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ONE PROPERTY OF THE WEAK COVERGENCE OF OPERATORS ITERATIONS IN VON NEUMANN ALGEBRAS

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Conditions are given for *-weak convergence of iterations for an ultraweak continuous fuctional in von Neumann algebra to imply norm convergence.

Let M be a von Neumann algebra [5], acting on a separable Hilbert space H. Let T be a contraction from M_* to M_*, so that $TM_{*+} \subset M_{*+}$. On the pre-conjugate to M space M_* there are two topologies selected: the weak, or the $\sigma(M_*, M)$ topology, and the strong topology of the convergence in the norm of the space M_*.

Let now $T = \alpha_*$, where α be an automorphism of the algebra M. We will say that T in M_* is *mixing*, if for all $x \in M_*^0$ and $A \in M$, the following condition is valid:

$$\lim_{n\to\infty} \langle T^n x, A\rangle = 0,$$

where

$$M_*^0 = \{y \in M_* : \, y(\mathbf{1}) = 0\}.$$

We will say that a positive contraction T in M_* is *completely mixing*, if for all $x \in M_*^o$ the following condition is valid:

$$\lim_{n\to\infty} \|T^n x\| = 0.$$

The following theorem is valid:

Theorem. *Let T be a pre-conjugate operator to an automorphism α of a von Neumann algebra M for which there is no invariant normal state. Then, for $x \in M_*$, the weak convergence of $T^n x$ implies the strong convergence of $T^n x$. In particular, if T is mixing, then T is completely mixing.*

$\lhd$ Let us denote by $|T^n x|$ the sum

$$(T^n x)_+ + (T^n x)_-, \text{ where } T^n x = (T^n x)_+ - (T^n x)_-$$

is the Hahn decomposition of the functional $T^n x$ [4]. The sequence $\{|T^n x|\}_{n=1}^{\infty}$ is $\sigma(M_*, M)$ pre-compact [4] and, therefore, the convex envelope of the set $\{|T^n x|\}_{n=1}^{\infty}$ is pre-compact as well. The sequence $\{A^n\,|x|\}_{n=1}^{\infty}$ is also pre-compact because it belongs to the convex envelope of the set $\{|T^n x|\}_{n=1}^{\infty}$.

Because T is pre-conjugate to an automorphism, then $|T^n x| = T^n\,|x|$. In fact, the support of $T(T^n x)_+$ is orthogonal to the support of $T(T^n x)_-$, $T(T^n x)_+ - T(T^n x)_- = T(T^n x) = T^{n+1}x$, and from the uniqueness of the Hahn decomposition [4] it follows that $|T^n x| = T^n\,|x|$.

Let $\overline{x}$ be $\sigma(M_*, M)$-limit point of the set $\{A^n|x|\}_{n=1}^{\infty}$. Then the functional $\overline{x}$ will be T-invariant. In fact,

$$T\overline{x} = \lim_{n_\gamma \to \infty} \sum_{k=1}^{n_\gamma} \left\langle T^k x, y \right\rangle$$

$$= \lim_{n_\gamma \to \infty} \left[n_\gamma^{-1} \cdot \sum_{k=0}^{n_\gamma - 1} \left\langle T^k x, y \right\rangle - n_\gamma^{-1} \cdot \langle x, y \rangle + n_\gamma^{-1} \cdot \langle T^{n_\gamma} x, y \rangle \right] = \overline{x}.$$

It is easy to see that $\overline{x} \geqslant 0$ and, therefore, from the conditions of the theorem it follows that $\overline{x} = 0$. Now we know that the only weakly limit point of the set $\{A^n|x|\}_{n=1}^{\infty}$ is the point $\overline{x} = 0$. Therefore

$$0 = \lim_{n\to\infty} \|A^n|x|\| = \lim_{n\to\infty} (A^n|x|)(\mathbf{1}) = \lim_{n\to\infty} (T^n|x|)(\mathbf{1}) = \lim_{n\to\infty} \|T^n|x|\|,$$

because $(T^n|x|)(\mathbf{1}) = (T^m|x|)(\mathbf{1})$ for all $n, m \in \mathbb{N}$. The theorem is proven. ▷

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