tau} [S^{p^\alpha}(RG)S(PH)] = S^{p^\tau}(RG)S(PH)$ for every limit ordinal τ. In fact, given an element x in the left hand-side, hence, by [3], $x \in (\sum_{i=1}^{m} r_i g_i)S(PH) = (\sum_{i=1}^{n} r'_i g'_i)S(PH) = \ldots$, where $r_i \in R^{p^\alpha}$, $\sum_{i=1}^{m} r_i = 1$, $g_i \in G^{p^\alpha}$; $r'_i \in R^{p^\beta}$, $\sum_{i=1}^{n} r'_i = 1$, $g'_i \in G^{p^\beta}$; $\alpha < \beta < \tau$ and β is arbitrary but a fixed ordinal. Thus we can write

$$\sum_{i=1}^{m} r_i g_i = \left(\sum_{i=1}^{n} r'_i g'_i\right)\left(\sum_{i=1}^{n} f_i h_i\right) = \sum_i \sum_j r'_i f_j g'_i h_j,$$

whenever $f_i \in P$ with $\sum_{i=1}^{n} f_i = 1$ and $h_i \in H$.

Writing $\sum_{i,j} r'_i f_j g'_i h_j$ in canonical form, we may presume without loss of generality that the following relations hold:

$$r'_1 f_1 \neq 0,\ r'_1 f_2 = \ldots = r'_1 f_n = 0;\ r'_2 f_2 \neq 0,\ r'_2 f_1 = r'_2 f_3 = \ldots = r'_2 f_n = 0; \ldots;$$
$$r'_s f_s \neq 0,\ r'_s f_1 = \ldots = r'_s f_{s-1} = r'_s f_{s+1} = \ldots = r'_s f_n = 0$$

for some $s \in \mathbb{N}$, and all other ring products are not zero. Of course, these ring dependencies are indeed correct and well-chosen, because if in addition $r'_1 f_1 = 0$ we detect that $0 = r'_1(f_1 + \ldots + f_n) = r'_1$ which is a contradiction. Moreover, we note that $r'_1 f_1 = r'_1(f_1 + \ldots + f_n) = r'_1$, $\ldots$, $r'_s f_s = r'_s(f_1 + \ldots + f_n) = r'_s$.

Now, let us assume for difficulty that the following additional group ratios hold (if not, the things are easy): $g'_2 h_2 = g'_3 h_3 = \ldots = g'_{s-1} h_{s-1}$ such that $r'_2 f_2 + r'_3 f_3 + \ldots + r'_{s-1} f_{s-1} = 0$, i. e. these elements do not lie in the support.

A crucial fact is that, since the supports of the elements in the group ring are finite while the set $\{\alpha < \beta < \tau : \beta \geqslant \omega\}$ is infinite, all given relations are assumed of the above types presented. We mention that all other variants, even when there is no zero divisors, are identical or have a simple interpretation.

The canonical records imply

$$r_1 = r'_1 f_1,\ g_1 = g'_1 h_1;\ r_2 = r'_{s+1} f_1,\ g_2 = g'_{s+1} h_1;\ r_3 = r'_{s+2} f_2,\ g_3 = g'_{s+2} h_2; \ldots;$$
$$r_k = r'_{s+1} f_2,\ g_k = g'_{s+1} h_2;\ r_{k+1} = r'_{s+2} f_1,\ g_{k+1} = g'_{s+2} h_1; \ldots;\ r_s = r'_s f_s,\ g_s = g'_s h_s; \ldots;$$
$$r_n = r'_n f_n,\ g_n = g'_n h_n; \ldots;\ r_{m-2} = r'_{n-2} f_{n-1},\ g_{m-2} = g'_{n-2} h_{n-1};$$
$$r_{m-1} = r'_{n-1} f_n,\ g_{m-1} = g'_{n-1} h_n;\ r_m = r'_n f_1,\ g_m = g'_n h_1.$$

Therefore, we get that, $r_1 \in \bigcap_{\beta<\tau} R^{p^\beta} = R^{p^\tau}, \ldots, r_m \in R^{p^\tau}$, hence $r'_1 \in R^{p^\tau}, \ldots, r'_n \in R^{p^\tau}$ since

$$r'_1 = r'_1 f_1 = r_1, \ldots, r'_s = r'_s f_s = r_s,$$
$$r'_{s+1} = r'_{s+1} f_1 + \ldots + r'_{s+1} f_n = r_2 + \ldots,$$
$$\ldots,$$
$$r'_n = r'_n f_1 + \ldots + r'_n f_n = r_m + \ldots + r_n,$$

where $m = n^2 - s + 2 - s(n-1) = n^2 - sn + 2$. Besides,

$$g_1 \in \bigcap_{\beta<\tau} (G^{p^\beta} H) = G^{p^\tau} H, \ldots, g_m \in G^{p^\tau} H.$$

Thus we can write $g_1 = g_{\tau 1}a_1, \ldots, g_m = g_{\tau m}a_m$ where $g_{\tau 1}, \ldots, g_{\tau m} \in G^{p^\tau}$ and $a_1, \ldots, a_m \in H$. Since $g_1 g_2^{-1} \in G^{p^\tau}$, whence $a_1 a_2^{-1} \in G^{p^\tau}$, we shall presume that $a_1 = a_2$ because $g_{\tau 1}a_1 = g'_{\tau 1}a_2$ for some $g'_{\tau 1} \in G^{p^\tau}$. By the same token we may produce also for the other pairs of indices (i, j) such that $g_i g_j^{-1} \in G^{p^\tau}$. Besides, $g_2 g_k^{-1} = h_1 h_2^{-1} \in H$, hence $g_{\tau 1} g_{\tau k}^{-1} \in H$. The same procedure can be done for the other pairs of distinct indexes with this property as well.

We observe that $\sum\limits_{i=1}^{m} r_i g_i = \left(\sum\limits_{i=1}^{n} r'_i g_{\tau u_i}\right)\left(\sum\limits_{i=1}^{n} f_i b_i\right)$, where for $1 \leqslant i \leqslant n$ we have $b_i = a_{u_i}$ or $b_i = a_{u_i} g_{\tau v_i} g_{\tau w_i}^{-1} \in H$ for some appropriate permutations u_i, v_i, w_i of the indexes $1, \ldots, n$ so that $g_{\tau 2} b_2 = g_{\tau 3} b_3 = \ldots = g_{\tau(s-1)} b_{s-1}$, and eventually $r_i = r_{u_i}$.

When $m > n$ it may be possible that $\sum\limits_{i=1}^{m} r_i g_i = \left(\sum\limits_{i=1}^{m} r_i g_{\tau i}\right) a$ for some $a \in H$.

Since $\sum\limits_{i=1}^{m} r_i g_i \in S(RG)$, there exists a group member from the sum which member belongs to G_p. By a reason of symmetry the same should be valid even for $\sum\limits_{i=1}^{n} r'_i g'_i$ and $\sum\limits_{i=1}^{n} f_i h_i$. So, with no harm of generality, we may suppose that: $g_1, \ldots, g_l \in G_p$, $r_1 + \ldots + r_l - 1$ belongs to the nil-radical of R; $G_p \not\ni g_{l+1} \in g_{l+2}G_p \in \ldots \in g_m G_p$, $r_{l+1} + r_{l+2} + \ldots + r_m$ lies in the nil-radical of R; $l \in \mathbb{N}$. Analogously $g'_1, \ldots, g'_k \in G_p$, $r_1 + \ldots + r_k - 1$ belongs to the nil-radical of R; $G_p \not\ni g'_{k+1} \in g'_{k+2}G_p \in \ldots \in g'_n G_p$, $r_{k+1} + r_{k+2} + \ldots + r_n$ lies in the nilradical of R and $h_1, \ldots, h_k \in H_p$, $f_1 + \ldots + f_k - 1$ is in the nilradical of R; $H_p \not\ni h_{k+1} \in h_{k+2}H_p \in \ldots \in h_n H_p$, $f_{k+1} + f_{k+2} + \ldots + f_n$ is in the nilradical of R; $n \in \mathbb{N}$.

Because, for any ordinal δ, we know that $(G^{p^\delta}H)_p = G_p^{p^\delta} H_p$, we will presume that $g_{\tau 1} \in G_p^{p^\tau}$ and $a_1 \in H_p$. Moreover, by what we have already proved,

$$g_{l+1} g_{l+2}^{-1} \in (G^{p^\tau}H)_p = G_p^{p^\tau} H_p,\ \ldots,\ g_{l+1} g_m^{-1} \in G_p^{p^\tau} H_p,\ \ldots,\ g_{l+2} g_m^{-1} \in G_p^{p^\tau} H_p \ \text{ etc.}$$
$$g'_{k+1} g'^{-1}_{k+2} \in G_p^{p^\tau} H_p,\ \ldots,\ g'_{k+1} g'^{-1}_n \in G_p^{p^\tau} H_p,\ \ldots,\ g'_{k+2} g'^{-1}_n \in G_p^{p^\tau} H_p \ \text{ etc.}$$

Similarly for $h_{k+1} h_{k+2}^{-1} \in H_p$, …, $h_{k+1} h_n^{-1} \in H_p$, …, $h_{k+2} h_n^{-1} \in H_p$ etc.

Furthermore, $b_i = a_{u_i} g_{\tau v_i} g_{\tau w_i}^{-1} \in H_p$ for $i = 1, \ldots, k$ or $b_i \in b_j H_p$ for $k + 1 \leqslant i \neq j \leqslant n$.

Finally, it is apparent that $\sum\limits_{i=1}^{n} r'_i g_{\tau u_i} \in \bigcap\limits_{\alpha<\tau} S(R^{p^\alpha} G^{p^\alpha}) = S(R^{p^\tau} G^{p^\tau}) = S^{p^\tau}(RG)$ and $\sum\limits_{i=1}^{n} f_i b_i \in S(PH)$. That is why, it is easily checked that $x \in S^{p^\tau}(RG)S(PH)$. Thereby, the wanted equality is true, as expected.

«p-isotype». Exploiting [1],

$$S(PH) \cap S^{p^\alpha}(RG) = S(PH) \cap S(R^{p^\alpha} G^{p^\alpha}) = S(P(H \cap G^{p^\alpha})) = S(PH^{p^\alpha}) = S^{p^\alpha}(PH).$$

So, the proof is completed in all generality. ▷

References

1. *Danchev P.* Commutative group algebras of cardinality $\aleph_1$ // Southeast Asian Bull. Math.—2001 / 2002.—V. 25, № 4.—P. 589–598.
2. *Danchev P.* Quasi-completeness in commutative modular group algebras // Ricerche Mat.—2002.—V. 51, № 2.—P. 319–327.
3. *Danchev P.* Quasi-closed primary components in abelian group rings // Tamkang J. Math.—2003.—V. 34, № 1.—P. 87–92.
4. *Danchev P.* Torsion completeness of Sylow p-groups in semisimple group rings // Acta Math. Sinica.—2004.—V.20, № 5.—P. 893–898.

5. *Danchev P.* A nice basis for $S(FG)/G_p$ // Atti Sem. Mat. Fisico Univ. Modena.—2005.—V. 53, № 1.—P. 3–11.
6. *Danchev P.* On a decomposition formula in commutative group rings // Miskolc Math. Notes.—2005.—V. 6, № 2.—P. 153–159.
7. *Fuchs L.* Infinite Abelian Groups. V. I, II.—M.: Mir, 1974, 1977.
8. *Hill P., Ullery W.* A note on a theorem of May concerning commutative group algebras // Proc. Amer. Math. Soc.—1990.—V. 110, № 1.—P. 59–63.
9. *May W.* The direct factor problem for modular abelian group algebras // Contemp. Math.—1989.—V. 93.—P. 303–308.

Received by the editors September 2, 2005.

Dr. Danchev Peter. V.
Plovdiv, Bulgaria, Plovdiv State University «Paissii Hilendarski»
E-mail: `pvdanchev@yahoo.com`