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ON IDEAL OF COMPACT OPERATORS IN REAL FACTORS

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Dedicated to the memory of
Professor Yuri A. Abramovich

In the present paper the real ideals of relatively compact operators of W^*-algebras are considered. A description (up to isomorphism) of real two-sided ideal of relatively compact operators of the complex W^*-factors is given.

1. Introduction

It is well known that the set of relatively completely continuous operators in a Hilbert space forms a two-sided uniformly closed ideal. Analogous classes of such operators for operator algebras, in particular, for the von Neumann algebras, has been studies in classical papers of M. Sonis, V. Ovchinnikov, H. Halpern and V. Kaftal. Let now $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed *-subalgebra M with identity element $\mathbf{1}$ in $B(H)$ is called a W^*-algebra. Let $P(M)$ be the set of all projections of M, I be the ideal of all operators with the finite range projection relatively to M, $J = \overline{I}$ be the ideal of compact operators relatively to M. It is known [2], that I and J are proper iff M is infinite; and that J is the maximal two-sided ideal of M without infinite projections. The compact operators relative to M were defined by Sonis [6] (in the case of the algebras with Segal measure, i. e. for finite W^*-algebras) as the operators which send bounded sets into relatively compact sets. In the paper [4] it has been introduced and considered an analogous notion of finiteness and compactness in purely infinite W^*-algebras. In the present paper we will introduce and consider the ideal of compact operators relative to a real W^*-algebra. Similarly to the complex case, it has been given a description and classification (up to isomorphism) of the real two-sided ideal of the relatively compact operators.

2. Preliminary Information

A real *-subalgebra R with $\mathbf{1}$ in $B(H)$ is called a *real W^*-algebra* if it is closed in the weak operator topology and $R \cap iR = \{0\}$. A real W^*-algebra R is called a *real factor* if its center $Z(R)$ contains only elements of the form $\{\lambda\mathbf{1}\}$, $\lambda \in \mathbb{R}$. We say that a real W^*-algebra R is of the type $\mathrm{I_{fin}}$, I_∞, II_1, II_∞, or III_λ $(0 \leqslant \lambda \leqslant 1)$ if the enveloping W^*-algebra $U(R) = R + iR$ has the corresponding type in the ordinary classification of W^*-algebras [1].

A linear mapping α with $\alpha(x^*) = \alpha(x)^*$ of the algebra R into itself is called an **-automorphism* if $\alpha(xy) = \alpha(x)\alpha(y)$; an **-antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$; *involutive* if $\alpha^2(x) = \alpha(\alpha(x)) = x$, for all $x, y \in R$.

A *trace* on a (complex or real) W^*-algebra N is an additive and positively homogenus function τ on the set N^+ of positive elements of N with values in $[0, +\infty]$, satisfying the following condition: $\tau(uxu^*) = \tau(x)$, for an arbitrary unitary u and x in N.

The trace τ is said to be *finite* if $\tau(\mathbf{1}) < +\infty$; *semifinite* if given any $x \in N^+$ there is a nonzero $y \in N^+$, $y \leqslant x$ with $\tau(y) < +\infty$.

Let $R \subset B(H)$ be a real W^*-algebra, $M = R + iR$ be the *enveloping* W^**-algebra* for R. Let τ be a semifinite trace on R. Subspace K of H with $K\eta R$, i. e. $P_K \in R$, is called *finite* relative to τ if $\tau(P_K) < \infty$, where P_K projection of H on K; *compact* relative to τ if K is an approximate of the bounded sets from relatively finite subspaces.

Real operator x of H (i. e. $x \in R$) is called *real compact* relative to τ if it is the operator mapping bounded sets into relatively compact sets.

3. Compact Operators in Semifinite Real Factor

Let $I(R)$ be the set of all relatively compact operators of R.

Theorem 1. *Let R be a semifinite real factor. Then $I(R)$ is a unique* (*non zero*) *uniformly closed two-sided ideal of R.*

$\lhd$ Let $I(R)$ be a uniformly closed two-sided ideal in R. By Proposition 3 of [5] it contains all bounded operators with the finite metric range. Because $I(R)$ is uniformly closed, from Proposition 2 of [5] it follows that $\sigma_\infty(R,\tau) \subset I(R)$, where $\sigma_\infty(R,\tau)$ is the set of all relatively completely continuous operators in R. Let us prove that $I(R) \subset \sigma_\infty(R,\tau)$. Let $x \in J$. Because $x^* \in J$ and $\mathrm{Re}(x)$, $\mathrm{Im}(x) \in J$, without the loss of generality we can say that x is Hermitian. Let now $\{e_\lambda\}$ be the family of the spectral projections for the operator x, and let the interval $\triangle = (\alpha, \beta)$ be without zero. Then $P_\triangle = P_\beta - P_\alpha \in I(R)$, and by Proposition 3 of [5] the metric range of $P_\triangle$ is finite. Because $\lambda_k P_{\triangle_k} \to x$ uniformly, from Proposition 2 of [5] is follows that $x \in \sigma_\infty(R,\tau)$. The theorem is proven. $\rhd$

Theorem 2. *Let R be a semifinite real factor, $U = R + iR$ is its enveloping factor. Let $I(U)$ be a unique* (*nonzero*) *uniformly closed two-sided ideal of U. Then*

$$I(U) = I(R) + iI(R).$$

$\lhd$ Since $I(R)$ is a uniformly closed two-sided ideal, then $I(R) + iI(R)$ is also a uniformly closed two-sided ideal. In fact, let $\{c_n = a_n + ib_n\}$ be a Cauchy sequence in $I(R) + iI(R)$, i. e. $\|c_n - c_m\| \to 0$ as $n, m \to \infty$. Then $\|(a_n - a_m) + i(b_n - b_m)\| \to 0$ as $n, m \to \infty$. Using the lemma 1.1.3 (iii) from [1] we have

$$\max\{\|a_n - a_m\|, \|b_n - b_m\|\} \leqslant \|(a_n - a_m) + i(b_n - b_m)\|,$$

therefore, $\|a_n - a_m\| \to 0$ and $\|b_n - b_m\| \to 0$ as $n, m \to \infty$. Thus, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $I(R)$, hence they converge to a and b in $I(R)$ respectively. Thus, $c_n = a_n + ib_n \to a + ib$ in $I(R) + iI(R)$, which is uniformly closed. Now, if $x = a + ib \in U$, $y = c + id \in I(R) + iI(R)$, then $xy = (ac - bd) + i(ad + bc) \in I(R) + iI(R)$. Similarly, $yx \in I(R) + iI(R)$. Therefore, $I(R) + iI(R)$ is a uniformly closed two-sided ideal of U and we have proved that $I(R) + iI(R) \subset I(U)$.

Now, since for $x \in I(U)$ we have $x = a + ib$, $a, b \in R$, let $I(U) = A + iB$, for some $A, B \subset R$. But, for $a \in A, b \in B$ we have ab, $ba \in I(U)$. Therefore, ab, $ba \in A$, whence $A = B$, i. e. $I(U) = A + iA$. Then $I(R) \subset A$ as $A, I(R) \subset R$. Let $\{a_n\}$ be a Cauchy sequence in $A \subset I(U)$. Since $I(U)$ is uniformly closed, $\{a_n\}$ converges to $a \in I(U)$. But, R is also uniformly closed, therefore, $a \in R$. Then $a \in A$. Now, let $x \in A$, $y \in R$. Since $I(U)$ is a two-sided ideal of U, xy, $yx \in I(U)$, i. e. xy, $yx \in A$ as $xy, yx \in R$. Therefore, A is a uniformly closed two-sided ideal of R with $I(R) \subset A$. Then by Theorem 1 we have $A = I(R)$. This completes the proof. $\triangleright$

4. Real ideals of compact operators of factors of type III_λ ($\lambda \neq 1$)

Let us recall [3] the notion of the *crossed product* of a W^*-algebra by a locally compact topological group by its *-automorphism. Let N be a (complex or real) W^*-algebra in $B(H)$, $\gamma : G \to \mathrm{Aut}(M)$ be a group homomorphism such that each map $g \to \gamma_g$ is strongly continuous. Let $L_2(G, H)$ be the Hilbert space of all H-valued square integrable functions on G. We consider an *-algebra $U \subset B(L_2(G, H))$ generated by operators of the form $\pi_\gamma(a)$ ($a \in M$) and $u(g)$ ($g \in G$), where

$$(\pi_\gamma(a)\xi)(h) = \gamma_h^{-1}(a)\xi(h), \quad (u(g)\xi)(h) = \xi(g^{-1}h), \quad \xi = \xi(h) \in L_2(G, H),\ g, h \in G.$$

The algebra U is called *crossed product* of M by G, and denoted by $W^*(M, G)$ (or $M \times_\gamma G$). Moreover, there exists a canonical embedding $\pi_\gamma : M \to \pi_\gamma(M) \subset U$. Each element $x \in U$ has the form: $x = \sum_{g \in G} \pi_\gamma(x(g))u(g)$, where $x(\cdot)$ is a M-valued function on G.

Let θ be an *-automorphism of N. For the action $\{\theta^n\}$ of the group $\mathbb{Z}$ on N we denoted by $W^*(\theta, N)$ (or $N \times_\theta \mathbb{Z}$) the crossed product of N by θ.

Now, let R be a factor of type III_λ ($\lambda \neq 1$). Then ([7]), either there exist a real factor F of type II_∞ and an automorphism θ of F such that R is isomorphic to the crossed product $F \times_\theta \mathbb{Z}$ or there exist a complex factor N of a type II_∞ and an antiautomorphism σ of N such that R is isomorphic to $((N \oplus N^{\mathrm{op}}) \times_\sigma \mathbb{Z}, \beta)$, where N^{op} is the opposite W^*-algebra for N, $\beta(x, y) = (y, x)$, for all $x, y \in N$.

In the first case let $I(R)$ be the norm closure of $\mathrm{span}\{x \in R^+ : E(x) \in I(F)\}$, where $E : R \to F$ is a unique faithful normal conditional expectation. In the second case let $I(R)$ be the norm closure of $\mathrm{span}\{x \in R^+ : E(x) \in I(N \oplus N^{\mathrm{op}})\}$, where E is a unique faithful normal conditional expectation from M to $N \oplus N^{\mathrm{op}}$.

If we now apply Theorem 1 and use the scheme of proof of Theorem 6.2 from [4] then we prove a real analogue of the theorem of Halpern–Kaftal:

Theorem 3. *In each case $I(R)$ is a unique (non zero) uniformly closed two-sided ideal of R.*

Similarly to Theorem 2 we can prove the following:

Theorem 4. *Let M be an injective factor of type III_λ, $0 < \lambda < 1$, R and Q be non isomorphic real factors with the some enveloping factor M, i. e. $R + iR = Q + iQ = M$. If $I(M)$ is a (nonzero) uniformly closed two-sided ideal of M then*

$$I(M) = I(R) + iI(R), \qquad I(M) = I(Q) + iI(Q),$$

where $I(R)$ and $I(Q)$ are non isomorphic unique uniformly closed two-sided ideals of R and Q respectively.

5. Main Result

Let M be a factor, α be an involutive *-antiautomorphism of M. Then ([1]) the set $R = \{x \in M : \alpha(x) = x^*\}$ is a real factor and the enveloping W^*-algebra $U(R)$ of R coincides with M, and conversely, given an arbitrary real factor R there exists a unique involutive

*-antiautomorphism α_R of the W^*-algebra $U(R)$ such that $R = \{x \in U(R) : \alpha(x) = x^*\}$. Moreover, two real factors R_1 and R_2 are *-isomorphic if and only if the enveloping factors $U(R_1)$ and $U(R_1)$ are *-isomorphic and the involutive *-antiautomorphisms α_{R_1} and α_{R_2} are conjugate, i. e. $\alpha_{R_1} = \theta \cdot \alpha_{R_2} \cdot \theta^{-1}$, for some *-automorphism θ.

It is known ([1]) that

in a factor of type I_n, n even, there exists a unique conjugacy class of involutive *-antiautomorphisms;

in a factor of type I_n, n odd or $n = \infty$, there exist exactly two conjugacy classes of involutive *-antiautomorphisms;

in an injective factor of type II_1, II_∞, or III_1 there exists a unique conjugacy class of involutive *-antiautomorphisms;

in an injective factor of type III_λ, $0 < \lambda < 1$, there exist exactly two conjugacy classes of involutive *-antiautomorphisms.

Hence, and from Theorems 1 and 3 we obtain the following result:

Theorem 5. *Let M be a factor. Then the following assertions are true:*

(1) *if M has type I_n, n even, then in M there exist two* (*non zero*) *uniformly closed two-sided real ideals up to isomorphisms;*

(2) *if M has type I_n, n odd or $n = \infty$, then in M there exist three* (*non zero*) *uniformly closed two-sided real ideals up to isomorphisms;*

(3) *if M is an injective factor of type II_1 or type II_∞ then in M there exist two* (*non zero*) *uniformly closed two-sided real ideals up to isomorphisms;*

(4) *if M is an injective factor of type III_λ* $(0 < \lambda < 1)$ *then in M there exist three* (*non zero*) *uniformly closed two-sided real ideals up to isomorphisms.*

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