*Владикавказский математический журнал*
*Июль–сентябрь, 2004, Том 6, Выпуск 3*

*УДК* **517.5**

# A CACCIOPPOLI TYPE INEQUALITY

*To Yu. G. Reshetnyak*
*on the occasion of his*
*75th birthday*

**M. S. Alborova**

A Caccioppoli type inequality for solutions of the quasi-elliptic equations is ostablished.

## 1. Notations and basic definitions

By $x = (x_1, \dots, x_n)$ we denote a point in the $n$-dimensional Euclidean space $\mathbb{R}^n$. We make use of the standard notations: for any $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \geqslant 0$ integers,

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad b \cdot \alpha = b_1\alpha_1 + \dots + b_n\alpha_n, \quad x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n},$$

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_n} \cdot \ldots \cdot \partial x_n^{\alpha_n}}.$$

We give an $n$-tuple $l = (l_1, \dots, l_n)$, $l_i \geqslant 1$ integers, fixed throughout the paper.

Now we consider a quasi-homogeneous transformation group of $\mathbb{R}^n$:

$$H_t(x) = \Big(t^{\frac{l^*}{l_1}} x_1, \dots, t^{\frac{l^*}{l_n}} x_n\Big), \quad t \in \mathbb{R}^+,$$

where ${l^*}^{-1} = \frac{1}{n}\sum_{i=1}^n \frac{1}{l_i}$.

Instead of the ordinary (homogeneous of degree 1) Euclidean distance we shall use the distance (quasi-homogeneous of degree 1):

$$r_l : \mathbb{R}^n/0 \to \mathbb{R}^+, \quad r_l(H_t(x)) = t r_l(x), \quad x \in \mathbb{R}^n.$$

Let $B_\rho^l(x)$ denotes the ball:

$$B_\rho^l(x) = \{y : r_l(x, y) \leqslant \rho\}$$

and $\omega_\rho^l$ denotes the volume of the unit-ball, i. e. $\omega_\rho^l = \mathcal{L}^n(B_1^l)$.

We shall also use the lemma (cf. [3]):

***Lemma.*** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $e \subset \Omega$ be compact. Then there exists a function $\xi(x) \in C_0^\infty(\Omega,[0,1])$ such that $\xi(x)=1$ in $e$ and*

$$|D^\alpha \xi(x)| \leqslant k_\alpha / \delta(e)^{l^* \sum_{i=1}^n \frac{\alpha_i}{l_i}}$$

*for all multi-indexes $\alpha=(\alpha_1,\dots,\alpha_n)$, where $\delta(e)=\inf\limits_{\substack{x\in e\\ y\in\partial\Omega}} r_l(x,y)$.*

Note that we have the elementary estimate

$$|D^\beta(\xi^{pm})| \leqslant c_3 \xi^{pm-|\beta|}\cdot \rho^{-l^*\sum_{i=1}^n \frac{\beta_i}{l_i}}.$$

The function $\xi(x)$ is called the cut-off function (the homogeneous case, cf. [4]).

Let $\mathcal{F}$ be the set of all polynomials $\mathcal{P}(x)=\sum_{|\alpha:l|\leqslant 1} c_\alpha x^\alpha$, where $|\alpha:l|=\frac{\alpha_1}{l_1}+\dots+\frac{\alpha_n}{l_n}$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Define the space $L_p^l=\{f:\, D^\alpha f\in L_p(\Omega),\, |\alpha:l|=1\}$, where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index with $|\alpha:l|=\sum_{i=1}^n \frac{\alpha_i}{l_i}=1$. The norm in $L_p^l$ is given by (cf. [1, 2])

$$\|f\|_{L_p^l(\Omega)}=\sum_{|\alpha:l|=1}\left(\int\limits_\Omega |D^\alpha f|^p dx\right)^{1/p}.$$

## 2. The main result

We will now consider a weak solution $u\in L_p^l(\Omega)$ of the equation

$$\sum_{|\alpha:l|=1}\int\limits_\Omega a_\alpha(x,D^\alpha u)D^\alpha\varphi\,dx=0 \tag{1}$$

for all $\varphi\in C_0^\infty(B_\rho^l(x_0))$.

The coefficients $a_\alpha(x,D^\alpha u)$, $|\alpha:l|=1$, are defined on $\Omega$. We impose the following structure conditions:

**(A1)** $|A(x,D^\alpha u)-A(x,D^\alpha v)|\leqslant a_0\sum_{|\alpha:l|=1}|D^\alpha u-D^\alpha v|^{p-1}$;

**(A2)** $(A(x,D^\alpha u)-A(x,D^\alpha v))D^\alpha(u-v)\geqslant \beta\sum_{|\alpha:l|=1}|D^\alpha(u-v)|^p$, $\beta\geqslant 2$;

**(A3)** $|A(x,D^\alpha u)-A(\tilde{x},D^\alpha u)|\leqslant k[r_l(x-\tilde{x})]^s(1+|D^\alpha u|)$,

where $x$, $\widetilde{x}\in\Omega$.

Under these assumptions we prove the following result.

***Theorem*** (A Caccioppoli type Inequality)**.** *Let $B_\rho^l(x_0)\subset\Omega$, with $\rho\leqslant 1$. Consider an arbitrary solution $u\in L_p^l(\Omega)$ of* (1)*, $p>1$, where the structure conditions* (A1), (A2) *and* (A3) *are valid, and an arbitrary polynomial $\mathcal{P}\in\mathcal{F}$. Then there holds:*

$$\int\limits_{B^l_{\rho/2}(x_0)} |D^\alpha u - D^\alpha\mathcal{P}|^p\,dx \leqslant c_1\Bigg(\sum_{|\alpha:l|=1}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\rho^{-pl^*\sum_{i=1}^n\frac{\alpha_i-\gamma_i}{l_i}} \times\int\limits_{B^l_\rho(x_0)}|D^\gamma(u-\mathcal{P})|^p\,dx+(1+\mathrm{P})^{\frac{p-1}{p}}\rho^{s\frac{p}{p-1}}\omega^l_\rho\Bigg). \tag{2}$$

*Here* $\mathrm{P}=\sum_{|\alpha:l|=1}|D^\alpha\mathcal{P}^k(x_0)|$.

(The homogeneous case, cf. [5].)

### 3. Proof of the main theorem

We consider a test function $\varphi = \xi^{p|\alpha|}(u - \mathcal{P})$ in (1), where $\xi$ is a cut-off function, $\xi \in C_0^\infty(B_\rho^l(x_0))$. We have

$$\sum_{|\alpha:l|=1} \int_{B_\rho^l(x_0)} a_\alpha(x, D^\alpha u) D^\alpha(u - \mathcal{P}) \xi^{p|\alpha|}\, dx$$
$$= - \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho^l(x_0)} a_\alpha(x, D^\alpha u) D^\gamma(u - \mathcal{P}) D^{\alpha-\gamma}(\xi^{p|\alpha|})\, dx.$$

By the definition of $\varphi$ we further have

$$- \sum_{|\alpha:l|=1} \int_{B_\rho^l(x_0)} a_\alpha(x, D^\alpha \mathcal{P}) D^\alpha(u - \mathcal{P}) \xi^{p|\alpha|}\, dx = \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho^l(x_0)} a_\alpha(x, D^\alpha \mathcal{P})$$
$$\times D^\gamma(u - \mathcal{P}) D^{\alpha-\gamma} \xi^{p|\alpha|}\, dx - \sum_{|\alpha:l|=1} \int_{B_\rho^l(x_0)} a_\alpha(x, D^\alpha \mathcal{P}) D^\alpha \varphi dx.$$

We note

$$0 = \sum_{|\alpha:l|=1} \int_{B_\rho^l(x_0)} a_\alpha(x_0, D^\alpha \mathcal{P}) D^\alpha \varphi\, dx.$$

Combining these three equations, we arrive at the inequality

$$\sum_{|\alpha:l|=1} \int_{B_\rho^l(x_0)} \left[a_\alpha(x, D^\alpha u) - a_\alpha(x, D^\alpha \mathcal{P})\right] D^\alpha(u - \mathcal{P}) \cdot \xi^{p\,|\alpha|} dx \leqslant I_1 + I_2 + I_3 \tag{3}$$

where $I_1$, $I_2$, $I_3$ are defined as follows:

$$I_1 = \left| \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho^l(x_0)} \big[a_\alpha(x, D^\alpha u) - a_\alpha(x, D^\alpha \mathcal{P})\big] D^\gamma(u - \mathcal{P}) D^{\alpha-\gamma} \xi^{p\,|\alpha|} dx \right|,$$
$$I_2 = \left| \sum_{|\alpha:l|=1} \int_{B_\rho^l(x_0)} \big(a_\alpha(x, D^\alpha \mathcal{P}) - a_\alpha(x_0, D^\alpha \mathcal{P})\big) D^\alpha(u - \mathcal{P}) \xi^{p\,|\alpha|} dx \right|,$$
$$I_3 = \left| \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho^l(x_0)} \big[a_\alpha(x, D^\alpha \mathcal{P}) - a_\alpha(x_0, D^\alpha \mathcal{P})\big] D^\gamma(u - \mathcal{P}) D^{\alpha-\gamma} \xi^{p\,|\alpha|} dx \right|.$$

Using (A1) we have

$$I_1 \leqslant a_0 \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho^l(x_0)} |D^\alpha u - D^\alpha \mathcal{P}\,|^{p-1} \cdot |D^\gamma(u - \mathcal{P})| \cdot |D^{\alpha-\gamma} \xi^{p\,|\alpha|}|\, dx$$

$$\leqslant a_0c_3 \sum_{|\alpha:l|=1}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\int\limits_{B^l_\rho(x_0)} (|D^\alpha u - D^\alpha\mathcal{P}\,|^{p-1}\xi^{p\,|\alpha|-(|\alpha|-|\gamma|)}\cdot\rho^{-l^*\sum_{i=1}^n\frac{\alpha_i-\gamma_i}{l_i}}$$

$$\times|D^\gamma(u-\mathcal{P})|\,dx \leqslant \frac{p-1}{p}\sum_{|\alpha:l|=1}\int\limits_{B^l_\rho(x_0)}|D^\alpha u - D^\alpha\mathcal{P}\,|^p\,\xi^{|\alpha|p}\,dx$$

$$+\frac{(c_3a_0)^p}{p}\sum_{|\alpha:l|=1}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\rho^{-pl^*\sum_{i=1}^n\frac{\alpha_i-\gamma_i}{l_i}}\int\limits_{B^l_\rho(x_0)}|D^\gamma(u-\mathcal{P})|^p\,dx.$$

To estimate $I_2$ we use (A3) and Young's inequality and obtain

$$I_2 \leqslant \sum_{|\alpha:l|=1}\int\limits_{B^l_\rho(x_0)}|a_\alpha(x,D^\alpha\mathcal{P})-a_\alpha(x_0,D^\alpha\mathcal{P})|\,|D^\alpha(u-\mathcal{P})|\,|\xi^{p\,|\alpha|}|\,dx$$

$$\leqslant\sum_{|\alpha:l|=1}k\int\limits_{B^l_\rho(x_0)}[r_l(x-x_0)]^s(1+|D^\alpha\mathcal{P}\,|)\,|D^\alpha(u-\mathcal{P})|\,|\xi^{p\,|\alpha|}|\,dx$$

$$\leqslant\sum_{|\alpha:l|=1}k^{\frac{p}{p-1}}\frac{p-1}{p}\int\limits_{B^l_\rho(x_0)}[r_l(x-x_0)]^{\frac{sp}{p-1}}(1+|D^\alpha\mathcal{P}\,|)^{\frac{p}{p-1}}\,dx$$

$$+\frac{1}{p}\sum_{|\alpha:l|=1}\int\limits_{B^l_\rho(x_0)}|D^\alpha(u-\mathcal{P})|^p\xi^{p\,|\alpha|}\,dx$$

$$\leqslant\sum_{|\alpha:l|=1}c_2k^{\frac{p}{p-1}}\frac{p-1}{p}(1+\mathrm{P})^{\frac{p}{p-1}}\int\limits_{B^l_\rho(x_0)}[r_l(x-x_0)]^{\frac{sp}{p-1}}\,dx$$

$$+\frac{1}{p}\sum_{|\alpha:l|=1}\int\limits_{B^l_\rho(x_0)}D^\alpha(u-\mathcal{P})|^p\xi^{p\,|\alpha|}|\,dx$$

$$\leqslant c_2k^{\frac{p}{p-1}}\frac{p-1}{p}(1+\mathrm{P})^{\frac{p}{p-1}}\rho^{\frac{sp}{p-1}}\omega^l_\rho+\frac{1}{p}\sum_{|\alpha:l|=1}\int\limits_{B^l_\rho(x_0)}|D^\alpha(u-\mathcal{P})|^p\xi^{p\,|\alpha|}\,dx.$$

Finally, we estimate $I_3$ using (A3):

$$I_3\leqslant\left|\sum_{|\alpha:l|=1}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\int\limits_{B^l_\rho(x_0)}\big[a_\alpha(x,D^\alpha\mathcal{P})-a_\alpha(x_0,D^\alpha\mathcal{P})\big]\,D^\gamma(u-\mathcal{P})D^{\alpha-\gamma}\xi^{p\,|\alpha|}dx\right|$$

$$\leqslant k\sum_{|\alpha:l|=1}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\int\limits_{B^l_\rho(x_0)}|r_l(x-x_0)|^s(1+|D^\alpha p|))|D^\gamma(u-\mathcal{P})|\cdot|D^{\alpha-\gamma}\xi^{p|\alpha|}|\,dx$$

$$\leqslant kc_3\sum_{|\alpha:l|=1}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\int\limits_{B^l_\rho(x_0)}|r_l(x-x_0)|^s(1+|D^\alpha\mathcal{P}|)D^\gamma(u-\mathcal{P})|$$

$$\times\,\xi^{p|\alpha|-(|\alpha|-|\gamma|)}\rho^{-l^*\sum_{i=1}^n\frac{\alpha_i-\gamma_i}{l_i}}\,dx$$

$$\leqslant \frac{1}{p} \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \rho^{-pl^* \sum_{i=1}^{n} \frac{\alpha_i-\gamma_i}{l_i}}$$

$$\times \int\limits_{B_\rho^l(x_0)} |D^\gamma(u-\mathcal{P})|^p\,dx + c_2(kc_3)^{\frac{p}{p-1}} \frac{p-1}{p} \rho^{\frac{sp}{p-1}} (1+\mathrm{P})^{\frac{p}{p-1}} \omega_\rho^l.$$

Combining these estimates and applying (A2) to the left hand side of (3) we arrive at

$$\beta \sum_{|\alpha:l|=1} \int\limits_{B_\rho^l(x_0)} |D^\alpha u - D^\alpha \mathcal{P}|^p \xi^{p\|\alpha|}\,dx \leqslant \frac{p-1}{p} \sum_{|\alpha:l|=1} \int\limits_{B_\rho^l(x_0)} |D^\alpha u - D^\alpha \mathcal{P})\xi^{p|\alpha|}\,dx$$

$$+ \frac{(c_3 a_o)^p}{p} \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \rho^{-pl^* \sum_{i=1}^{n} \frac{\alpha_i-\gamma_i}{l_i}} \int\limits_{B_\rho^l(x_0)} |D^\gamma(u-\mathcal{P})|^p\,dx + c_2 k^{\frac{p}{p-1}} \frac{p-1}{p}$$

$$\times (1+\mathrm{P})^{\frac{p}{p-1}} \rho^{\frac{sp}{p-1}} \omega_\rho^l + \frac{1}{p} \sum_{|\alpha:l|=1} \int\limits_{B_\rho^l(x_0)} |D^\alpha(u-\mathcal{P})|^p \xi^{p|\alpha|}\,dx + \frac{1}{p} \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \rho^{-pl^* \sum_{i=1}^{n} \frac{\alpha_i-\gamma_i}{l_i}}$$

$$\times \int\limits_{B_\rho^l(x_0)} |D^\gamma(u-\mathcal{P})|^p\,dx + c_2(kc_3)^{\frac{p}{p-1}} \frac{p}{p-1} \rho^{\frac{s(p-1)}{p}} (1+\mathrm{P})^{\frac{p}{p-1}} \omega_\rho^l.$$

Thus, we have got the desired inequality (2):

$$\int\limits_{B_{\rho/2}^l(x_0)} |D^\alpha u - D^\alpha \mathcal{P}|^p\,dx \leqslant c_1 \Bigg( \sum_{|\alpha:l|=1} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \rho^{-pl^* \sum_{i=1}^{n} \frac{\alpha_i-\gamma_i}{l_i}}$$

$$\times \int\limits_{B_\rho^l(x_0)} |D^\gamma(u-\mathcal{P})|^p\,dx + (1+\mathrm{P})^{\frac{p-1}{p}} \rho^{\frac{sp}{p-1}} \omega_\rho^l \Bigg).$$

## References


1. *Бесов О. В., Ильин В. П., Никольский С. М.* Интегральные представления функций и теоремы вложения.—М.: Наука, 1975.—480 с.
2. *Водопьянов С. К.* Анизотропные пространства дифференцируемых функций и квазиконформные отображения // Тезисы докл. XI Всесоюзной школы по теории операторов в функциональных пространствах. Ч 2.—Челябинск, 1986.—С. 23.
3. *Алборова М. С., Водопьянов С. К.* Устранимые особенности для ограниченных решений квазиэллиптических уравнений / Новосибирск, 1987.—44 с. Деп. в ВИНИТИ 04.02.87, № 804-В87.
4. *Решетняк Ю. Г.* О понятии емкости в теории функций с обобщенными производными // Сиб. мат. журн.—1969.—Т. 10, № 5.—С. 1109–1139.
5. *Duzaar F., Gastel A, Grotowski J. F.* Optimal partial regularity for nonlinear elliptic systems of higher order // J. Math. Sci. Univ. Tokyo.—2001.—V. 8.—P. 463–499.


*Статья поступила 5 апреля 2004 г.*

Алборова Мира Сослановна, к. ф.-м. н.
г. Владикавказ, Институт прикладной математики
и информатики ВНЦ РАН