Владикавказский математический журнал
Апрель–июнь, 2003, Том 5, Выпуск 2

УДК **517.98**

ON NEVEU DECOMPOSITION AND ERGODIC TYPE THEOREMS FOR SEMI-FINITE VON NEUMANN ALGEBRAS

G. Ya. Grabarnik, A. A. Katz

Some ergodic type theorems for automorphisms of semi-finite von Neumann algebras are considered. Neveu decomposition is employed in order to prove stochastical convergence. This work is a generalization of authors results from [5] to the case of semi-finite von Neumann algebras.

1. Introduction and Notations

This work is devoted to some results concerning ergodic type theorems for semi-finite von Neumann algebras. The first results in this field were obtained by Sinai and Anshelevich [17] and Lance [14]. Developments of the subject are reflected in the monographs of Jajte [7] and Krengel [13].

The notion of a weakly wandering set (in commutative context) was introduced by Hajian and Kakutani [9] in order to establish conditions which are equivalent to the existence of finite invariant measures. The non-commutative case was first considered by Jajte [7], and later, for the case of finite von Neumann algebras, by Grabarnik and Katz [5] and Katz [2].

In section 2 we consider Neveu decomposition which gives a characterization of the existence of the invariant measures in terms of a weakly wandering operator.

Section 3 is devoted to a presentation of the Krengel's Stochastic Ergodic Theorem for the actions of an automorphism on semi-finite von Neumann algebra [4].

In section 4 we consider a multiparametric version of the Stochastic Ergodic Theorem [5, 2].

Remark 1. The Multiparametric Superadditive Stochastic Ergodic Theorem will be separately presented in the forthcoming paper [6].

We use the following notations: everywhere below M is assumed to be a *σ-finite von Neumann algebra* with *semi-finite faithful normal trace* τ (*semi-finite algebra*), M_* is a *predual* of M, and M^* is the *Banach dual space* to M.

1 denotes the unit of M. For $\rho \in M_*$, the support of ρ will be denoted by $S(\rho)$.

Let α be an automorphism of algebra M, and let α_* be an operator acting in M_*, to which α is conjugated.

By A^n (A^n_*) we denote the *Česaro average* of α (α_*).

2. Neveu Decomposition and the Weakly Wandering Operator

Definition 1. An operator $h \in M_+^1$ is said to be a *weakly wandering operator*, if

$$\|A^n h\| \to 0 \text{ when } n \to \infty.$$

The following theorem is valid:

Theorem 1. *Let M, α and τ be as defined above. The following conditions are equivalent:*

(i) *There exists an α_*-invariant normal state ρ on M with support $S(\rho) = E$, $\tau(E) < \infty$, such that the support of every α_*-invariant normal state μ is less then or equal to E; in symbols*

$$S(\mu) \leqslant E.$$

(ii) *E is the maximal projection such that for every projection $P \leqslant E$, $P \in M$,*

$$\inf_n \tau(\alpha^n P) > 0.$$

(iii) *There exists a weakly wandering operator $h_0 \in M_+$ with support*

$$S(h_0) = \mathbf{1} - E$$

such that the support of every weakly wandering operator is less then or equal to $\mathbf{1} - E$.

It follows immediately from the theorem, that:

Corollary 1 (Neveu Decomposition). *Let α be an automorphism of von Neumann algebra M with α-invariant semi-finite normal trace τ. Then there exist projections E_1 and E_2,*

$$E_1 + E_2 = \mathbf{1} \tag{1}$$

such that:

(i) *There exists an α_*-invariant normal state ρ with support $S(\rho) = E_1$,*

(ii) *There exists a weakly wandering operator $h \in M$ with $S(h) = E_2$.*

3. Stochastic Ergodic Theorem

The space M_* of normal functionals on von Neumann algebra M with α-invariant semi-finite normal trace τ is naturally identified with the space $L_1(M, \tau)$ of locally measurable operators, each affiliated to M and integrable with modulus. Action α' is defined as an operator conjugated to α with respect to duality:

$$\tau(\alpha' X \cdot y) = \tau(X \cdot \alpha y) \qquad (X \in L_1(M, \tau),\ y \in M).$$

Definition 2. A sequence $\{X_n\}$ of measurable operators is said to *converge stochastically* to operator X_0, if for every $\varepsilon > 0$,

$$\tau(\{|X_n - X_0| > \varepsilon\}) \to 0 \text{ when } n \to \infty.$$

Theorem 2 (Stochastic Ergodic Theorem). *Let α be an automorphism of von Neumann algebra M with α-invariant semi-finite normal trace τ. Then for $X \in L_1(M, \tau)$, the Česaro averages $A'^n X$ converge stochastically to $\widetilde{X} \in L_1(M, \tau)$. The limit $\widetilde{X}$ is α'-invariant and*

$$E_2 \widetilde{X} E_2 = 0 \tag{2}$$

(*where* E_2 *is a projection from Neveu decomposition* (1)).

To prove the Theorem (2), we need the following variant of non-commutative Individual Ergodic Theorem:

Theorem 3 (Individual Ergodic Theorem). *Let* M *be a von Neumann algebra with* α-*invariant semi-finite normal trace* τ, $\tau(\mathbf{1}) = 1$. *Let* α *be an automorphism of* M, ρ *be a normal faithful state on* M,

$$\rho \circ \alpha = \rho.$$

Then for every $\mu \in M_*$ *there exists an* α_*-*invariant normal functional* $\overline{\mu}$ *such that for every* $\varepsilon > 0$ *there exists a projection* $E \in M$ *with* $\tau(\mathbf{1} - E) < \varepsilon$ *and*

$$\sup_{\substack{x \in EM_+E \\ x \neq 0}} |(A_*^n \mu - \overline{\mu})(x)/\tau(x)| \to 0 \text{ when } n \to \infty.$$

Let $(H_\rho, \pi_\rho, \mathfrak{M})$ be a representation of algebra M constructed by a faithful normal state ρ. Then $\mathfrak{M}$ is a von Neumann algebra isomorphic to M. Let $\widehat{\alpha}$ be an image of automorphism α and $\widehat{\alpha}'$ be an associated transformation on $\mathfrak{M}'$:

$$(\widehat{\alpha} X \cdot Y \Omega, \Omega) = (X \cdot \widehat{\alpha}' Y \Omega, \Omega), \; X \in \mathfrak{M}, \; Y \in \mathfrak{M},$$

where Ω is a bicyclic vector with $(X\Omega, \Omega) = \rho(X)$, $X \in \mathfrak{M}$.

The following theorem is a variant of the Maximal Hopf Lemma.

Theorem 4 (Maximal Hopf Lemma). *Let* $\mu \in \mathfrak{M}$ *be a Hermitian functional and* $\varepsilon > 0$ *be such that* $\|\mu\| \cdot \varepsilon^{-1} < 1$. *Then, for a fixed* N *there exists a projection* $E \in \mathfrak{M}$, $\rho(E^\perp) < \|\mu\| \cdot \varepsilon^{-1}$ *such that*

$$\sup_{\substack{x \in E\mathfrak{M}_+E \\ x \neq 0}} |(A^n(\widehat{\alpha}_*, \mu)(x)/\rho(x)| < \varepsilon, n = 1, 2, \ldots, N.$$

4. Multiparametric Stochastic Ergodic Theorem (the case of d-commuting automorphisms)

Now we will consider the case of d-commuting automorphisms. Let $d \geqslant 1$ be a natural number and $\mathbb{V} = \{0, 1, 2, \ldots\}^d$ be an additive semigroup of d-dimentional vectors with natural coordinates. For $u = (u_i)$, $v = (v_i) \in \mathbb{V}$, relation $u \geqslant v$ ($u > v$) means $u_i \geqslant v_i$ ($u_i > v_i$) for $i = 1, \ldots, d$. By $[u, v[$ we denote the set $\{w \in \mathbb{V} : u \leqslant w < v\}$. For the finite set B let $card(B)$ or $|B|$ means the number of elements of B. For $n = (n_1, \ldots, n_d) \in \mathbb{V}$ let

$$\pi(n) = \prod_{v=1}^{d} n_v = |[0, n[| \, .$$

For $n \in \mathbb{V}$ and operators $\beta_1, \beta_2, \ldots, \beta_d$,

$$\beta_n = \beta_1^{n_1} \beta_2^{n_2} \ldots \beta_d^{n_d}; \;\; S_n = \sum_{u \in [o, n[} \beta_u; \;\; A_n = \pi(n)^{-1} S_n;$$

expression $n \to \infty$ means that n_v tends to infinity independently for $v = 1, 2, \dots, d$. Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be automorphisms of algebra M.

DEFINITION 3. An operator $h \in M_+^1$ is called a *weakly wandering* if

$$\|A^n h\|_\infty \to 0 \text{ when } n \to \infty.$$

DEFINITION 4. A multisequence $\{X_n\}_{n \in \mathbb{V}}$ of measurable operators affiliated with M is said to *converge stochastically* to operator X_0, if for every $\varepsilon > 0$,

$$\tau(\{|X_n - X_0| > 0\}) \to 0$$

holds when the multiindex $n \to \infty$.

The following theorem is valid:

Theorem 5. *Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be commuting automorphisms on von Neumann algebra M with faithful normal semi-finite trace τ. The following conditions are equivalent:*

(i) *There exists an $\alpha_{*,i}$-invariant normal state ρ on M with support E such that the support of every normal state does not exceed E $(i = 1, 2, \dots, d)$.*

(ii) *There exists a weakly wandering operator $h_0 \in M_+$ with support $\mathbf{1} - E$ such that the support of every weakly wandering operator does not exceed $\mathbf{1} - E$.*

Moreover,

$$E = \bigwedge_{i=1}^{d} E_i; \quad \mathbf{1} - E = \bigvee_{i=1}^{d} (\mathbf{1} - E_i),$$

where E_i is the «maximal» support of the invariant normal states of the automorphism α_i, $i = 1, 2, \dots, d$. The following Stochastic Multiparametric Ergodic Theorem is valid:

Theorem 6 (Stochastic Multiparametric Ergodic Theorem). *Let α_i be automorphisms of semi-finite von Neumann algebra M with semi-finite weight τ, $i = 1, 2, \dots, d$. Then for $X \in L_1(M, \tau)$, the averages $A_{*n}X$ converge stochastically to $\overline{X} \in L_1(M, \tau)$, where $n = (n_1, n_2, \dots, n_d)$. The limit $\overline{X}$ is α_{*i}-invariant and*

$$\widetilde{E}\overline{X}\widetilde{E} = 0,$$

where

$$\widetilde{E} = \bigvee_{i=1}^{d} (\mathbf{1} - E_i),$$

and E_i are projections that were constructed by Theorem 5.

The proof of the above theorem is based on the following:

Theorem 7. *Let M be a semi-finite von Neumann algebra, α_i be automorphisms of algebra M, $i = 1, 2, \dots, d$; τ be a normal semi-finite α_i-invariant trace and ρ be a faithful normal α_i-invariant $(i = 1, 2, \dots, d)$ state on M. Then for every $\mu \in M_*$ there exists an α_i-invariant functional $\overline{\mu}$ such that for every $\varepsilon > 0$ there exists a projection*

$$E \in M, \quad \tau(E^\perp) < \varepsilon;$$

moreover, $\|A_*^n - \overline{\mu}\|_1 \to 0$ and

$$\sup_{\substack{x \in EM_+E \\ x \neq 0}} |(A_*^n \mu - \overline{\mu})(x)/\tau(x)| \to 0 \text{ when the multiindex } n \to \infty.$$

Let P_i *be a map:*

$$\nu \to \lim_{k\to\infty} A_{*i}^k \nu.$$

The map P_i *is a projection on the set of* α_{*i}*-stationary points and*

$$\overline{\mu} = P_d \cdot P_{d-1} \cdot \dots \cdot P_1 \mu.$$

References

1. *Akcoglu M., Sucheston L.* A stochastic ergodic theorem for superadditive precesses // Ergodic Theory and Dynamical Systems.—1983.—V. 3.—P. 335–344.
2. *Cunze J. P., Dang-Nqoc N.* Ergodic theorems for non-commutative dynamical systems // Inventiones Mathematicae.—1978.—V. 46.—P. 1–15.
3. *Goldstein M. S., Grabarnik G. Ya.* Almost sure convergence theorems in von Neumann algebras // Israel J. Math.—1991.—V. 76.—P. 161–182.
4. *Dixmier J.* Les algebres d'operateurs dans l'espace hilbertien (algebres de von Neumann).—Paris: Gauthier-Villar, 1969.—367 s.
5. *Grabarnik G. Ya., Katz A. A.* Ergodic type theorems for finite von Neumann algebras // Israel J. Math.—1995.—V. 90.—P. 403–422.
6. *Grabarnik G. Ya., Katz A. A.* On multiparametric superadditive stochastic ergodic theorem for semi-finite von Neumann algebras / to appear.
7. *Jajte R.* Strong limit theorem in non-commutative probability // Lecture Notes in Math.—V. 1110.—Berlin: Spring-Verlag, 1985.—162 p.
8. *Jajte R.* On the existence of invariant states in W^*-algebras // Bull. Polish Acad. Sci.—1986.—V. 34.—P. 617–624.
9. *Hajian A., Kakutani S.* Weakly wandering sets and invariant measures // Trans. Amer. Math. Soc.—1964.—V. 110.—P. 131–151.
10. *Katz A. A.* Ergodic type theorems in von Neumann algebras.—Ph. D. Thesis.—Pretoria: University of South Africa, 2001.—84 p.
11. *Kingman J. F. C.* Subadditive ergodic theory // Annals of Probability.—1973.—V. 1.—P. 883–909.
12. *Kovacs I., Szücs J.* Ergodic type theorem in von Neumann algebras // Acta Scientiarum Mathematicarum (Szeged).—1966.—V. 27.—P. 233–246.
13. *Krengel U.* Ergodic Theorems de Greuter.—Berlin, 1985.
14. *Lance E. C.* Ergodic theorems for convex sets and operator algebras // Inventiones Mathematicae.—1976.—V. 37.—P. 201–214.
15. *Petz D.* Ergodic theorems in von Neumann algebras // Acta Scientiarum Mathematicarum (Szeged).—1983.—V. 46.—P. 329–343.
16. *Segal I. E.* A non-commutative extension of abstract integration // Archiv der Math.—1953.—V. 57.—P. 401–457.
17. *Синай Я. Г., Аншелевич В. В.* Некоторые проблемы некоммутативной эргодической теории // Успехи мат. наук.—1976.—Т. 32.—С. 157–174.
18. *Takesaki M.* Theory of Operator Algebras. I.—Berlin: Springer-Verlag, 1979.—vii+415 p.
19. *Yeadon F. J.* Convergence of measurable operators // Math. Proc. Cambridge Philos. Soc.—1973.—V. 74.—P. 257–269.
20. *Yeadon F. J.* Ergodic theorems for semi-finite von Neumann algebras, I // J. London Math. Soc.—1977.—V. 16.—P. 326–332.
21. *Yeadon F. J.* Ergodic theorems for semi-finite von Neumann algebras, II // Math. Proc. Cambridge Philos. Soc.—1980.—V. 88.—P. 135–147.

Статья поступила 11 апреля, 2003

Dr. Genady Ya. Grabarnik,
IBM T.J. Watson Research Center, 19 Skyline Dr., Hawthorne, NY 10510, USA.
E-mail: genady@us.ibm.com

Alexander A. Katz, Ph.D.
Department of Mathematics & CS, St. John's University, 300 Howard Ave.,
Staten Island, NY 10301, USA.
E-mail: katza@stjohns.edu