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THE STIELTJES MOMENT PROBLEM IN VECTOR LATTICES

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1. Introduction

The following problem for the first time was posed in the famous memoir by Thomas Stieltjes [1] devoted to continued fractions: given a real sequence $(s_k)_{k=0}^{\infty}$, find a nondecreasing function σ on a positive half-line $\mathbb{R}_+$ such that

$$\int_0^\infty t^k \, d\sigma(t) = s_k \quad (k := 0, 1, \dots).$$

He had called it the moment problem having in mind an obvious mechanical interpretation. Since then the moment problem has been developed in different directions as one of the most attractive and important areas of modern analysis. The extended moment problem is posed analogously on the whole real line and called the Hamburger moment problem, while the moment problem posed on a line segment is called the *Hausdorff moment problem*.

The whole history of the moment problem is quite well known and there is no need in recalling. It should be only noted that several authors explored vector-valued statements of the moment problem, see [2–5]. In the present short talk we will briefly outline the Stieltjes moment problem in vector lattices.

2. Two examples

First of all we consider two examples justifying the statement of the moment problem in vector lattices

2.1. The first example concerns a stochastic setting of the moment problem. Suppose that the moment sequence depends on a measurable parameter. More precisely, let (Ω, Σ, ν) be a measure space and $s_n : \Omega \to \mathbb{R}$ be a measurable function for each $n \in \mathbb{N}$. Assume that the sequence is positive in the Stieltjes sense, i.e. the inequalities

$$\sum_{k,l=0}^{n} \sigma_k \sigma_l s_{k+l}(\omega) \geqslant 0, \qquad \sum_{k,l=0}^{n} \sigma_k \sigma_l s_{k+l+1}(\omega) \geqslant 0 \quad ((\sigma_k)_{k=0}^{\infty} \subset \mathbb{R}^n;\ n := 0, 1, \dots)$$

hold almost everywhere in Ω. Then, as is well-known, for almost every $\omega \in \Omega$ the Stieltjes moment problem has a solution μ_ω. Now a new question arises: is the function $\mu_{(\cdot)}(A) : \omega \mapsto \mu_\omega(A)$ measurable for each $A \in \Sigma$ or not? This new problem is actually the old one but set in the space of measurable functions $L^0(\nu) := L^0(\Omega, \Sigma, \nu)$. Indeed, if we define a vector measure by assigning $A \mapsto \mu_{(\cdot)}(A)$ $(A \in \Sigma)$ then the problem is to find a vector measure $\mu : \Sigma \to L^0(\nu)$ such that

$$s_k(\cdot) = \int_0^\infty t^k \, dm(t) \quad (k := 0, 1, \dots).$$

2.2. The second example concerns the spectral resolution of a self adjoint operator. Consider a positive self-adjoint operator A in a Hilbert space and suppose that $(e_\lambda)_{\lambda \in \mathbb{R}}$ is its spectral resolution. Then

$$I = \int_0^\infty de_\lambda, \quad A = \int_0^\infty \lambda \, de_\lambda.$$

Denote by $\mathbb{B}$ a Boolean algebra of orthogonal projections in Hilbert space under consideration and $(\mathbb{B})$ the space of all self-adjoint (not necessarily bounded) operators whose spectral resolutions take values in $\mathbb{B}$. Now we can set the following natural question:

Given a sequence $(A_k)_{k \in \mathbb{N}}$ of pairwise commuting positive self-adjoint operators in $(\mathbb{B})$ with $A_0 = I$, find a spectral resolution or a spectral measure $\mu : \mathcal{B}(\mathbb{R}_+) \to \mathbb{B}$ such that

$$A^k = \int_0^\infty \lambda^k \, d\mu(\lambda) \quad (k := 0, 1, \dots).$$

3. Vector lattices

We recall the basic notion from the theory of vector lattice (= Riesz spaces), see [6, 7].

3.1. An *ordered vector space* over is a pair $(E, \leqslant)$, where E is a real vector space and $\leqslant$ is an order relation in E with the following conditions being fulfilled:

(1) $x \leqslant y \& u \leqslant v \to x + u \leqslant y + v$ $(x, y, v, u \in E)$;

(2) $x \leqslant y \to \lambda x \leqslant \lambda y$ $(x, y \in E;\ 0 \leqslant \lambda \in \mathbb{R}$.

Thus, inequalities in an ordered vector spaces can be summed and multiplied by positive reals.

3.2. *Vector lattice* is an ordered vector space which is a lattice. Thus, in any vector lattice E there exist least upper bound $\sup\{x_1, \dots, x_n\} := x_1 \vee \dots \vee x_n$ and greatest lower bound $\inf\{x_1, \dots, x_n\} := x_1 \wedge \dots \wedge x_n$ for an arbitrary finite subset $\{x_1, \dots, x_n\} \subset E$. In particular, every element $x \in E$ has the *positive part* $x^+ := x \vee \mathbf{0}$, the *negative part* $x^- := (-x)^+ := -x \wedge \mathbf{0}$, and the *module* $\mid x \mid := x \vee (-x)$.

3.3. *Disjointness* $\perp$ in a vector lattice E is introduced by

$$\perp := \{(x, y) \in E \times E : \mid x \mid \wedge \mid y \mid = \mathbf{0}\}.$$

A *band* in E is a set of the form

$$M^\perp := \{x \in E : (\forall y \in M)\, x \perp y\},$$

where $M \subset E$. The set of all bands ordered by inclusion is a complete Boolean algebra $\mathfrak{B}(E)$ with the following Boolean operations:

$$L \wedge K := L \cap K,\ L \vee K = (L \cup K)^{\perp\perp},\ L^* := L^{\perp} \quad (L, K \in \mathfrak{B}(E)).$$

The Boolean algebra $\mathfrak{B}(E)$ is called the *basis* of E.

3.4. An element $\mathbf{1} \in E$ is said to be (weak) *order unit* if $\{\mathbf{1}\}^{\perp\perp} = E$, i.e. if there is no nonzero element in E disjoint to $\mathbf{1}$. A positive element $e \in E$ is called a *fragment* or a *component* of the unit if $e \wedge (\mathbf{1} - e) = \mathbf{0}$. The set of all fragments of $\mathbf{1}$, denoted by $\mathbb{B}(\mathbf{1})$, is a Boolean algebra with lattice operations being induced from E. Moreover, $e^* := \mathbf{1} - e$ is the Boolean complement of e.

3.5. A vector lattice E is said to be Dedekind σ-complete if each order bounded countable set in E has supremum and infimum. In this case $\mathbb{B}(\mathbf{1})$, is a σ-complete Boolean algebra. A Dedekind σ-complete vector lattice E can be represented as a direct sum $\{e\}^{\perp} \oplus \{e\}^{\perp\perp}$ for every $e \in E$. The corresponding projection P_e onto the band $\{e\}^{\perp\perp}$ (parallel to $\{e\}^{\perp}$) is called a *band projection* and can be calculated by

$$P_e x = \sup\{x \wedge (ne) : n \in \mathbb{N}\} \quad (x \in E^+).$$

4. The Freudenthal spectral theorem

The Freudenthal spectral theorem is one of the most powerful tools in the theory of vector lattices, and can be interpreted as one of the first solutions to the moment problem in vector lattices. In the next two sections E is a Dedekind σ-complete vector lattice with a weak order unit $\mathbf{1}$. The Boolean algebra of all components of $\mathbf{1}$ will be denoted by $\mathbb{B} = \mathbb{B}(\mathbf{1})$.

4.1. A resolution of unity in $\mathbb{B}$ is a mapping $e : \mathbb{R} \to \mathbb{B}$ such that

(1) $e(\lambda) \leqslant e(\mu)$ for $\lambda \leqslant \mu$;

(2) $\bigvee_{\lambda\in\mathbb{R}} e(\lambda) = \mathbf{1},\ \bigwedge_{\mu\in\mathbb{R}} e(\mu) = \not\vdash$;

(3) $\bigvee_{\mu<\lambda} e(\mu) = e(\lambda)\ (\lambda \in \mathbb{R})$.

To each $x \in E$ we assign a resolution of identity $(e^x_\lambda)_{\lambda\in\mathbb{R}}$ in $\mathbb{B}$ by setting $e^x_\lambda := P_{c(\lambda)}\mathbf{1}$, where $c(\lambda) = (\lambda\mathbf{1} - x)^+$. This resolution of identity is called the spectral function of x.

4.2. Now, define the Stieltjes integral with respect to an arbitrary resolution of identity $e : \mathbb{R} \to \mathbb{B}$. Let $f : \mathbb{R} \to \mathbb{R}$ be an uniformly continuous function. Take a partition of real axis $\Lambda := (\lambda_k)_{k\in\mathbb{Z}}$,

$$-\infty \leftarrow \lambda_n < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_n \to +\infty$$

and compose the integral sum

$$\sum_{-\infty}^{+\infty} f(t_n)(e(\lambda_{n+1}) - e(\lambda_n)),$$

where $\lambda_n < t_n < \lambda_{n+1}$. It is clear that there exists an order limit for integral sums as partitions are refined. This limit is called the Stieltjes integral of f with respect to a resolution of identity $e(\cdot)$ and denoted by

$$\int_{\mathbb{R}} f(\lambda)\, de^x_\lambda := \int_{-\infty}^{+\infty} f(\lambda)\, de^x_\lambda := \underset{\delta(\Lambda)\to 0}{o\text{-lim}} \sum_{-\infty}^{+\infty} f(t_n)(e(\lambda_{n+1}) - e(\lambda_n)).$$

Soundness of the above definitions can be easily verified.

The Freudenthal spectral theorem (1936). *For every $x \in E$ the integral representation holds:*

$$x = \int_{-\infty}^{\infty} \lambda \, de_\lambda^x.$$

4.3. A spectral measure is a σ-continuous Boolean homomorphism μ from $\mathcal{B}(\mathbb{R})$ to (E); here σ-continuity means that for any sequence of pairwise disjoint elements $(A_n) \subset \mathcal{B}(\mathbb{R})$ we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigvee_{k=1}^{\infty} \mu(A_k).$$

Theorem (J .D. M. Wright [8]). *Each spectral resolution $(e_\lambda^x)_{\lambda \in \mathbb{R}}$ has a unique extension to a spectral measure, i.e. there exists a unique spectral measure $\mu_x : \mathcal{B}(\mathbb{R}) \to (E)$ such that $e_\lambda^x = \mu_x(-\infty, \lambda)$ $(\lambda \in \mathbb{R})$. Moreover,*

$$\int_{\mathbb{R}} f(\lambda) \, de_\lambda^x = \int_{\mathbb{R}} f(\lambda) \, d\mu(\lambda).$$

4.4. In E one can introduce a unique partial multiplication so that $\mathbf{1}$ is a neutral element. It can be easily seen that if $x \geqslant 0$ and all x^n exist then the spectral measure μ_x is a solution to the Stieltjes moment problem in E for the sequence $\mathbf{1}, x, x^2, \dots, x^n, \dots$, i.e.

$$x^k = \int_0^{\infty} \lambda^k \, de_\lambda^x = \int_0^{\infty} \lambda^k \, d\mu(\lambda) \quad (k := 0, 1, \dots).$$

5. The main results

5.1. A sequence $(s_k)_{k=0}^{\infty}$ in E is said to be positive if

$$\sum_{k,\, l=0}^{n} \sigma_k \sigma_l s_{k+l} \geqslant 0 \quad ((\sigma_k)_{k=0}^{\infty} \subset \mathbb{R};\ n := 0, 1, \dots),$$

and positive in Stieltjes sense if

$$\sum_{k,l=0}^{n} \sigma_k \sigma_l s_{k+l} \geqslant 0, \ \sum_{k,l=0}^{n} \sigma_k \sigma_l s_{k+l+1} \geqslant 0 \quad ((\sigma_k)_{k=0}^{\infty}\, n := 0, 1, \dots).$$

5.2. Theorem. *Given a sequence $(s_k)_{k=0}^{\infty} \subset E$, there exists a positive measure $\mu : \mathcal{B}(\mathbb{R}) \to E$ which solves the Stieltjes moment problem if and only if the sequence is positive in Stieltjes sense.*

5.3. Theorem. *For every positive in Stieltjes sense sequence $(s_k)_{k=0}^{\infty}$ there exists a sequence of pairwise disjoint principal band projections $(\pi)_{k=0}^{\infty}$ and a band projection π_h*

with $\pi_h \circ \pi_k = 0$ $(k := 0, 1, \dots)$ and $\pi_h + \sum_{k=0}^{\infty} \pi_k = I_E$ for which the following statements hold:

(1) *for the sequence $(\pi_0 s_k)_{k=0}^{\infty}$ in $\pi_0 E$ the Stieltjes moment problem has a unique solution;*

(2) *for the sequence $(\pi s_k)_{k=0}^{\infty}$ in πE solution to the Stieltjes moment problem is not unique for whatever nonzero band projection $\pi \leqslant \pi_h$;*

(3) *for the sequence $(\pi_n s_k)_{k=0}^{\infty}$ in $\pi_n E$ the Stieltjes moment problem has a unique solution which can be represented as a linear combination of n disjoint spectral measures, whatever $n \in \mathbb{N}$.*

5.4. The band projections in Theorem 5.3 can be described explicitly. Let $\mathcal{P}(\mathbb{R}^+)$ be the vector space of all polynomials defined on $\mathbb{R}^+$. We introduce a positive linear operator $U : \mathcal{P}(\mathbb{R}^+) \to E$ defined by

$$U(p) := \sum_{k=0}^{n} a_k s_k, \ p(u) = \sum_{k=0}^{n} a_k u^k.$$

Fix an arbitrary complex number $\lambda \in \mathbb{C} \backslash \mathbb{R}^+$ and consider the function $R_\lambda(u) = \Re 1/(u - \lambda)$.

We define the following vector in E:

$$a := \inf\{U(p - q) : p, q \in \mathcal{P}(\mathbb{R}^+), \ q \leqslant R_\lambda \leqslant p\}.$$

Then π_h coincides with the band projection onto the band $\{a\}^{\perp\perp}$. For an arbitrary $(\sigma_k)_{k=0}^{n}$ in $\mathbb{R}$ we put

$$E(\sigma_0, \dots, \sigma_n) = \left\{ \sum_{k,l=0}^{n} s_{k+l} \sigma_k \sigma_l \right\}^{\perp\perp}.$$

Let $E_n = \bigcap \{E(\sigma_0, \dots, \sigma_n) : \sigma_0^2 + \dots + \sigma_n^2 > 0\}$. Then π_0 coincides with the projection onto the band $\bigcap_{n+0}^{\infty} E_n$ and π_n coincides with the projection onto the band $E_{n-1} \cap E_n^{\perp}$.

5.5. Analogous results are true for the Hamburger and Hausdorff moment problems. The main difficulty is to find an appropriate measure extension in vector lattices, see [9, 10].

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